THE LIOUVILLE THEOREM FOR A CLASS OF FOURIER MULTIPLIERS AND ITS CONNECTION TO COUPLING

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Abstract. The classical Liouville property says that all bounded harmonic functions in $\mathbb{R}^n$, i.e. all bounded functions satisfying $\Delta f = 0$, are constant. In this paper we obtain necessary and sufficient conditions on the symbol of a Fourier multiplier operator $m(D)$, such that the solutions f to $m(D)f = 0$ are Lebesgue a.e. constant (if f is bounded) or coincide Lebesgue a.e. with a polynomial (if f is polynomially bounded). The class of Fourier multipliers includes the (in general non-local) generators of Lévy processes. For generators of Lévy processes, we obtain necessary and sufficient conditions for a strong Liouville theorem where f is positive and grows at most exponentially fast. As an application of our results above we prove a coupling result for space-time Lévy processes.

The classical Liouville property for the Laplace operator $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on $\mathbb{R}^n$ can be stated in the following way:

$$f \in L^\infty(\mathbb{R}^n) \ \&\ \Delta f = 0 \implies f \equiv c \quad \text{almost everywhere.} \tag{1}$$

Of course, Δf has to be interpreted in a suitable sense since f lacks regularity for a pointwise interpretation. Either one uses a mollifying argument based on convolution along with the fact that convolution and Δ commute or, as we do here, understands Δf as a Schwartz distribution, i.e. Δf is defined as the continuous linear form $\varphi \mapsto \langle f, \Delta\varphi\rangle$ for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ (the smooth functions with compact support) or $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the rapidly decreasing smooth functions).

Recently, Alibaud *et al.* [1] and, independently, two of us [3] gave proofs providing necessary and sufficient conditions ensuring an analogue of the Liouville property (1) for infinitesimal generators of Lévy processes. The proof in [1] combines harmonic analysis and further methods from group theory, while the approach in [3] uses mainly probabilistic arguments; the latter proof also yields the strong Liouville property where, in the appropriate analogue to (1), $f \in L^\infty(\mathbb{R}^n)$ is relaxed to $f \geq 0$ and at most exponential growth at infinity. Sufficient conditions for a 'polynomial' Liouville property (if f is polynomially bounded, then f coincides a.e. with a polynomial) are due to Kühn [14].

In the present paper, we give a very short and purely analytic proof for both the Liouville property and the polynomial Liouville property for Lévy generators and – as it turns out – a much larger class of Fourier multiplier operators. In fact, the necessary and sufficient condition for the Liouville property is that $\xi = 0$ is the only zero of the multiplier $m(\xi)$. For generators of Lévy processes we refine the strong Liouville result proved in [3] and we establish a further probabilistic interpretation of the Liouville property for Lévy generators in terms of coupling and space-time harmonic functions.

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Notation. Most of our notation is standard or self-explanatory. We write $\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = (2\pi)^{-n}\int_{\mathbb{R}}^n e^{-ix\cdot\xi}\varphi(x)\,dx$ and $\mathcal{F}^{-1}u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}u(\xi)\,d\xi$ for the Fourier and the inverse Fourier transform. We denote by $(T_h\varphi)(x) := \varphi(x-h)$ the shift by $h \in \mathbb{R}^n$, $\widetilde{\varphi}(x) := \varphi(-x)$ is the reflection at the origin, and $\Lambda(x) := (1+|x|^2)^{1/2}$ is the standard weight function.

1. The proof of the Liouville theorem from [3] revisited

We will need a few notions from probability theory and stochastic processes, which can be found in Sato [18] or Jacob [10], [11] and [12], but the essential ingredient is the structure of the infinitesimal generator, see below. A **Lévy process** is a stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^n$ and sample paths $t \mapsto X_t(\omega)$ which are for almost all ω right-continuous with finite left-hand limits; moreover the random variables $X_{t_k} - X_{t_{k-1}}$, $0 = t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$, are stochastically independent (independent increments) and each increment $X_{t_k} - X_{t_{k-1}}$ has the same distribution as $X_{t_k - t_{k-1}}$ (stationary increments). The fact that we are looking at a process with independent and stationary increments means that the distribution of X_t (for any fixed $t > 0$) characterizes the whole process; moreover, X_t is necessarily infinitely divisible, so that its characteristic function (inverse Fourier transform) is of the form

$$\mathbb{E}e^{i\xi\cdot X_t} = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^n,\ t > 0, \tag{2}$$

where the **characteristic exponent** $\psi(\xi)$ is uniquely given by the Lévy–Khintchine formula

$$\psi(\xi) = -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int\limits_{0<|y|<1}\left(1 - e^{iy\cdot\xi} + iy\cdot\xi\right)\nu(dy) + \int\limits_{|y|\geq 1}\left(1 - e^{iy\cdot\xi}\right)\nu(dy). \tag{3}$$

The so-called Lévy triplet (b, Q, ν) comprising a vector $b \in \mathbb{R}^n$, a symmetric, positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$, and a measure ν such that $\int_{y\neq 0}\min\{1, |y|^2\}\,\nu(dy) < \infty$, characterizes ψ uniquely. For example, taking $b = 0, Q = \mathrm{id}, \nu \equiv 0$ we get Brownian motion with its characteristic exponent $\psi(\xi) = \frac{1}{2}|\xi|^2$, and the choice $b = 0, Q = 0, \nu(dy) = c_\alpha|y|^{-n-\alpha}\,dy$ with a suitable constant c_α yields a rotationally symmetric α-stable process with characteristic exponent $|\xi|^\alpha$, $0 < \alpha < 2$.

Since Lévy processes are Markov processes, their transition behaviour can be described by a transition semigroup $\mathcal{P}_t f(x) = \mathbb{E}f(X_t + x)$ which, in turn, is uniquely characterized by the infinitesimal generator $\mathcal{L}f := \frac{d}{dt}\mathcal{P}_t f\big|_{t=0}$ (in the Banach space $C_\infty(\mathbb{R}^n)$ of all continuous functions vanishing at infinity, say). Now the key point is the following observation:

Fact. *The infinitesimal generator $\mathcal{L} = \mathcal{L}_\psi$ of a Lévy process with characteristic exponent $\psi(\xi)$ is on $C_c^\infty(\mathbb{R}^n)$ a* ***Fourier multiplier operator*** *with symbol $-\psi(\xi)$, i.e.*

$$\mathcal{L}_\psi\varphi(x) = -\psi(D)\varphi(x) = \mathcal{F}^{-1}_{\xi\to x}\left(-\psi(\xi)\widehat{\varphi}(\xi)\right), \quad \varphi \in C_c(\mathbb{R}^n), \tag{4}$$

and, combining this with (3),

$$\begin{aligned}\mathcal{L}_\psi\varphi(x) = {}& b\cdot\nabla\varphi(x) + \frac{1}{2}\nabla\cdot Q\nabla\varphi(x)\\ &+ \int_{0<|y|<1}\left(\varphi(x+y) - \varphi(x) - y\cdot\nabla\varphi(x)\right)\nu(dy) + \int_{|y|\geq 1}\left(\varphi(x+y) - \varphi(x)\right)\nu(dy).\end{aligned} \tag{5}$$

Note that we get $\mathcal{L}_\psi = \frac{1}{2}\Delta$, if $\psi(\xi) = \frac{1}{2}|\xi|^2$ (Brownian motion) and $\mathcal{L}_\psi = -(-\Delta)^{\alpha/2}$, if $\psi(\xi) = |\xi|^\alpha$ (stable process).

Our first aim is to give a purely analytic proof of the following result.

Theorem 1 (Liouville; [1], [3]). *Let ψ be the characteristic exponent of a Lévy process and denote by $\psi(D)$ the corresponding Fourier multiplier operator. Suppose $f \in L^\infty(\mathbb{R}^n)$ is such that $\psi(D)f = 0$ as a distribution, i.e.*

$$\langle f, \widetilde{\psi}(D)\varphi\rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{6}$$

If $\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}$, then $f \equiv$ const Lebesgue almost everywhere.

Conversely, if $f \equiv$ const Lebesgue almost everywhere for every $f \in L^\infty(\mathbb{R}^n)$ satisfying (6), then $\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}$.

A formal proof of the implication

$$\begin{aligned} &\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\} \\ &\quad\implies \quad \text{every bounded solution of } \psi(D)f = 0 \text{ is a.e. constant} \end{aligned}$$

is very easy. Indeed, if $\psi(D)f = 0$, then $\psi(\eta)\widehat{f}(\eta) \equiv 0$. Hence

$$\operatorname{supp} \widehat{f} \subset \big\{\eta \in \mathbb{R}^d \mid \ \psi(\eta) = 0\big\} = \{0\}.$$

Then $\widehat{f} = \sum_{|\alpha|\le N} c_\alpha \partial^\alpha \delta$ for some $N \in \mathbb{Z}_+$ and $c_\alpha \in \mathbb{C}$. Hence f is a polynomial. If it is bounded, it has to be constant. This proof can be made rigorous if ψ is C^∞-smooth (see [3, Theorem 3.2]). However, ψ may be continuous but nowhere differentiable (see [3, Remark 3.3]), in which case defining the product of ψ and the distribution $\widehat{f}$ is by no means trivial.

Our analytic proof of Theorem 1 is based on the following standard result, which is known from the proof of Wiener's Tauberian theorem, cf. Rudin [17, Theorem 9.3].

Theorem 2. *If $f \in L^\infty(\mathbb{R}^n)$, Y is a linear subspace of $L^1(\mathbb{R}^n)$, and*

$$f * g = 0 \quad \text{for every } g \in Y, \tag{7}$$

then the set

$$Z(Y) := \bigcap_{g\in Y} \{\xi \in \mathbb{R}^n \mid \widehat{g}(\xi) = 0\} \tag{8}$$

contains the support of the tempered distribution $\widehat{f}$.

Proof of Liouville's Theorem 1. Since ψ is the characteristic exponent of a Lévy process, $\widetilde{\psi}(D)$ (as well as $\psi(D)$) is a continuous operator from $C_c^\infty(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, see e.g. [19, Lemma 3.4] or the discussion in Example 6 below. Thus, the dual pairing in (6) is well-defined.

Suppose $\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}$. Notice that $T_y\varphi \in C_c^\infty(\mathbb{R}^n)$ for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and all $y \in \mathbb{R}^n$. Therefore, we see from (6) that

$$\left(f * \widetilde{\widetilde{\psi}(D)\varphi}\right)(y) = \langle f, T_y\widetilde{\psi}(D)\varphi\rangle = \langle f, \widetilde{\psi}(D)\,(T_y\varphi)\rangle = 0 \quad \text{for all } y \in \mathbb{R}^n.$$

Hence

$$f * \widetilde{\widetilde{\psi}(D)\varphi} = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

It is easy to see that

$$\begin{aligned}\bigcap_{\varphi\in C_c^\infty(\mathbb{R}^n)}\left\{\eta\in\mathbb{R}^n \mid \widehat{\widetilde{\psi}(D)\varphi}(\eta)=0\right\} &= \bigcap_{\varphi\in C_c^\infty(\mathbb{R}^n)}\left\{\eta\in\mathbb{R}^n \mid \widehat{\widetilde{\psi}(D)}\varphi(-\eta)=0\right\}\\ &= \bigcap_{\varphi\in C_c^\infty(\mathbb{R}^n)}\{\eta\in\mathbb{R}^n \mid \psi(\eta)\widehat{\varphi}(-\eta)=0\}\\ &= \{\eta\in\mathbb{R}^n \mid \psi(\eta)=0\}=\{0\}.\end{aligned}$$

Applying Wiener's theorem (Theorem 2) with

$$Y:=\left\{\widetilde{\widetilde{\psi}(D)\varphi} \mid \varphi\in C_c^\infty(\mathbb{R}^n)\right\}\subset L^1(\mathbb{R}^n),$$

we conclude that the support of the tempered distribution $\widehat{f}$ is contained in $\{0\}$. Therefore, there exist $N\in\mathbb{N}_0$ and $c_\alpha\in\mathbb{C}$ ($\alpha\in\mathbb{N}_0^n$, $|\alpha|\le N$) such that

$$\widehat{f}=\sum_{|\alpha|\le N}c_\alpha\partial^\alpha\delta_0,$$

see, e.g. [17, Thorems 6.24 and 6.25]. Inverting the Fourier transform, and using the assumption that f is bounded, shows that $f\equiv c_0$ Lebesgue a.e.

For the converse direction, we assume the contrary, i.e. that the zero-set $\{\eta\in\mathbb{R}^n \mid \psi(\eta)=0\}$ contains some $\gamma\neq 0$. Then $f(x):=e^{i\gamma\cdot x}$ satisfies $\psi(D_x)e^{i\gamma\cdot x}=e^{i\gamma\cdot x}\psi(\gamma)=0$ for all $x\in\mathbb{R}^n$. Thus, f is a non-constant solution, and we are done. We note in passing that since $\psi(-\gamma)=\overline{\psi(\gamma)}$, $-\gamma\in\{\psi=0\}$, and we can even get a real-valued solution:

$$2\psi(D_x)\cos(\gamma\cdot x)=\psi(D_x)\left(e^{i\gamma\cdot x}+e^{-i\gamma\cdot x}\right)=e^{i\gamma\cdot x}\psi(\gamma)+e^{-i\gamma\cdot x}\psi(-\gamma)=0. \qquad \square$$

Remark 3. With a bit more effort, see [3] or [1], one can show that all bounded solutions in the converse direction of Theorem 1 are necessarily periodic. Since $\{\psi=0\}$ is a closed subgroup of the additive group $(\mathbb{R}^n,+)$, the periodicity group of all bounded solutions is given by the orthogonal subgroup $\{\psi=0\}^{[\perp]}:=\{x\in\mathbb{R}^n \mid e^{i\gamma\cdot x}=1,\ \forall\gamma\in\{\psi=0\}\}$. The proof in [1] actually shows that a Lévy generator $\psi(D)$ has the Liouville property if, and only if, $\{\psi=0\}^{[\perp]}=\mathbb{R}^n$.

The above proof of Theorem 1 extends without change to Fourier multiplier operators that map $C_c^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Theorem 4. *Let $m\in C(\mathbb{R}^n)$ be such the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n)\ni\varphi\mapsto\widetilde{m}(D)\varphi:=\mathcal{F}^{-1}(\widetilde{m}\widehat{\varphi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Suppose $f\in L^\infty(\mathbb{R}^n)$ is such that $m(D)f=0$ as a distribution, i.e.

$$\langle f,\,\widetilde{m}(D)\varphi\rangle=0\quad\text{for all } \varphi\in C_c^\infty(\mathbb{R}^n). \tag{9}$$

If $\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}\subset\{0\}$, then $f\equiv const$ Lebesgue almost everywhere.

Conversely, if $f\equiv const$ Lebesgue almost everywhere for every complex-valued $f\in L^\infty(\mathbb{R}^n)$ satisfying (9), then $\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}\subset\{0\}$. If $m(\eta)=0$ implies that $m(-\eta)=0$, then it is enough to consider real-valued $f\in L^\infty(\mathbb{R}^n)$ satisfying (9).

Remark 5. If $\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}\subsetneqq\{0\}$, i.e. $\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}=\emptyset$, then $f\equiv 0$ is the only constant solution of $m(D)f=0$.

Example 6. Let us give a few examples of multipliers satisfying the key assumption of Theorem 4. The following multipliers κ are such that $\kappa(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

a) κ is a linear combination of terms of the form ab, where a is the Fourier transform of a finite Borel measure on $\mathbb{R}^n$, and all partial derivatives $\partial_\xi^\alpha b$ with $|\alpha| \leq n+1$ are polynomially bounded.

Indeed: Let $b_N := b\Lambda^{-2N}$, $N \in \mathbb{R}^n$. For a sufficiently large N, all partial derivatives $\partial_\xi^\alpha b_N$ with $|\alpha| \leq n+1$ belong to $L^1(\mathbb{R}^n)$. Then $x^\alpha \mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n)$, $|\alpha| \leq n+1$. Hence

$$\begin{aligned} &\mathcal{F}^{-1}(b_N),\ |x_j|^{n+1}\mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n),\ j=1,\dots,n \\ &\implies (1+|x|)^{n+1}\mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n) \implies \mathcal{F}^{-1}(b_N) \in L^1(\mathbb{R}^n). \end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. We have $b_N(D)\varphi = (2\pi)^{-n}\mathcal{F}^{-1}(b_N) * \varphi$, and it follows from Young's inequality

$$\|(\mathcal{F}^{-1}b_N) * \varphi\|_{L^1} \leq \|\mathcal{F}^{-1}b_N\|_{L^1}\|\varphi\|_{L^1}$$

that $b_N(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Since the differential operator $\Lambda^{2N}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into itself, and $(ab)(D) = a(D)b_N(D)\Lambda^{2N}(D)$, it is left to show that $a(D)$ maps $L^1(\mathbb{R}^n)$ into itself. Since $a = \mathcal{F}\mu$ for a finite Borel measure μ,

$$(a(D)g)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} g(x-y)\,\mu(dy), \quad x \in \mathbb{R}^n, \qquad g \in \mathbb{R}^n,$$

and

$$\|a(D)g\|_{L^1} \leq (2\pi)^{-n}\mu(\mathbb{R}^n)\|g\|_{L^1} \quad \text{for all} \quad g \in L^1(\mathbb{R}^n).$$

A particular example is the characteristic exponent of a Lévy process

$$\begin{aligned} \psi(\xi) &= -ib\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int\limits_{0<|y|<1}\left(1-e^{iy\cdot\xi}+iy\cdot\xi\right)\nu(dy) + \int\limits_{|y|\geq 1}\left(1-e^{iy\cdot\xi}\right)\nu(dy) \\ &=: \psi_0(\xi) - \int\limits_{|y|\geq 1} e^{iy\cdot\xi}\,\nu(dy). \end{aligned}$$

The last term in the above formula is the Fourier transform of a finite Borel measure. The smoothness of ψ_0 follows immediately from the differentiation lemma for parameter-dependent integrals and the observation that the integrand of the formally differentiated function $\partial_\xi^\alpha\psi_0$ can be bounded by const $|y|^2$, which is ν-integrable over $\{y \in \mathbb{R}^n :\ 0 < |y| < 1\}$. This bound also shows that $|\partial^\alpha\psi_0(\xi)| \leq c_0(1+|\xi|^2)$ if $|\alpha| = 0$, $|\partial^\alpha\psi_0(\xi)| \leq c_1(1+|\xi|)$ if $|\alpha| = 1$, and $|\partial^\alpha\psi_0(\xi)| \leq c_\ell$ if $|\alpha| = \ell \geq 2$, thus we get polynomial boundedness of ψ_0 and its derivatives. A fully worked-out proof can be found in [3, Lemma 4] as well as [10, Lemma 3.6.22, Theorem 3.7.13].

b) Any κ of the form

$$\kappa(\xi) = \sum_{|\alpha|=0}^{2s} c_\alpha \frac{i^{|\alpha|}}{\alpha!}\xi^\alpha + \int\limits_{0<|y|<1}\left[1-e^{iy\cdot\xi}+\sum_{|\alpha|=0}^{2s-1}\frac{i^{|\alpha|}}{\alpha!}y^\alpha\xi^\alpha\right]\nu(dy) + \int\limits_{|y|\geq 1}\left(1-e^{iy\cdot\xi}\right)\nu(dy)$$

with $s \in \mathbb{N}$, $c_\alpha \in \mathbb{R}$, and a measure ν on $\mathbb{R}^n\backslash\{0\}$ such that $\int_{y\neq 0}\min\{1, |y|^{2s}\}\ \nu(ds) < \infty$. (As usual, for any $\alpha \in \mathbb{N}_0^n$ and $\xi \in \mathbb{R}^n$, we define $\alpha! := \prod_1^n \alpha_k!$ and $\xi^\alpha := \prod_1^n \xi_k^{\alpha_k}$.) The proof of this assertion goes along the lines of Part a).

Functions of this type appear naturally in positivity questions related to generalized functions, see e.g. Gelfand & Vilenkin [7, Chapter II.4] or Wendland [22]. Some authors call the function $-\kappa$ (under suitable additional conditions on c_α's) a **conditionally positive definite function**. Note that $s=1$ is just the Lévy–Khintchine formula (3).

2. The Liouville theorem for polynomially bounded functions

We are now going to show that our argument used in the proof of Theorems 1 and 4 extends to polynomially bounded functions f in (6).

To simplify the presentation, we use the function $\Lambda(x) = (1+|x|^2)^{1/2}$ as well as the following function spaces. Let $\beta \geq 0$, and

$$L^1_\beta(\mathbb{R}^n) := \left\{ g \in L^1(\mathbb{R}^n) \mid \|g\|_{L^1_\beta} := \int_{\mathbb{R}^n} |g(x)|\Lambda(x)^\beta \, dx < \infty \right\},$$

$$L^\infty_{-\beta}(\mathbb{R}^n) := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \mid \|f\|_{L^\infty_{-\beta}} := \left\|\Lambda^{-\beta} f\right\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \Lambda(x)^{-\beta}|f(x)| < \infty \right\}.$$

Obviously, $L^\infty_{-\beta}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n) \subset L^1_\beta(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, and $L^1_\beta(\mathbb{R}^n)$ is a convolution algebra; $A_\beta := \left\{ c_1\delta_0 + c_2 g \mid c_1, c_2 \in \mathbb{R}, g \in L^1_\beta(\mathbb{R}^n) \right\}$ is $L^1_\beta(\mathbb{R}^n)$ with a unit attached, cf. Rudin [17, 10.3(d), 11.13(e)].

We need the following analogue of Theorem 2 for the pair $(L^1_\beta(\mathbb{R}^n), L^\infty_{-\beta}(\mathbb{R}^n))$.

Theorem 7. *If $f \in L^\infty_{-\beta}(\mathbb{R}^n)$, Y is a linear subspace of $L^1_\beta(\mathbb{R}^n)$, and*

$$f * g = 0 \quad \text{for every } g \in Y, \tag{10}$$

then the set

$$Z(Y) := \bigcap_{g\in Y} \{\xi \in \mathbb{R}^n \mid \widehat{g}(\xi) = 0\} \tag{11}$$

contains the support of the tempered distribution $\widehat{f}$.

Proof. Pick any $\xi_0 \in \mathbb{R}^n \setminus Z(Y)$. There exists some $g \in Y$ such that $\widehat{g}(\xi_0) = 1$. Since $\widehat{g}$ is continuous, there is a neighbourhood $V = V(\xi_0)$ of ξ_0 such that $|\widehat{g}(\xi) - 1| < 1/2$ for all $\xi \in V$.

To prove the theorem, it is sufficient to show that $\widehat{f} = 0$ in V, or, equivalently, that $\langle \widehat{f}, \widehat{v} \rangle = 0$ for every $v \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform $\widehat{v}$ has its support in V. Since

$$\langle \widehat{f}, \widehat{v} \rangle = (2\pi)^{-n} \langle f, \widetilde{v} \rangle = (2\pi)^{-n} (f * v)(0),$$

it is sufficient to prove that $f * v(0) = 0$.

Take $\varphi \in C^\infty_c(V)$ such that $0 \leq \varphi \leq 1$, and $\varphi(\xi) = 1$ for every ξ in the support of $\widehat{v}$. Since $\mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$, there exists an element $u \in A_\beta$ such that

$$\widehat{u} = \varphi \widehat{g} + 1 - \varphi.$$

It is easy to see that $\operatorname{Re} \widehat{u}(\xi) > 1/2$ for all $\xi \in \mathbb{R}^n$. Then there exists some $w \in A_\beta$ such that $\widehat{w} = 1/\widehat{u}$, see, e.g. [5, Theorems 1.41 and 2.11] or [6, Theorem 1.3]. Hence

$$\widehat{v} = \widehat{w}\,\widehat{u}\,\widehat{v} = \widehat{w}\,\varphi\,\widehat{g}\,\widehat{v} = \widehat{g}\,\widehat{w}\,\varphi\,\widehat{v}.$$

So, $v = g * G$ for some $G \in L^1_\beta(\mathbb{R}^n)$.([1]) Iterating the standard Peetre inequality $\Lambda(x-y) \leq \sqrt{2}\Lambda(x)\Lambda(y)$, we see that

$$\Lambda(x-y)^\beta \leq 2^\beta \Lambda(x)^\beta \Lambda(y-z)^\beta \Lambda(z)^\beta \quad \text{for all } x, y, z \in \mathbb{R}^n.$$

([1])The equality $v = g * G$ with $G \in L^1(\mathbb{R}^n)$ is derived in the proof of [17, Theorem 9.3]. Unfortunately, this version does not seem sufficient for the proof of the equality $f*(g*G) = (f*g)*G$ when $f \in L^\infty_{-\beta}(\mathbb{R}^n)$.

This yields

$$\begin{aligned}&\int_{\mathbb{R}^n} |f(x-y)| \left(\int_{\mathbb{R}^n} |g(y-z)||G(z)|\,dz\right) dy\\ &\quad\le 2^\beta\Lambda(x)^\beta \int_{\mathbb{R}^n} |f(x-y)|\Lambda(x-y)^{-\beta} \left(\int_{\mathbb{R}^n} |g(y-z)|\Lambda(y-z)^\beta\Lambda(z)^\beta|G(z)|\,dz\right) dy\\ &\quad\le 2^\beta\Lambda(x)^\beta \|f\|_{L^\infty_{-\beta}}\|g\|_{L^1_\beta}\|G\|_{L^1_\beta} < \infty,\end{aligned}$$

and the Fubini–Tonelli theorem implies $f*(g*G) = (f*g)*G$. Finally,

$$f*v = f*(g*G) = (f*g)*G = 0*G = 0. \qquad \square$$

We can now state and prove the Liouville theorem for polynomially bounded functions.

Theorem 8 (Liouville property for polynomially bounded functions)**.** *Let $m \in C(\mathbb{R}^n)$ be such the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \varphi \mapsto \widetilde{m}(D)\varphi := \mathcal{F}^{-1}(\widetilde{m}\widehat{\varphi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_\beta(\mathbb{R}^n)$. Suppose $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e.

$$\langle f, \widetilde{m}(D)\varphi\rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{12}$$

If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then f coincides Lebesgue a.e. with a polynomial of degree at most $\lfloor\beta\rfloor$.

Conversely, if every complex-valued $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ satisfying (12) *coincides Lebesgue a.e. with a polynomial, then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$. If $m(\eta) = 0$ implies that $m(-\eta) = 0$, then it is enough to consider real-valued $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ satisfying* (12)*.*

Proof. If we replace in the proof of Theorem 2 $\psi \rightsquigarrow m$, $L^1(\mathbb{R}^n) \rightsquigarrow L^1_\beta(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n) \rightsquigarrow L^\infty_{-\beta}(\mathbb{R}^n)$, we can follow the argument line-by line up to the point where we get

$$\widehat{f} = \sum_{|\alpha|\le N} c_\alpha \partial^\alpha \delta_0.$$

Again, we invert the Fourier transform and use the polynomial boundedness of f to see that f coincides Lebesgue a.e. with a polynomial of degree less or equal than $\lfloor\beta\rfloor$. The converse statement follows from that in Theorem 4. Indeed, if every $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ satisfying (12) coincides Lebesgue a.e. with a polynomial, then the same is true for any such f in $L^\infty(\mathbb{R}^n) \subseteq L^\infty_{-\beta}(\mathbb{R}^n)$. Since the only polynomials contained in $L^\infty(\mathbb{R}^n)$ are constants, one can apply the converse statement in Theorem 4. $\square$

Example 9. In this example we discuss the multipliers from Example 6 in the setting of Theorem 8. The following conditions ensure that $\kappa(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_\beta(\mathbb{R}^n)$.

a) κ is a linear combination of terms of the form ab, where $a = \mathcal{F}\mu$, μ is a finite Borel measure on $\mathbb{R}^n$ such that $\int \Lambda^\beta(y)\mu(dy) < \infty$, and all partial derivatives $\partial_\xi^\alpha b$ with $|\alpha| \le \lfloor n+\beta\rfloor + 1$ are polynomially bounded.

Indeed: Let $b_N := b\Lambda^{-2N}$, $N \in \mathbb{R}^n$. For a sufficiently large N, all partial derivatives $\partial_\xi^\alpha b_N$ with $|\alpha| \le \lfloor n+\beta\rfloor + 1$ belong to $L^1(\mathbb{R}^n)$. Then $x^\alpha\mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n)$, $|\alpha| \le \lfloor n+\beta\rfloor + 1$. Hence

$$\begin{aligned}&\mathcal{F}^{-1}(b_N),\ |x_j|^{\lfloor n+\beta\rfloor+1}\mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n),\ j = 1,\dots,n\\ &\quad\implies (1+|x|)^{\lfloor n+\beta\rfloor+1}\mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n) \implies \mathcal{F}^{-1}(b_N) \in L^1_\beta(\mathbb{R}^n).\end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n) \subset L^1_\beta(\mathbb{R}^n)$. We have $b_N(D)\varphi = (2\pi)^{-n}\mathcal{F}^{-1}(b_N) * \varphi$, and it follows from Young's and Peetre's inequalities

$$\begin{aligned}\|(\mathcal{F}^{-1}b_N) * \varphi\|_{L^1_\beta} &= \left\|\Lambda^\beta\big((\mathcal{F}^{-1}b_N) * \varphi\big)\right\|_{L^1} \leq 2^{\beta/2}\left\|(\Lambda^\beta\mathcal{F}^{-1}b_N) * (\Lambda^\beta\varphi)\right\|_{L^1}\\ &\leq 2^{\beta/2}\|\Lambda^\beta\mathcal{F}^{-1}b_N\|_{L^1}\|\Lambda^\beta\varphi\|_{L^1} = 2^{\beta/2}\|\mathcal{F}^{-1}b_N\|_{L^1_\beta}\|\varphi\|_{L^1_\beta}\end{aligned}$$

that $b_N(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_\beta(\mathbb{R}^n)$. Since the differential operator $\Lambda^{2N}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into itself, and $(ab)(D) = a(D)b_N(D)\Lambda^{2N}(D)$, it is left to show that $a(D)$ maps $L^1_\beta(\mathbb{R}^n)$ into itself. Using Peetre's inequality again, we deduce from

$$(a(D)g)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} g(x-y)\,\mu(dy), \quad x \in \mathbb{R}^n,\ g \in \mathbb{R}^n,$$

that

$$\begin{aligned}&(2\pi)^n\|a(D)g\|_{L^1_\beta(\mathbb{R}^n)}\\ &\leq \int_{\mathbb{R}}^n \Lambda^\beta(x)\int_{\mathbb{R}}^n |g(x-y)|\,\mu(dy)\,dx \leq 2^{\beta/2}\int_{\mathbb{R}}^n |g(z)|\Lambda^\beta(z)\,dz\int_{\mathbb{R}}^n \Lambda^\beta(y)\,\mu(dy)\\ &= 2^{\beta/2}\int_{\mathbb{R}}^n \Lambda^\beta(y)\,\mu(dy)\,\|g\|_{L^1_\beta}.\end{aligned}$$

With a bit more effort, one can actually show that $\int \Lambda^\beta(y)\,\mu(dy) < \infty$ is also a necessary condition for $a(D)$ to map $C_c^\infty(\mathbb{R}^n)$ into $L^1_\beta(\mathbb{R}^n)$ (see [4, Theorem 3] for a proof).

A particular example is the characteristic exponent $\kappa = \psi$ of a Lévy process such that the Lévy measure has finite moments of order β (cf. Example 6a)).

b) Any κ of the form

$$\kappa(\xi) = \sum_{|\alpha|=0}^{2s} c_\alpha\frac{i^{|\alpha|}}{\alpha!}\xi^\alpha + \int_{0<|y|<1}\left[1 - e^{iy\cdot\xi} + \sum_{|\alpha|=0}^{2s-1}\frac{i^{|\alpha|}}{\alpha!}y^\alpha\xi^\alpha\right]\nu(dy) + \int_{|y|\geq 1}\left(1 - e^{iy\cdot\xi}\right)\nu(dy)$$

with $s \in \mathbb{N}$, $c_\alpha \in \mathbb{R}$, and a measure ν on $\mathbb{R}^n \setminus \{0\}$ such that $\int_{0<|y|<1}|y|^{2s}\,\nu(dy) + \int_{|y|\geq 1}|y|^\beta\,\nu(dy) < \infty$.

It is not difficult to extend Theorem 8 from solutions of the equation $m(D)f = 0$ to solutions of $m(D)f = p$, where p is a polynomial.

Corollary 10. *Let $m \in C(\mathbb{R}^n)$ be as in Theorem 8, and let p be a polynomial. Suppose $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ is such that*

$$\langle f,\, \widetilde{m}(D)\varphi\rangle = \langle p,\, \varphi\rangle \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{13}$$

If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then f coincides Lebesgue a.e. with a polynomial of degree at most $\lfloor\beta\rfloor$. Conversely, if there exists $f \in L^\infty_{-\beta}(\mathbb{R}^n)$ satisfying (13), and every such f coincides Lebesgue a.e. with a polynomial, then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$.

Proof. Pick $k \in \mathbb{N}$ such that $2k$ is greater than the degree of p. Then $\Delta^k p = 0$. Set $m_k(\xi) := |\xi|^{2k}m(\xi)$; clearly, $m_k \in C(\mathbb{R}^n)$. Since Δ^k maps $C_c^\infty(\mathbb{R}^n)$ continuously into itself, $\widetilde{m_k}(D) = \widetilde{m}(D)\Delta^k$ maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_\beta(\mathbb{R}^n)$. Moreover,

$$\langle f,\, \widetilde{m}_k(D)\varphi\rangle = \langle f,\, \widetilde{m}(D)(\Delta^k\varphi)\rangle = \langle p,\, \Delta^k\varphi\rangle = \langle\Delta^k p,\, \varphi\rangle = \langle 0,\, \varphi\rangle = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Now we can apply Theorem 8 with m_k in place of m. Note that $\{m_k = 0\} = \{m = 0\} \cup \{0\}$, i.e. $\{m_k = 0\} \subset \{0\}$ if, and only if, $\{m = 0\} \subset \{0\}$. □

3. The Liouville theorem for slowly growing functions

Theorem 7 covers bounded functions f, while Theorem 8 is about functions whose growth is compared with the growth of a polynomial. This leaves a gap where f grows slower than a polynomial, e.g. at a logarithmic scale. To deal with this case, we need the notion of a measurable, **locally bounded, submultiplicative function**, i.e. a locally bounded measurable function $h : \mathbb{R}^n \to (0, \infty)$ satisfying

$$h(x + y) \leq ch(x)h(y) \quad \text{for some } c \geq 1 \text{ and all } x, y \in \mathbb{R}^n.$$

Without loss of generality, we will always assume that $h \geq 1$, otherwise we would replace $h(x)$ by $h(x)+1$. Typical examples of submultiplicate functions are $(1+|x|)^\beta$, $\Lambda(x)^\beta$, $e^{\alpha|x|^\beta}$ for $\beta \in [0, 1]$, and $\log^\beta(|x| + e)$ for $\beta \geq 0$. Observe that every submultiplicative function is exponentially bounded. For further details see [18, Section 25] or [13, Section II.§1].

Fix some locally bounded submultiplicative h that satisfies, in addition,

$$\lim_{|x|\to\infty} \Lambda^{-k}(x)h(x) = 0 \quad \text{for some } k \in \mathbb{N}. \tag{14}$$

The condition (14) implies the so-called **GRS (Gelfand–Raikov–Shilov)-condition**, see [2] and [6] for details. We replace the pair $(L^1_\beta(\mathbb{R}^n), L^\infty_{-\beta}(\mathbb{R}^n))$ by

$$L^1_h(\mathbb{R}^n) := \left\{ g \in L^1(\mathbb{R}^n) \mid \|g\|_{L^1_h} := \int_{\mathbb{R}^n} |g(x)|h(x)\, dx < \infty \right\},$$
$$L^\infty_{h^{-1}}(\mathbb{R}^n) := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \mid \|f\|_{L^\infty_{h^{-1}}} := \|h^{-1}f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} h(x)^{-1}|f(x)| < \infty \right\}$$

and use, instead of A_β,

$$A_h := \{c_1\delta_0 + c_2 g \mid c_1, c_2 \in \mathbb{R}, g \in L^1_{\tilde{h}}(\mathbb{R}^n)\}.$$

The family A_h is a convolution algebra and, due to the GRS-condition, an element of $u \in A_h$ is invertible if, and only if, it is invertible in A_1, where A_1 is the algebra with $h = 1$. One can replace the Peetre inequality by $h(x - y) \leq ch(x)h(-y)$, which is a direct consequence of the submultiplicativity of h, and then get by iteration $h(x - y) \leq c^2h(x)h(-(y-z))h(-z)$. This allows one to repeat the arguments in the proof of Theorem 7 and arrive at a version of this theorem with $L^1_\beta \rightsquigarrow L^1_{\tilde{h}}$ and $L^1_{-\beta} \rightsquigarrow L^\infty_{h^{-1}}$. If $\widetilde{m}(D)$ maps $C^\infty_c(\mathbb{R}^n)$ into $L^1_h(\mathbb{R}^n)$, one can apply this version of Theorem 7 with

$$Y := \left\{ \widetilde{\widetilde{m}(D)\varphi} \mid \varphi \in C^\infty_c(\mathbb{R}^n) \right\} \subset L^1_{\tilde{h}}(\mathbb{R}^n).$$

This results in the following analogue of Theorem 8 for slowly growing functions.

Theorem 11 (Liouville property for slowly growing functions)**.** *Let $m \in C(\mathbb{R}^n)$ be such the Fourier multiplier operator*

$$C^\infty_c(\mathbb{R}^n) \ni \varphi \mapsto \widetilde{m}(D)\varphi := \mathcal{F}^{-1}(\widetilde{m}\widehat{\varphi})$$

maps $C^\infty_c(\mathbb{R}^n)$ into $L^1_h(\mathbb{R}^n)$. Suppose $f \in L^\infty_{h^{-1}}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e.

$$\langle f, \widetilde{m}(D)\varphi\rangle = 0 \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n). \tag{15}$$

If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then f coincides Lebesgue a.e. with a polynomial $p \in L^\infty_{h^{-1}}(\mathbb{R}^n)$.

Conversely, if f coincides Lebesgue a.e. with a polynomial for every $f \in L^\infty(\mathbb{R}^n)$ satisfying (15), then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$.

Theorem 11 can be used for functions of the form $h(x) = \Lambda^\beta(x)\log(e+|x|)^\alpha$ for $\alpha, \beta \geq 0$. If we use $h(x) = \log(e + |x|)$, we see that in the setting of Theorem 11 every solution $m(D)f = 0$ is a.e. constant.

We want to point out that without the boundedness condition (14) the function f is not necessarily a tempered distribution. For example, if we choose $h(x) = e^{a|x|^\gamma}$, then f need not be a tempered distribution, and our method above would not work. In the next section we discuss functions, which might not define a tempered distributions, but are positive. The condition that $\tilde{m}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into $L_h^1(\mathbb{R}^n)$ is essential for Theorem 11. Recently we learned from M. Kwasnicki [8] that there is a multiplier given by a Lévy process, i.e. $m = \psi$, admitting a very slowly growing non-constant function u such that $\psi(D)u = 0$. Note that this multiplier does not satisfy the mapping property required for Theorem 11.

4. The Liouville theorem for rapidly growing functions

We will now turn to the case where the solution of $\psi(D)f = 0$ is locally bounded and positive. It is well known that

$$f \geq 0, \quad \Delta f = 0 \quad \implies \quad f \equiv \text{const.}$$

This is usually called the **strong Liouville property**. One cannot expect this property to hold for general Fourier multiplier operators discussed in Sections 2 and 3 or even for higher order partial differential operators. Indeed,

$$f(x) := |x|^2 = x_1^2 + \cdots + x_n^2$$

is a non-constant nonnegative polynomially bounded solution of the equation $\Delta^2 f = 0$. Here, $\Delta^2 = \psi(D)$, $\psi(\xi) = |\xi|^4$, and $\{\xi \in \mathbb{R}^n \mid \psi(\xi) = 0\} = \{0\} = \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\}$ (cf. Theorem 13 below). So, we consider in this section Fourier multiplier operators $\psi(D)$ that generate positivity preserving operator semigroups $\left\{e^{-t\psi(D)}\right\}_{t\geq 0}$, i.e. such that ψ are the characteristic exponents of Lévy processes, see Section 1 and Example 6. Even within this class of operators, the Laplacian is a special case, and the strong Liouville property does not hold for more general second order partial differential operators $\mathcal{L}$ without restrictions on the growth rate of a solution f of the equation $\mathcal{L}f = 0$. Let

$$\mathcal{L}u(x) = \frac{1}{2}\nabla \cdot Q\nabla u(x) + b \cdot \nabla u(x),$$

where $Q \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix, and $b \in \mathbb{R}^n \setminus \{0\}$. Then $\mathcal{L}f = 0$ has non-constant nonnegative solutions. Indeed, if $Q \neq 0$, there exists $c_0 \in \mathbb{R}^n$ such that $c_0 \cdot Qc_0 > 0$. Since $b \neq 0$, there exists c in a neighbourhood of c_0 such that $c \cdot Qc > 0$ and $b \cdot c \neq 0$. Let

$$f(x) := e^{\tau c\cdot x}, \quad \text{where} \quad \tau = -\frac{2b\cdot c}{c \cdot Qc} \neq 0.$$

Then $f > 0$ is non-constant and

$$\begin{aligned}\mathcal{L}f(x) &= \frac{1}{2}\nabla \cdot Q\nabla e^{\tau c\cdot x} + b\cdot\nabla e^{\tau c\cdot x} = \left(\frac{1}{2}\tau^2 c\cdot Qc + \tau b\cdot c\right)e^{\tau c\cdot x}\\ &= \tau\left(\frac{1}{2}\tau c\cdot Qc + b\cdot c\right)e^{\tau c\cdot x} = 0.\end{aligned}$$

If $Q = 0$, it is sufficient to take any $c \in \mathbb{R}^n \setminus \{0\}$ such that $b \cdot c = 0$. Then

$$f(x) := e^{c\cdot x}$$

is positive, non-constant, and

$$\mathcal{L}f(x) = b \cdot \nabla e^{c\cdot x} = (b \cdot c)e^{c\cdot x} = 0.$$

It follows from the above that one needs to put appropriate boundedness restrictions on f. Note also that while local boundedness of f is sufficient to ensure that $\langle f, \widetilde{\psi}(D)\varphi\rangle$ is well-defined when $\psi(D)$ is a local operator, i.e. a partial differential operator, one needs boundedness restrictions on f to ensure that $\psi(D)f$ has a meaning when $\psi(D)$ is non-local.

Our current proof is a refinement of our result in [3, Theorem 17], and we focus here on the role of the upper bound. The key ingredient in the proof of [3, Theorem 17] is the equivalence of $-\psi(D)f = 0$ and $e^{-t\psi(D)}f = f$, which is used to get a Choquet representation of all positive solutions $f \geq 0$. In order to define $\psi(D)f$ or $e^{-t\psi(D)}f$ as a distribution, we need a bound $f \leq g$ and an integrability condition on the measure ν appearing in the Lévy–Khintchine representation (3) of ψ, see the discussion in [3]. Here we will concentrate on the role of the upper bound g in the proof of the strong Liouville property which was glossed-over in our presentation in [3, Theorem 17]; this explains, in particular, in which directions ψ can be extended from $\mathbb{R}^n$ into $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$.

Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function. We need to describe the directional growth behaviour of g. For $\omega \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}$, let

$$g_\omega(r) := g(r\omega), \quad r \geq 0.$$

Set

$$\beta(\omega) := \inf_{r>0} \frac{\ln g_\omega(r)}{r} = \lim_{r\to\infty} \frac{\ln g_\omega(r)}{r}.$$

Since $\ln g_\omega$ is subadditive, the infimum is, in fact, a limit; moreover, the function $\beta : \mathbb{S}^{n-1} \to \mathbb{R}$ is continuous, see [9, Theorem 7.13.2].

Applying [13, Chapter II, Theorem 1.3] to the function $v(t) := g_\omega(\ln t)$, $t > 0$, we conclude that

$$g_\omega(r) \geq e^{\beta(\omega)r} \quad \text{for all } r > 0, \tag{16}$$

and that for every $\epsilon > 0$, there is some $r_\epsilon > 0$ such that

$$g_\omega(r) \leq e^{(\beta(\omega)+\epsilon)r} \quad \text{for all } r > r_\epsilon. \tag{17}$$

Let

$$\Pi_g := \left\{\xi \in \mathbb{R}^n \mid \xi \cdot \omega \leq \beta(\omega) \quad \text{for all } \omega \in \mathbb{S}^{n-1}\right\}.$$

Example 12. a) If $g(x) = (1 + |x|)^\lambda$ with $\lambda \geq 0$, or $g(x) = e^{\alpha|x|^\gamma}$ with $\alpha \geq 0$, $\gamma \in [0, 1)$, then $\beta(\omega) \equiv 0$ and $\Pi_g = \{0\}$.

b) If $g(x) = e^{\alpha|x|}$ with $\alpha > 0$, then $\beta(\omega) \equiv \alpha$ and

$$\Pi_g = \{\eta \in \mathbb{R}^n \mid |\eta| \leq \alpha\}.$$

c) If $g(x) := \max\{e^{x_1}, 1\}$, then $\beta(\omega) = \max\{\omega_1, 0\}$, and it is easy to see that

$$\begin{aligned}\Pi_g &= \left\{\xi \in \mathbb{R}^n \mid \xi_1\omega_1 + \sum_{j=2}^{d} \xi_j\omega_j \leq \max\{\omega_1, 0\} \quad \text{for all } \omega \in \mathbb{S}^{n-1}\right\}\\ &= \{\xi = (\xi_1, 0, \dots, 0) \in \mathbb{R}^n \mid \xi_1\omega_1 \leq \max\{\omega_1, 0\} \quad \text{for all } \omega_1 \in [-1, 1]\}\\ &= \{\xi = (\xi_1, 0, \dots, 0) \in \mathbb{R}^n \mid \xi_1 \in [0, 1]\},\end{aligned}$$

i.e. Π_g is the one-dimensional interval $[0, 1] \times \{0\} \times \cdots \times \{0\}$.

Theorem 13 (strong Liouville property)**.** *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be the characteristic exponent of a Lévy process with Lévy triplet (b, Q, ν). Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function such that $\int_{|y|\geq 1} g(y)\,\nu(dy) < \infty$. Every measurable, positive and g-bounded ($0 \leq f \leq g$) weak solution f of the equation $\psi(D)f = 0$ is constant if, and only if,*

$$\{\xi \in \mathbb{R}^n \mid \psi(\xi) = 0\} = \{0\} = \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\} \cap \Pi_g. \tag{18}$$

Proof. Sufficiency of (18). Assume that the first equality in (18) holds, $0 \leq f \leq g$, and $\psi(D)f = 0$. Using Choquet's theorem, just as in the proof of [3, Theorem 17], we get the following representation of f: there exists a measure ρ with support in $E := \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\}$ such that

$$f(x) = \int_E e^{x\cdot\xi}\,\rho(d\xi).$$

We will show that $\operatorname{supp}\rho \subset \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\} \cap \Pi_g$; thus, the second equality in (18) proves that $f \equiv \rho(\{0\})$ a.s.

Suppose there exists some $\xi^0 \in \operatorname{supp}\rho \setminus \Pi_g$. This means that there is some $\omega^0 \in \mathbb{S}^{n-1}$ such that $\xi^0 \cdot \omega^0 > \beta(\omega^0)$. Take any $0 < \epsilon < \frac{1}{2}(\xi^0 \cdot \omega^0 - \beta(\omega^0))$ and consider the open ball $B_\epsilon := B_\epsilon(\xi^0)$ of radius ϵ centred at ξ^0. Let $x = r\omega^0$, $r > 0$. Then

$$x \cdot \xi = x \cdot \xi^0 + x \cdot (\xi - \xi^0) \geq (\xi^0 \cdot \omega^0)r - r|\xi - \xi^0| > (\xi^0 \cdot \omega^0 - \epsilon)r \quad \text{for all } \xi \in B_\epsilon,$$

and there exists some $r_0 > 0$, depending only on g, ω_0, and ϵ (see (17)), such that

$$\frac{e^{x\cdot\xi}}{g_{\omega^0}(r)} > \frac{e^{(\xi^0\cdot\omega^0-\epsilon)r}}{g_{\omega^0}(r)} \geq \frac{e^{(\xi^0\cdot\omega^0-\epsilon)r}}{e^{(\beta(\omega^0)+\epsilon)r}} = e^{(\xi^0\cdot\omega^0-\beta(\omega^0)-2\epsilon)r} \quad \text{for all } r \geq r_0.$$

Since $\xi^0 \in \operatorname{supp}\rho$, we know that $\rho(B_\epsilon) > 0$. Thus,

$$\begin{aligned}\frac{f(x)}{g(x)} = \frac{\int_E e^{x\cdot\xi}\,\rho(d\xi)}{g(x)} \geq \frac{\int_{B_\epsilon} e^{x\cdot\xi}\,\rho(d\xi)}{g_{\omega^0}(r)} &> \frac{e^{(\xi^0\cdot\omega^0-\beta(\omega^0)-2\epsilon)r} g_{\omega^0}(r)\rho(B_\epsilon)}{g_{\omega^0}(r)} \\ &= e^{(\xi^0\cdot\omega^0-\beta(\omega^0)-2\epsilon)r}\rho(B_\epsilon) \to \infty \quad \text{as } r \to \infty.\end{aligned}$$

This contradicts the bound $0 \leq f \leq g$, and we conclude that $\operatorname{supp}\rho \setminus \Pi_g = \emptyset$, hence $\operatorname{supp}\rho \subset \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\} \cap \Pi_g$.

Necessity of (18). Suppose every measurable, positive and g-bounded weak solution f of $\psi(D)f = 0$ is constant. If the first equality in (18) does not hold, then there exists $\xi^0 \in \{\xi \in \mathbb{R}^d : \psi(\xi) = 0\} \setminus \{0\}$. Since $\psi(-\xi^0) = \overline{\psi(\xi^0)} = 0$ and $\psi(0) = 0$, one has $-\xi^0, 0 \in \{\xi \in \mathbb{R}^d : \psi(\xi) = 0\}$. Let $e_{\pm\xi^0}(x) := e^{\pm i\xi^0\cdot x}$. Then $\psi(D)e_{\pm\xi^0} = \psi(\pm\xi^0)e_{\pm\xi^0} = 0$ and $\psi(D)1 = \psi(0) = 0$. Hence

$$f(x) := \frac{1}{2}\big(1 + \cos(\xi^0 \cdot x)\big)$$

is a non-constant measurable, positive and g-bounded weak solution f of $\psi(D)f = 0$. This contradiction proves the first equality in (18).

Suppose now the second equality in (18) does not hold, i.e. there exists a nonzero $\theta \in \{\eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0\} \cap \Pi_g$. From the definition of Π_g we see that, for every $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$ and $r \geq 0$,

$$f(x) := e^{\theta\cdot x} = e^{(\theta\cdot\omega)r} \leq e^{\beta(\omega)r} \leq g_\omega(r) = g(r\omega) = g(x),$$

see (16). So, f is non-constant, positive and g-bounded. Since $\psi(D)f = \psi(-i\theta)f \equiv 0$, we get again a contradiction. □

The following result shows that the polynomial f appearing in Theorem 8 has degree at most 1 in the case where $n = 1$ and m is the characteristic exponent of a Lévy process.

Theorem 14. *Let $\psi : \mathbb{R} \to \mathbb{C}$ be the characteristic exponent of a Lévy process with Lévy triplet (Q, b, ν). Suppose there exists a $\beta \geq 0$ such that $\int_{|y|\geq 1} |y|^\beta \, \nu(dy) < \infty$. If $\{\xi \in \mathbb{R} \mid \psi(\xi) = 0\} = \{0\}$ and $f \in L^\infty_{-\beta}(\mathbb{R})$ is a weak solution of the equation $\psi(D)f = 0$, then $f(x) = c_1 x + c_0$ with some constants c_1 and c_0.*

Proof. Since $\psi(D)$ maps real-valued functions onto real-valued ones (see (5)), the real and the imaginary parts of any solution of $\psi(D)f = 0$ satisfy the same equation. Hence we can assume without loss of generality that f is real-valued.

It follows from Theorem 8 that f is a polynomial of degree at most $[\beta]$. In fact, the degree of f is actually less than or equal to 2 (see [15]), i.e.

$$f(x) = c_2 x^2 + c_1 x + c_0$$

with some constants $c_2, c_1, c_0 \in \mathbb{R}$.

Suppose $c_2 \neq 0$. Since $-f$ is also a solution, we can assume without loss of generality that $c_2 > 0$.

Since $\psi(0) = 0$, we have $\psi(D)C = 0$ for every constant C, and hence $\psi(D)(f + C) = 0$. Choosing a sufficiently large $C \geq 0$, we get $f + C \geq 0$. Then Theorem 13 implies that the quadratic polynomial $f + C$ is constant, which contradicts the assumption $c_2 \neq 0$. So, $c_2 = 0$. □

The above result does not hold in the multi-dimensional case $n \geq 2$. Note that in the case $n = 1$, $f'' = 0 \implies f(x) = ax + b$, while in the case $n = 2$, $\Delta f = 0$ has polynomial solutions of any degree, e.g. $\mathrm{Re}(x_1 + i x_2)^k$, $k \in \mathbb{N}$.

5. Coupling

In this section we want to establish a connection between our Livouville theorem (Theorem 1), the coupling property of Lévy processes, and the notion of space-time harmonic functions. Throughout, $(X_t)_{t\geq 0}$ is a Lévy process starting at 0 with characteristic exponent ψ, see Section 1.

We write $X_t^x := X_t + x$ to indicate that the Lévy process starts at the position $X_0^x = x \in \mathbb{R}^n$. The **(exact) coupling** property of a Lévy process (or a general Markov process) says that for *any two* starting points $x, y \in \mathbb{R}^n$ the trajectories of the processes $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$ meet with probability one in finite time; this is the **coupling time** $\tau = \tau^{x,y}$. Because of the Markov property, both processes $(X_t^x)_{t\geq 0}$ and $(X_t^y)_{t\geq 0}$, run independently from each other on the same probability space, move together and cannot (statistically) be distinguished from each other. Coupling techniques provide powerful tools to study the regularity of the semigroup $x \mapsto \mathcal{P}_t f(x) = \mathbb{E} f(X_t^x)$, the existence of invariant (stationary) measures for $(X_t^x)_{t\geq 0, x\in\mathbb{R}^n}$ and many further properties, see the discussion in [21]. It was shown in [20, Theorem 4.1] that a Lévy process has the coupling property if, and only if, the transition probability $\mathbb{P}(X_t \in dy) = p_t(dy)$ has an absolutely continuous component.

If $(X_t^x)_{t\geq 0, x\in\mathbb{R}^n}$ is a Lévy process (or a general Markov process with generator $\mathcal{A}_x$), the space-time process $((s + t, X_t^x))_{t\geq 0, (s,x)\in[0,\infty)\times\mathbb{R}^n}$ is again a Lévy process (resp., Markov process), and its semigroup is given by $\mathcal{Q}_t u(s, x) = \mathbb{E}(s + t, X_t^x)$. Thus, the infinitesimal generator is of the form $\frac{d}{ds} - \psi(D_x)$ (resp. $\frac{d}{ds} + \mathcal{A}_x$), and we are naturally led to the notion of space-time harmonic functions and the space-time Liouville property.

Definition 15. Let $(X_t^x)_{t\geq 0, x\in\mathbb{R}^n}$ be a Lévy process.

a) A function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is **space-time harmonic**, if one has $f(s, x) = \mathbb{E}^x f(s + t, X_t) = \mathbb{E} f(s + t, X_t + x)$ for *every* $t, s \geq 0$ and *every* $x \in \mathbb{R}^n$.

b) The process has the **space-time Liouville property**, if every measurable and bounded space-time harmonic function is constant.

Remark 16. a) The space-time Liouville property is known to be equivalent to the exact coupling property for Markov processes, see [21, Theorem 4.1, p. 205].
b) We call a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying $f(s, x) = \mathbb{E}^x f(s + t, X_t)$ for every $t \geq 0$ and $(s, x) \in \mathbb{R}^{n+1}$ also space-time harmonic.
c) Notice that the notions of 'space-time harmonicity' and 'exact coupling' are pointwise defined notions, which do not allow for exceptional sets. This is the main difficulty when we want to compare our notion of the Liouville property and the space-time Liouville property.

For a Lévy triplet (b, Q, ν) with Lévy process $X = (X_t)_{t\geq 0}$ in $\mathbb{R}^n$ and a vector $\eta \in \mathbb{R}^n$ we introduce the notation

$$b^\eta := \eta \cdot b + \int \eta \cdot y \left(\mathbb{1}_{(0,1)}(|\eta \cdot y|) - \mathbb{1}_{(0,1)}(|y|) \right) \nu(dy).$$

This is the vector in the Lévy triplet of $\eta \cdot X_t$, see [18, Proposition 11.10]. We need a few auxiliary lemmas preparing the proof of the main result of this section, Theorem 22.

Lemma 17. *Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be the characteristic exponent of a Lévy process with Lévy triplet (b, Q, ν). If $\xi \in \mathbb{R}^n$ is such that $\psi(\xi) \in i\mathbb{R}$, then $\psi(\xi) = -ib^\xi$.*

Proof. Let

$$\mathbb{E}_\xi := \left\{ y \in \mathbb{R}^n \mid 1 - e^{iy\cdot\xi} = 0 \right\} = \{ y \in \mathbb{R}^n \mid y \cdot \xi \in 2\pi\mathbb{Z} \},$$
$$\mathbb{E}^0_\xi := \{ y \in \mathbb{R}^n \mid y \cdot \xi = 0 \} = \{\xi\}^\perp.$$

Since $\operatorname{Re}\left(1 - e^{iy\cdot\xi}\right) \geq 0$, and $\operatorname{Re}\left(1 - e^{iy\cdot\xi}\right) = 0$ if, and only if, $1 - e^{iy\cdot\xi} = 0$, it follows from $\psi(\xi) \in i\mathbb{R}$ and (3) that $\nu(\mathbb{R}^n \setminus \mathbb{E}_\xi) = 0$.

If $y \in \mathbb{E}_\xi \setminus \mathbb{E}^0_\xi$, then $|y \cdot \xi| \geq 2\pi > 1$. So, $\mathbb{1}_{(0,1)}(|\xi \cdot y|) = 0$ if $y \in \mathbb{E}_\xi$. Hence

$$\begin{aligned}
\psi(\xi) &= -ib \cdot \xi + \int_{\mathbb{R}^n} \left(1 - e^{iy\cdot\xi} + iy \cdot \xi\right) \mathbb{1}_{(0,1)}(|y|)\, \nu(dy) \\
&= -ib \cdot \xi + \int_{\mathbb{E}_\xi} \left(1 - e^{iy\cdot\xi} + iy \cdot \xi\right) \mathbb{1}_{(0,1)}(|y|)\, \nu(dy) \\
&= -ib \cdot \xi + \int_{\mathbb{E}_\xi} \left(1 - e^{iy\cdot\xi} + iy \cdot \xi\right) \left(\mathbb{1}_{(0,1)}(|y|) - \mathbb{1}_{(0,1)}(|\xi \cdot y|) \right) \nu(dy) \\
&= -ib \cdot \xi + i \int_{\mathbb{E}_\xi} y \cdot \xi \left(\mathbb{1}_{(0,1)}(|y|) - \mathbb{1}_{(0,1)}(|\xi \cdot y|) \right) \nu(dy) \\
&= -ib \cdot \xi + i \int_{\mathbb{R}^n} y \cdot \xi \left(\mathbb{1}_{(0,1)}(|y|) - \mathbb{1}_{(0,1)}(|\xi \cdot y|) \right) \nu(dy) = -ib^\xi.
\end{aligned}$$

□

Lemma 18. *Let $n, d \in \mathbb{N}$, $\psi_1 : \mathbb{R}^d \to \mathbb{C}$ and $\psi_2 : \mathbb{R}^n \to \mathbb{C}$ be characteristic exponents of two Lévy processes with Lévy triplets (b_1, Q_1, ν_1) and (b_2, Q_2, ν_2). Then*

$$\left\{ (\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^n \mid \psi_1(\eta) + \psi_2(\xi) = 0 \right\} = \left\{ \eta \in \psi_1^{-1}(i\mathbb{R}),\ \xi \in \psi_2^{-1}(i\mathbb{R}) \;\middle|\; b_1^\eta + b_2^\xi = 0 \right\}.$$

Proof. Since $\operatorname{Re}\psi_i \geq 0$ for $i = 1, 2$ (see (3)), a necessary condition for $\psi_1(\eta) + \psi_2(\xi) = 0$ is $\operatorname{Re}\psi_1(\eta) = 0$ and $\operatorname{Re}\psi_2(\xi) = 0$. In this case, $\psi_1(\eta) = -ib_1^\xi$ and $\psi_2(\xi) = -ib_2^\xi$ (see Lemma 17), and the equality $\psi_1(\eta) + \psi_2(\xi) = 0$ is equivalent to $b_1^\eta + b_2^\xi = 0$. □

Remark 19. Assume that $d = 1$ and $\psi_1(\eta) = -ib\eta$ for some $b \in \mathbb{R} \setminus \{0\}$. We see easily that the set $\{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \psi_1(\eta) + \psi_2(\xi) = 0\}$ is equal to $\{(0,0)\}$ if, and only if, $\psi_2^{-1}(i\mathbb{R}) = \{0\}$.

Lemma 20. *Let $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a space-time harmonic function for the Lévy process $(X_t)_{t\geq 0}$. Then there exists a unique extension $\tilde{f} : (-\infty,\infty) \times \mathbb{R}^n \to \mathbb{R}$, which is still space-time harmonic.*

Proof. Assume that $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is such that

$$f(s,x) = \mathbb{E} f(s+t, x+X_t) \quad \text{for all } s,t \geq 0,\ x \in \mathbb{R}^n.$$

We define for $s > 0$ and $x \in \mathbb{R}^n$

$$f(-s,x) := \mathbb{E} f(0, x+X_s).$$

Let $s > t > 0$. By the Markov property and the stationary increments property of X_t we obtain that

$$\begin{aligned}\mathbb{E} f(-s+t, x+X_t) = \int \mathbb{E}\left(f(0,(x+X_t)+y)\right)\Big|_{y=X_{s-t}} d\mathbb{P} &= \mathbb{E} f(0,(x+X_t)+(X_s-X_t)) \\ &= \mathbb{E} f(0,x+X_s) =: f(-s,x).\end{aligned}$$

Now let $t > s > 0$. Again by the Markov property and the stationarity of the increments we see that

$$\mathbb{E} f(-s+t, x+X_t) = \mathbb{E} f(t-s, x+(X_t-X_s)+X_s) = \mathbb{E} f(0,x+X_s) =: f(-s,x). \quad \square$$

Recall that a Lévy process has the strong Feller property – i.e. $x \mapsto \mathbb{E} u(x+X_t)$ is a continuous function for every bounded measurable u – if, and only if, the transition probability $\mathbb{P}(X_t \in dy)$ is absolutely continuous w.r.t. Lebesgue measure, cf. Jacob [10, Lemma 4.8.20].

Lemma 21. *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic exponent ψ_X and let $(S_t)_{t\geq 0}$ be an independent subordinator*(2)*with Laplace transform $\mathbb{E} e^{-xS_t} = e^{-tf_S(x)}$, $x \geq 0$. If both X and S are strong Feller processes, then the subordinated space-time process (S_t, X_{S_t}) is a strong Feller Lévy process; its characteristic exponent is given by*

$$f_S(-i\tau + \psi_X(\xi)), \quad (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{19}$$

Proof. The process $Y = (Y_t)_{t\geq 0} := ((S_t, X_{S_t}))_{t\geq 0}$ can be seen as subordination of the space-time Lévy process $((t,X_t))_{t\geq 0}$, from which we conclude that Y is indeed a Lévy process. Furthermore, Y_t has for every $t > 0$ a transition density $p_{Y_t}(s,x)\,ds\,dx$, which is given by

$$p_{Y_t}(s,x) = p_{X_s}(x) p_{S_t}(s) \mathbb{1}_{[0,\infty)}(s).$$

This implies that Y has the strong Feller property. That the symbol of Y is given by (19) is a direct consequence of the subordination of the process $((t,X_t))_{t\geq 0}$:

$$\mathbb{E}\left[e^{i\tau S_t + i\xi\cdot X_{S_t}}\right] = \mathbb{E}\left[\left(\mathbb{E} e^{i\tau r + i\xi X_r}\right)\Big|_{r=S_t}\right] = \mathbb{E}\left[e^{-(-i\tau+\psi_X(\xi))S_t}\right] = e^{-tf_S(-i\tau+\psi_X(\xi))}. \quad \square$$

Theorem 22. *Let $(X_t)_{t\geq 0}$ be a strong Feller Lévy process with characteristic exponent ψ_X. Then the following assertions are equivalent:*

a) $(X_t)_{t\geq 0}$ *has the (exact) coupling property,*
b) $(X_t)_{t\geq 0}$ *has the space-time Liouville property,*
c) $(t, X_t)_{t\geq 0}$ *has the Liouville property (as in Theorem 1),*
d) $\{\xi \in \mathbb{R}^n \mid \psi_X(\xi) \in i\mathbb{R}\} = \{0\}$.

(2)A **subordinator** is a one-dimensional Lévy process with increasing sample paths.

Proof. a)⇔b) is due to Thorisson [21, Theorem 4.5, p. 205].

c)⇔d) is due to Theorem 1, (the proof of) Lemma 21 for $f_s(\lambda) = \lambda$ and Remark 19.

b)⇒c): let u be a bounded measurable function such that $\left(\frac{d}{ds} - \psi_X(D_x)\right) u = 0$ in the sense of distributions. If $\mathcal{P}_t$ is the semigroup generated by $\frac{d}{ds} - \psi(D_x)$, we know from the relation between semigroup and generator that

$$\mathcal{P}_t u(s,x) = u(s,x) + \int_0^t \mathcal{P}_r \left(\frac{d}{ds} - \psi_X(D_x)\right) u(s,x)\, dr = u(s,x)$$

$t > 0$ and all $(s,x) \in \mathbb{R} \times \mathbb{R}^n$ in the sense of distributions. Since $u(s,x) = \mathbb{E}u(s+t, x+X_t)$ does not depend on $t > 0$, we have

$$u(s,x) = \int_0^\infty \underbrace{\mathbb{E}u(s+r, x+X_r)}_{=u(s,x)} \mathbb{P}(S_t \in dr) = \mathbb{E}u(s+S_t, x+X_{S_t}),$$

where $S = (S_t)_{t \geq 0}$ is a 1/2-stable subordinator.(3) By Lemma 21, (S_t, X_{S_t}) has again the strong Feller property and hence, we choose u to be continuous, and the equality $\mathcal{P}_t u(s,x) = u(s,x)$ holds pointwise for every $(s,x) \in \mathbb{R} \times \mathbb{R}^n$; in particular, u is a bounded and continuous function. Setting $f(t,x) := u(t,x)$ it is clear that f is space-time harmonic, hence constant.

c)⇒b): We have to show that any space-time harmonic function $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is constant. Fix such an f and construct, as in Lemma 20, its unique extension to the negative real line; this extension is still space-time harmonic. By a similar argument as before we see that $f(s,x) = \mathbb{E}f(s+S_t, x+X_{S_t})$ pointwise for every $(s,x) \in \mathbb{R} \times \mathbb{R}^n$, where $S = (S_t)_{t\geq 0}$ is again a 1/2-stable subordinator. We conclude that f is continuous. The symbol of the process (S_t, X_{S_t}) is given by $\sqrt{i\tau - \psi_X(\xi)}$ by Lemma 21. As $(t, X_t)_{t\geq 0}$ has the Liouville property, we know that $i\tau - \psi_X(\xi) = 0$ if, and only if, $(\tau,\xi) = (0,0)$, from which we conclude that $\sqrt{i\tau - \psi_X(\xi)} = 0$ if, and only if, $(\tau,\xi) = (0,0)$. In view of Theorem 1, $(S_t, X_{S_t})_{t\geq 0}$ has the Liouville property; therefore, f is a.e. constant, and, as f is continuous, f is constant. □

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(3)A 1/2-stable subordinator is a subordinator with Laplace exponent $f_S(x) = \sqrt{x}$. The transition probability of the random variable S_t, $t > 0$, is given by the Lévy distribution $r \mapsto t(2\pi)^{-1/2} r^{-3/2} e^{-t^2/2r}$, $r > 0$, see [18, Example 40.14].

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