Lévy processes and function spaces

René L. Schilling

The Nottingham Trent University, Mathematics Department, Burton Street Nottingham NG1 4BU, U.K., email: `rls@maths.ntu.ac.uk`

With the exception of Brownian motions, Lévy processes are typical examples of jump processes. Many aspects of Lévy processes, their potential theory, and their path behaviour are well-known. Lévy processes have discontinuous, càdlàg paths, the distribution of the jumps and their semimartingale decomposition are well understood. But *how càdlàg* are the paths really? The question of the *smoothness* of the *per se* discontinuous paths was only recently asked. The first results in this direction were the papers by Z. Ciesielski, G. Kerkyacharian, and B. Roynette [6] (for stable and Gaussian processes) and by B. Roynette [12] (for Brownian motions). These authors proposed to use a tool familiar to most analysts who have to *measure smoothness*: function spaces, in particular *Besov spaces*. Here we give a full characterization of smoothness of Lévy processes in terms of Besov and Triebel-Lizorkin spaces, based on the papers [14, 15, 16]

1. Some notions from probability theory. A *Lévy process* $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^n$ is a stochastic process with stationary and independent increments that is continuous in probability. Lévy processes can be described in terms of their Fourier transforms,

$$\mathbb{E}^0(e^{i\langle\xi,X_t\rangle}) = e^{-t\psi(\xi)}, \qquad t\geq 0,\ \xi\in\mathbb{R}^n, \tag{1}$$

where the *characteristic exponent* $\psi:\mathbb{R}^n\to\mathbb{C}$ is given by the following Lévy-Khinchine-formula

$$\psi(\xi) = i\langle\ell,\xi\rangle + \langle Q\xi,\xi\rangle + \int_{y\neq 0}\Big(1-e^{-i\langle\xi,y\rangle}-\frac{i\langle\xi,y\rangle}{1+|y|^2}\Big)\,\nu(dy). \tag{2}$$

The triplet (ℓ,Q,ν) consisting of a vector $\ell\in\mathbb{R}^n$, a symmetric, positive semidefinite matrix $Q\in\mathbb{R}^{n\times n}$, and the Lévy measure ν on $\mathbb{R}^n\setminus\{0\}$, $\int_{y\neq 0}|y|^2/(1+|y|^2)\,\nu(dy)<\infty$, gives a one-to-one characterization of all possible characteristic exponents.

Since ψ gives a full characterization of the process $\{X_t\}_{t\geq 0}$, it contains all relevant information on the process, and many (probabilistic) properties of the process should be available through the (Fourier-analytic) study of the characteristic exponent. Such reasoning was the beginning of a success story dating back to S. Bochner [3, 4]. For an up-to-date account we refer to the books by J. Bertoin [2] and K. Sato [13]. For our purposes we recall the definition of certain *indices* that take into account the growth behaviour of ψ at infinity,

$$\beta = \beta_\infty = \inf\left\{\lambda>0\ :\ \lim_{|\xi|\to\infty}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\} \tag{3}$$

and at the origin

$$\text{(4)}\quad \beta_0 = \sup\left\{\lambda\geq 0\ :\ \lim_{|\xi|\to 0}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\};\qquad \delta_0 = \sup\left\{\lambda\geq 0\ :\ \liminf_{|\xi|\to 0}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\}.$$

The index β_∞ was introduced by R. Blumenthal and R. Getoor in their 1961 paper [5] in order to study Hölder properties of the paths of X_t in small time. Twenty years later, B. Pruitt proposed to study β_0 and δ_0. (His original definition was in terms of the triplet (ℓ, Q, ν). The above form can be found, e.g., in [15]). Provided that there is no dominating drift, he showed that

$$\lim_{t\to\infty} t^{-1/\lambda} \sup_{s\le t} |X_t - x| = 0 \text{ or } \infty, \qquad \text{a.s. } (\mathbb{P}^x) \tag{5}$$

according to $\lambda < \beta_0$ or $\lambda > \delta_0$. Note that this implies the bound

$$\sup_{s\le t} |X_s - x| \le c_\bullet (1+t)^{1/\lambda}, \quad \lambda < \beta_0, \tag{6}$$

with an a.s. finite random variable $c = c(\omega)$ depending on λ.
The limits (5) can be easily proved by a Borel-Cantelli argument and estimates of the type

$$\mathbb{P}^x(\sup_{s\le t} |X_s - x| > R) \;\le\; c_n\, t \sup_{|\epsilon|\le 1} \left|\psi\left(\frac{\epsilon}{R}\right)\right| \tag{7}$$

$$\mathbb{P}^x(\sup_{s\le t} |X_s - x| \le R) \;\le\; c_\kappa \frac{1}{t\sup_{|\epsilon|\le 1}\left|\psi\left(\frac{\epsilon}{4R\kappa}\right)\right|} \tag{8}$$

where in (8) we have to assume that $|\mathrm{Im}\,\psi(\xi)| \le \kappa \mathrm{Re}\,\psi(\xi)$, i.e. that there is no dominating deterministic drift. These estimates are implicit in [11], see also [15].

2. Function spaces. Function spaces are widely used in analysis, especially in the analysis of partial differential equations, in order to describe mapping properties of operators and smoothness of functions. Usually, smoothness of a function means *differentiability* or at least *continuity* properties which can be measured in scales of Sobolev/Bessel-potential spaces W_p^s, H_p^s or Hölder/Zygmund spaces $C^s, \mathcal{C}^s$. In the 1960s and 1970s *Besov-* B^s_{pq} and *Triebel-Lizorkin spaces* F^s_{pq} were developed that made it possible to give a unified and systematic approach to the above classical spaces. However, these new spaces have one additional feature: their scales go beyond continuity and allow us to describe the "smoothness" of, say, càdlàg functions. This is the application we have in mind here. In order to keep technicalities at a minimum, we restrict ourselves to Besov spaces.
Following H. Triebel [17, 18] we give a Fourier-analytic description of Besov spaces. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be a *smooth, dyadic partition of unity*, i.e., $\phi_j \in C^\infty(\mathbb{R}^n)$, $\mathrm{supp}\,\phi_0 \subset \{\xi : |\xi| \le 2\}$, $\mathrm{supp}\,\phi_j \subset \{\xi : 2^j \le |\xi| \le 2^{j+1}\}$, $\sup_{j,\xi} |2^{j|\alpha|} D^\alpha \phi_j| < \infty$ for all $\alpha \in \mathbb{N}_0^n$ and $\sum_j \phi_j \equiv 1$. Denote by Fu the Fourier transform of u. For $0 < p, q \le \infty$, $s \in \mathbb{R}$ we set

$$\|u|B^s_{pq}(\mathbb{R}^n)\| := \left(\sum_{j=0}^\infty 2^{jsq} \left\|F^{-1}(\phi_j F u)\right\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \tag{9}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). Clearly, for $u \in \mathcal{S}'(\mathbb{R}^n)$ (9) is well-defined and—in case it is finite—independent of the particular choice of the partition $\{\phi_j\}_{j\in\mathbb{N}}$.

Definition. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. The *Besov space* $B^s_{pq}(\mathbb{R}^n)$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that (9) is finite.

Equipped with the norms (9) Besov spaces are Banach spaces (quasi-Banach spaces if $p < 1$ or $q < 1$) that arise naturally as real interpolation spaces between the classical L^p-Sobolev spaces $W^m_p(\mathbb{R}^n)$. It is, therefore, not surprising that we have the following alternative description of Besov norms,

$$\|u|B^s_{pq}(\mathbb{R}^n)\| \sim \|u\|_{L^p(\mathbb{R}^n)} + \left(\int_0^\theta r^{-sq} \sup_{0<|h|\leq r} \|\Delta^M_h u\|^q_{L^p(\mathbb{R}^n)} \frac{dr}{r}\right)^{1/q} \tag{10}$$

(with the usual modification for $q = \infty$) whenever $n(\frac{1}{p} - 1)_+ < s < M$, $M \in \mathbb{N}$ suitably chosen, where $\Delta^M_h u$ is the M-fold repeated difference of step h, $\Delta_h u(x) = u(x+h) - u(x)$, and $\theta > 0$ is arbitrary. The $\sim$ in (10) indicates that (10) and (9) are equivalent (quasi-) norms for the admissible scope of s and M.

Remarks. **(A)** Replacing $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n; dx)$ through the space $L^p(\mathbb{R}^n; \rho_\alpha(x)\,dx)$ with weight function $\rho_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ gives polynomially *weighted Besov spaces* $B^s_{pq}(\mathbb{R}^n; \rho_\alpha)$. **(B)** A function u is said to be *locally* in $B^s_{pq}(\mathbb{R}^n; \rho_\alpha)$, if for some test-function $\phi \in C^\infty_c(\mathbb{R}^n)$ the product $\phi u \in B^s_{pq}(\mathbb{R}^n)$. **(C)** Besov spaces give a unified approach to various scales of function spaces. If, for example, $p \geq 1$ holds, then $B^s_{pp} = W^s_p$ ($s \neq$ integer, Sobolev-Slobodeckij scale "W"), $B^s_{22} = H^s_2 = W^s_2$ (Bessel-potential or Liouville scale "H"), or $B^s_{\infty\infty} = \mathcal{C}^s$ (Hölder-Zygmund scale). **(D)** One has the following analogue of the Sobolev embedding theorem: $B^{s+n/p}_{pq}(\mathbb{R}^n) \subset C^s(\mathbb{R}^n)$. In particular,

$$B^t_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \qquad \text{for all} \qquad t > \frac{n}{p}. \tag{11}$$

As usual "$\subset$" means continuous embedding. **(E)** The *smoothness* index s dominates the other two indices p, q in the following sense

$$B^{s+\epsilon}_{pq}(\mathbb{R}^n) \subset B^s_{pr}(\mathbb{R}^n) \qquad \text{for all} \qquad \epsilon > 0, \quad 0 < p, q, r \leq \infty. \tag{12}$$

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported càdlàg function and write $\Delta f(x) := f(x) - \lim_{y\uparrow x} f(y)$ for its jump at x. Then*

$$\Big(\sum_{x\in \mathrm{supp} f} |\Delta f(x)|^p\Big)^{1/p} \leq C_p \, \|f|B^{1/p}_{p\infty}(\mathbb{R})\| \tag{13}$$

holds true for all $0 < p < \infty$ with some $C_p > 0$ given by the norm-equivalence (10).

Proof. Choose $M = \left[\frac{1}{p}\right] + 1$ and observe that $\Delta^M_h f(x) = \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} f(x + kh)$. Therefore, we get for $h < 0$

$$\lim_{h\uparrow 0} \Delta^M_h f(x) = (-1)^M f(x) + \lim_{h\uparrow 0} \sum_{k=1}^M (-1)^{M-k} \binom{M}{k} f(x + kh) = (-1)^M (f(x) - f(x-)).$$

Since f is càdlàg and compactly supported, there are for a fixed $0 < \epsilon < 1$ at most $N(\epsilon) \in \mathbb{N}_0$ jumps of size $|\Delta f(x)| > \epsilon$; denote the corresponding sites by $x_1, x_2, \dots, x_{N(\epsilon)}$ and set $U_t(x_j) = \left[x_j, x_j + \frac{t}{2}\right)$, $t < t_0 < \theta$ so small that $f|U_t(x_j)$ is right continuous at x_j. Together with (10) we have

$$\begin{aligned} C_p \, \|f|B^{1/p}_{p\infty}(\mathbb{R})\|^p &\geq \sup_{0<t\leq\theta} \left(\frac{1}{t} \sup_{0<|h|\leq t} \int_{\mathbb{R}} |\Delta_h^M f(x)|^p \, dx \right) \\ &\geq \sup_{0<t\leq t_0/M} \left(\frac{1}{t} \sup_{-t<h\leq 0} \sum_{j=1}^{N(\epsilon)} \int_{U_t(x_j)} |\Delta_h^M f(x)|^p \, dx \right) \\ &\geq \sup_{0<t\leq t_0/M} \left(\frac{1}{2} \sum_{j=1}^{N(\epsilon)} \inf_{x\in U_t(x_j)} |\Delta_{-t}^M f(x)|^p \right) \\ &\geq \frac{1}{2} \sum_{j=1}^{N(\epsilon)} \liminf_{t\to 0} \inf_{x\in U_t(x_j)} |\Delta_{-t}^M f(x)|^p \end{aligned}$$

where the last estimate follows from Fatou's lemma. By definition, $x \in U_t(x_j)$ satisfies $x \geq x_j$, thus $f(x) \to f(x_j)$ as $x \to x_j$, whereas $x - kt \leq x_j + \frac{t}{2} - kt \leq x_j - \frac{t}{2} < x_j$, for all $k = 1, 2, \dots, M$. Hence $f(x - kt) \to f(x_j-)$ as $t \to 0$ (and, a fortiori, $x \to x_j$). Therefore,

$$\|f|B^{1/p}_{p\infty}\|^p \geq c'_p \sum_{j=1}^{N(\epsilon)} |f(x_j) - f(x_j-)|^p$$

and the assertion follows as $\epsilon \to 0$.

3. Lévy processes with paths in Besov spaces. We can now turn back to the question: *How smooth are the paths of Lévy processes*? An answer to this question would be to identify the sample paths as elements of certain Besov spaces. Since the paths of a Lévy process grow at a polynomial rate, there is no chance that $t \mapsto X_{t\vee 0} \in B^s_{pq}(\mathbb{R})$ *globally*. The natural spaces are, therefore, either localized spaces $B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ or weighted spaces. As we have already observed,

$$\sup_{s\leq t} |X_s(\omega) - x| \leq c(\omega)\,(1 + t^2)^{1/(2\lambda)}, \qquad \text{a.s. } \mathbb{P}^x \text{ for } \quad \lambda < \beta_0,$$

and the natural spaces to look at are polynomially weighted spaces.
To show the actual embedding, we have to prove that the norms (9) are finite. For this, set $\Omega_k := \{\omega \,:\, c(\omega) < k\}$ and observe that $\lim_{k\to\infty} \mathbb{P}^x(\Omega_k) = 1$. The major technical step is now contained in the following lemma.

Lemma 2. ([16, Lemma 2.5]) *Let* $\{X_t\}_{t\geq 0}$ *be as above. Then*

$$\int_{\Omega_k} \sup_{|h|\leq r} \left|\left[\Delta_h^M X_{\bullet\vee 0}(\omega)\right](t)\right|^p \, \mathbb{P}^x(d\omega) \leq C\, r\, (1 + t^2)^{p/(2\lambda)} \tag{14}$$

for $\lambda < \beta_0$ *and* $p > \beta_\infty$. *The constant* C *depends on* $p, \beta_\infty, \beta_0, k, \lambda, M$.

The proof of the above lemma uses essentially the estimate (7).
We are now in a position to state our main result. See the last section for further comments.

Theorem ([14] and [16]). *Let $\{X_t\}_{t\geq 0}$ be a Lévy process with characteristic exponent ψ and indices β_0, β_∞ as given above, and let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.*

(A) $\{t \mapsto X_{t\vee 0}\} \in L^p(\mathbb{R}; (1+t^2)^{-\mu/2})$ *almost surely* $\mathbb{P}^x$ *for all* $\mu > \frac{1}{\beta_0} + \frac{1}{p}$.

(B) $\{t \mapsto X_{t\vee 0}\} \in B^s_{pq}(\mathbb{R}; (1+t^2)^{-\mu/2})$ *almost surely* $\mathbb{P}^x$ *for all* $\mu > \frac{1}{\beta_0} + \frac{1}{p}$, $s > (\frac{1}{p} - 1)_+$ *and either* $q < \infty, s\max\{p, q, \beta_\infty\} < 1$ *or* $q = \infty, s\max\{p, \beta_\infty\} < 1$.

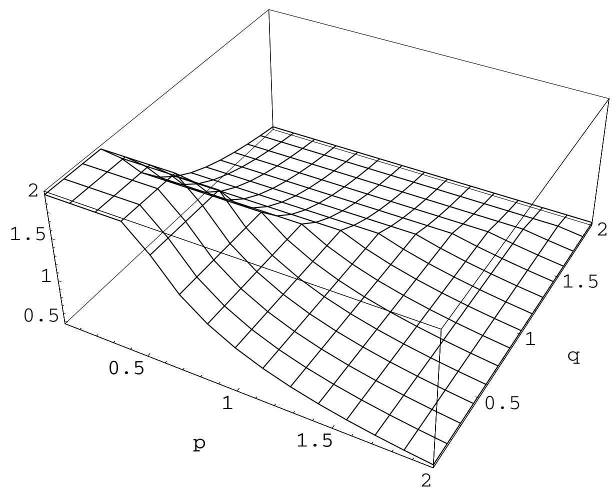
(C) $\{t \mapsto X_{t\vee 0}\} \in B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ *almost surely* $\mathbb{P}^x$ *for all* $s > (\frac{1}{p} - 1)_+$ *and either* $q < \infty, s\max\{p, q, \beta_\infty\} < 1$ *or* $q = \infty, s\max\{p, \beta_\infty\} < 1$.

(D) $\{t \mapsto X_{t\vee 0}\} \notin B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ *almost surely if either* $sp > 1$ *or* $sp = 1$ *and* $0 < q \leq 1$.

In order to prove (B) and (C) we have to check the finiteness of the norms (9). Having established Lemma 2 above, this is a straightforward calculation whenever $q \neq \infty$. The case $q = \infty$ is special, as it requires a Borel-Cantelli-trick. See [16] for details. The assertion (D) follows from embedding considerations: for $sp > 1$ Besov spaces are contained in spaces of *continuous* functions, hence cannot contain Lévy paths that are with probability 1 jump functions. The borderline case $ps = 1$ is, in general, difficult to answer. In the presence of scaling properties, e.g., for stable processes, Ciesielski, Kerkyacharian, and Roynette [6] showed a non-embedding result for $q = \infty$ and $\beta_\infty > 1$.

Corollary 1. *The assertions of the above Theorem hold for all* $s \geq 0$ *without the restriction* $s > (\frac{1}{p} - 1)_+$.

Proof. Use the fact that $B^s_{pq} \subset B^t_{pr}$ if $s > t$.



The graph shows the parameter s as a function of p and q. The area below the graph $s = \max\{p, q, \beta_\infty\}^{-1}$ represents the region of admissible parameters (s, p, q) for embedding into $B^{s,\mathrm{loc}}_{pq}$ with $q \neq \infty$.

For this picture we chose $\beta_\infty = 0.50$, i.e., the plateau occurs at $1/\beta_\infty = 2.00$.

Corollary 2. *Let $\{X_t\}_{t\geq 0}$ be as in the Theorem. Then*

$$\{t \mapsto X_{t\vee 0}(\omega)\} \in B^{1/p,\mathrm{loc}}_{p\infty}(\mathbb{R}) \qquad \textit{almost surely } \mathbb{P}^x \textit{ for all} \qquad p > \beta_\infty.$$

Proof. We refer to the proof in [16, Corollary 4.3, Case 1]. If we have the *strict* inequality $p > \beta_\infty$ and $q = \infty$, $s = 1/p$, we can find a $0 < \theta < 1$ such that $p\theta > \beta_\infty$ and $sp\theta = \theta < 1$. We can now proceed along the lines of the proof in [16]: instead of using Jensen's inequality for the finite measure $\langle t\rangle^{-p\mu}\,dt$, we use Jensen's inequality for the concave function $x \mapsto |x|^\theta$, $\theta < 1$, and the expectation $\mathbb{E}^x$. This yields $\mathbb{E}^x\left(|Y|^\theta\right) \leq (\mathbb{E}^x(|Y|))^\theta$. The other arguments in [14] need not be changed.

Corollary 3. *Let $\{X_t\}_{t\geq 0}$ be as in the Theorem and denote by $\Delta X_t := X_t - X_{t-}$ the jump at time t. Then*

$$\sum_{t\leq 1} |\Delta X_t|^p < \infty \qquad \textit{almost surely } \mathbb{P}^x \textit{ for all} \qquad p > \beta_\infty.$$

Proof. Combine Corollary 2 and estimate (13)

Remarks. (A) The above Theorem and its Corollaries generalize to all Feller processes that are generated by pseudo-differential operators cf. [14, 16] and the notes below. **(B)** The assertions of the Theorem and Corollary 1 remain valid for (weighted) spaces of Triebel-Lizorkin type F^s_{pq}, cf. [16].

4. Concluding remarks. Some remarks on the development of the above theorem seem to be in order. Originally, assertion (C) of the theorem was proved by Ciesielski, Kerkyacharian, Roynette [6] both for certain Gaussian and symmetric α-stable Lévy processes with index $\alpha > 1$ and for $q = \infty$. Their method used essentially the scaling property of stable processes and an atomic decomposition of Besov spaces. Subsequently, V. Herren generalized this result (using the same technique) to any Lévy process with index $\beta_\infty > 1$. In [14], the present author studied a class of Feller processes that are generated by pseudo-differential operators

$$(15) \qquad -p(x,D)u(x) = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle\xi,x\rangle}p(x,\xi)Fu(\xi)\,d\xi, \qquad u \in C_c^\infty(\mathbb{R}^n),$$

where $\xi \mapsto p(x,\xi)$ is given by a Lévy-Khinchine formula (depending on the parameter x)—cf. the talk of N. Jacob or [9]. (It is known that *every* Feller process whose generator has a domain containing the test functions $C_c^\infty(\mathbb{R}^n)$ is already of this type, cf. [7]; Lévy processes are exactly those processes where $p(x,\xi) = \psi(\xi)$, i.e., independent of x.) In the paper [14] assertion (C) is proved for this class of processes; as a by-product, (C) could be shown to hold for *all* Lévy processes, and the restriction $\beta_\infty > 1$ of [8] could be removed.

In [15, 16] we proved the global embedding for *Feller processes* (and, in particular, Lévy processes) as stated in the theorem above. The approach used in these papers is basically the one sketched here. The advantage of this approach is that it is more flexible and allows us to include also spaces of Triebel-Lizorkin type.

In the recently published paper [1], Orlicz spaces were used to describe the paths of Lévy processes, see also [10].

Acknowledgement. I am grateful to my colleague D. Kaye, Nottingham, for producing the 3-d plot included in this note.

References

[1] Braverman, M. and G. Samorodnitsky, Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes, *Stoch. Proc. Appl.* **78** (1998), 1–26.

[2] Bertoin, J., *Lévy Processes*, Cambridge University Press, Cambridge Tracts in Math. vol. **121**, Cambridge 1996.

[3] Bochner, S., Stochastic Processes, *Ann. Math.* **48** (1947), 1014–1061.

[4] Bochner, S., *Harmonic Analysis and the Theory of Probability*, University of California Press, California Monographs in Math. Sci., Berkeley, CA 1955.

[5] Blumenthal, R. M. and R. K. Getoor, Sample Functions of Stochastic Processes with Stationary Independent Increments, *J. Math. Mech.* **10** (1961), 493–516.

[6] Ciesielski, Z., Kerkyacharian, G. et B. Roynette, Quelques espaces fonctionnels associés à des processus gaussiens, *Studia Math.* **107** (1993), 171–204.

[7] Courrège, Ph., *Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principe du maximum*, Sém. Théorie du Potentiel (1965/66) exposé 2, 38 pp.

[8] Herren, V., Lévy type processes and Besov spaces, *Potential Analysis* **7** (1997), 689–704.

[9] Jacob, N., *Pseudo-differential operators and Markov processes*, Akademie Verlag, Mathematical Research vol. **94**, Berlin 1996.

[10] Norvaiša, R. and G. Samorodnitsky, Stable processes with sample paths in Orlicz spaces, *Ann. Probab.* **22** (1994), 1904–1929.

[11] Pruitt, W. E., The growth of random walks and Lévy processes, *Ann. Probab.* **9** (1981), 948–956.

[12] Roynette, B., Mouvement brownien et espaces de Besov, *Stoch. and Stoch. Rep.* **43** (1993), 221–260.

[13] Sato, K., *Additive Processes*, monograph to be published by Cambridge University Press, 1999.

[14] Schilling, R. L., On Feller processes with sample paths in Besov spaces, *Math. Ann.*, **309** (1997), 663–675.

[15] Schilling, R. L., Growth and Hölder conditions for the sample paths of Feller processes, *Probab. Theor. Rel. Fields* **112** (1998), 565–611.

[16] Schilling, R. L., Function Spaces as Path Spaces of Feller Processes, to appear in *Math. Nachr.*

[17] Triebel, H., *Theory of Function Spaces*, Birkhäuser, Monographs in Mathematics vol. **78**, Basel 1983.

[18] Triebel, H., *Theory of Function Spaces II*, Birkhäuser, Monographs in Mathematics vol. **84**, Basel 1992.