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Umarov, Sabir; Hahn, Marjorie; Kobayashi, Kei: *Beyond the triangle: Brownian Motion, Ito Calculus, and Fokker–Planck Equation — Fractional Generalizations.* World Scientific, Singapore 2018. xiii + 177 pp., £86.00, US-$ 98.00 (RRP). ISBN 978-981-3230-91-0.

This book is written as an introduction to fractional Fokker-Planck equations and the stochastic processes associated with them. The aim is *to develop a paradigm that embodies three objects fundamental to understanding* properties of solutions to SDEs which are not necessarily driven by Brownian motion. The authors hope that their book *makes the[se] properties more easily accessible to scientists, economists and other researchers* (Preface, p. viii). The book is geared towards such an audience, mathematicians and mathematicians-to-be seem not to be the principal target group.

The focus of the exposition is to explain the connection between i) the solutions to SDEs, ii) their random-walk approximation and iii) the Fokker-Planck-Kolmogorov (or master) equation of the solution process. These *three objects* are studied relative to the driving noise which is placed in the centre surround by the *triangle* with vertices i), ii), iii); the authors refer to this picture as their *paradigm.*

After a motivating introduction (Chapter 1), the tools of the trade are introduced in the next four chapters (pp. 11–82). Chapter 2 (pp. 11–24) introduces Itô's stochastic calculus for diffusions and SDEs driven by Brownian motion, with a focus on the connection of the coefficients of the SDE and the coefficients of the Kolmogorov forward resp. backward equation. Chapter 3 (pp. 25–40) deals with fractional derivatives on the real axis (in the sense of Riemann & Liouville, Caputo and Weyl, distributed order fractional derivatives corresponding to an integral mixture over the order of the earlier mentioned fractional derivatives) and with Riesz potentials and fractional Laplacians in $\mathbb{R}^n$. Chapter 4 (pp. 41–62) is about Lévy-type processes and their infinitesimal generators which are understood as pseudo differential operators. The chapter contains some basic material on classical pseudo differential operators (in the sense of Kohn, Nirenberg and Hörmander) as well as pseudo differential operators with singular symbols (following S. Umarov: *Introduction to fractional and pseudo-differential equations with singular symbols.* Springer, New York 2015). It is unfortunate that the authors do not point out that the idea to use pseudo differential operators in the study and the construction of stochastic processes is due to N. Jacob [starting from 1988, see, e.g. the monographs N. Jacob: *Pseudo-differential operators and Markov processes.* Mathematical Research **94**, Akademie-Verlag, Berlin 1996 and the three-volume treatise N. Jacob: *Pseudo-differential operators and Markov processes, 1,2,3.* Imperial College Press, London 2001–2005]. The chapter closes with some material on semigroups, pseudo differential operators on manifolds and – following Taira – generators of Markov processes on domains. Time-changes, the Skorokhod topology, semimartingales, stochastic integrals w.r.t. semimartingales, Lévy processes and (inverses of) subordinators are discussed in Chapter 5 (pp. 63–82).

The main part of the book are Chapters 6 and 7. The material of these chapters is mainly taken from original contributions of the authors since 2011. Chapter 6 is about stochastic calculus and SDEs for time-changed semimartingales; if the time-change is effected by an inverse subordinator, the authors also study approximations of the drivers using continuous-time random walks (CTRWs). Chapter 7 abandons the path-by-path picture of Chapter 6 and moves on to an averaged description leading to the Fokker-Planck-Kolmogorov equations. The following phenomena may appear: the driver of the SDE is a Lévy process (e.g. a subordinated Brownian motion) then the solution of the SDE is Markovian, and we get the classical Kolmogorov backward equation with a non-local ('fractional') operator. If the driver of the SDE is, say, an inverse-subordinated Brownian motion (that is, the time-change is the inverse of a subordinator of a certain type – typically a stable subordinator or a mixture of stable subordinators) we get a time-fractional Fokker-Planck equation, still with the Laplacian in space as we have time-changed a Brownian motion, but this can be generalized to Lévy generators. The authors also consider other non-Markovian drivers of the SDEs such as fractional Brownian motions and general Gaussian processes.

Most of the book is a compilation of definitions and results from other sources and often this is done in a rather unsystematic way: for example, Lévy processes are formally introduced in Chapter 5, while Chapter 4 is on their infinitesimal generators; another example is the discussion of pseudo differential operators which are introduced in unnecessary generality (on paracompact manifolds, although only subsets of $\mathbb{R}^n$ are used, with singular symbols although only 'rough' symbols are needed) while the important notion of negative definiteness (or conditional positive definiteness) is treated rather gentlemanlike: no word, for instance in connection with Courrèges theorem, that this property characterizes those pseudo differential operators which give rise to stochastic processes . Chapters 6 and 7 which are based on the authors' original papers are a bit more explicit when it comes to proofs, but these are mainly formal manipulations. The real difficulties (for example:

why is there no Itô integral for most fractional Brownian motions, and how can we fix it?) are swept under the carpet.

For a book with a mainly non-mathematical audience, too few and unconvincing applications are included. So it is really a *hope [...] that this book has stimulated you to think about the information and insights that might be gained from considering the interconnections in the paradigm [...] [mentioned above], rather than its vertices or center in isolation. This book should provide the beginning for many investigations and applications [...]* (Final Note, p. 160) – appropriately cast using the modal subjunctive 'should'.

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