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Bernard Roynette and Marc Yor: *Penalising Brownian Paths.* Springer, Lecture Notes in Mathematics vol. **1969**, Berlin 2009, xii + 275 pp., €53.45, US-$ 69.95, £44.99, ISBN 978-3-540-89698-2.

Penalising a stochastic process $X = (X_t, \mathcal{F}_t)_{t\geqslant 0}$ means to alter the (infinite dimensional) law $\mathbb{P}$ of X by appropriate weights in such a way that some of the properties of the original process change in a prescribed way. These changes can be quite radical, e.g. one can create Brownian-like stochastic processes whose overall maximum is finite or one could condition processes on sets which are of measure zero under $\mathbb{P}$. Among the most prominent penalisations are Doob's h-transform and Feynman-Kac transforms. In all concrete situations, X is either (d-dimensional) Brownian motion or a Bessel process. As usual, only the canonical version of these processes defined on $\Omega = C([0,\infty), \mathbb{R}^d)$.

The general penalisation set-up uses some $\mathbb{R}_+$-valued process $\Delta_t, t \geqslant 0$ on $(\Omega, \mathcal{F}_\infty)$, $\mathcal{F}_\infty = \bigvee_{t\geqslant 0} \mathcal{F}_t$, which need not be $\mathcal{F}_t$-adapted and which satisfies $0 < \mathbb{E}_x\Delta_t < \infty$. To penalise the laws $\mathbb{P}_x$, define

$$\mathbb{P}_x^{(t)}(\cdot) = \frac{\Delta_t}{\mathbb{E}_x\Delta_t} \cdot \mathbb{P}_x(\cdot) \quad \text{on} \quad \mathcal{F}_t,\ t > 0,$$

and study the (existence of the) limit $t \to \infty$ in a suitable way. This set-up reminds of Doob's h-transform, but the procedure is not projective, i.e. it does not define a new probability measure on Ω without passing to the limit $t \to \infty$. Often the process Δ_t is of the form $h(\Gamma_t)$. In this case one writes $\mathbb{P}_x^{(h,t)}$; if Γ_t is real-valued and increasing, it is often possible to show that $\mathbb{P}^{(h,t)} \xrightarrow{t\to\infty} \mathbb{Q}^{(h)}$ exists and defines a new probability measure $\mathbb{Q}^{(h)}$ on $(\Omega, \mathcal{F}_\infty)$. Clearly, $\mathbb{Q}^{(h)}(\Gamma_\infty \in dy) = h(y)\,dy$. Examples of Γ_t include (in one dimension) the supremum process $\sup_{s\leqslant t} X_s$, local time and more general positive continuous additive functionals, the number of downcrossings over a strip (a,b) in the state space.

Many of the penalisation theorems contained in the lecture notes volume under review take the following form:

Meta-Theorem of Penalisation: *Let $(X_t, \mathcal{F}_t, \mathbb{P}_x, t \geqslant 0, x \in \mathbb{R}^d)$ be Brownian motion and let h be a positive measurable function satisfying some additional integrability condition. Then there exists an $(\mathcal{F}_t, \mathbb{P}_x)$ martingale $M^{(h,x)}$*

such that, for every $s \geqslant 0$ and x

$$\mathbb{E}_x\left(\frac{h(\Gamma_t)}{\mathbb{E}_x h(\Gamma_t)}\,\middle|\,\mathcal{F}_s\right) \xrightarrow{t\to\infty} M_s^{(h,x)}$$

almost surely $(\mathbb{P}_x)$ and in $L^1(\mathbb{P}_x)$. Moreover,

$$\mathbb{E}_x\left(\Phi_s\cdot\frac{h(\Gamma_t)}{\mathbb{E}_x h(\Gamma_t)}\,\middle|\,\mathcal{F}_s\right) \xrightarrow{t\to\infty} \mathbb{E}_x\left(\Phi_s\cdot M_s^{(h,x)}\right)$$

holds for all bounded $\mathcal{F}_s$-measurable random variables Φ_s and

$$\int_\Omega \Phi_s\,d\mathbb{Q}_x^{(h)} = E_x\left(\Phi_s\cdot M_s^{(h,x)}\right)$$

induces a probability $\mathbb{Q}_x^{(h)}$ on the space $(\Omega,\mathcal{F}_\infty)$.

$(X_t,\mathbb{Q}_x^{(h)})$ is the *penalised process* or $\mathbb{Q}^{(h)}$*-process*. The meta-theorem of penalisation also asserts the existence of a positive martingale M_t^h; that is, whenever such a theorem holds, one obtains a new class of martingales. Among the examples which arise in this way are the famous Azéma-Yor martingales and the Kennedy martingale.

Various kinds of penalisations are investigated in the text under review. Chapter one (pp. 35–66) studies the Wiener measure, chapter two (pp. 67–130) is exclusively on Feynman-Kac penalisations and chapter three (pp. 131–224) is devoted to d-dimensional Bessel processes. A critical discussion and temporary conclusion looking *at what has or has not been achieved* (p. xi of the introduction) is contained in the last chapter, chapter four (pp. 225–260). This material is supplemented by a long introduction, Chapter 0 (pp. 1–34) which explains the idea of penalisation and contains also some of the meta-theorems of penalisation. All chapters are self-contained (even with separate bibliographies) and can be read individually and in any order. Substantial parts of the text are extensions and refinements of results from a loose series of papers which the authors have jointly written with P. Vallois (Zbl 1121.60027, Zbl 1121.60004, Zbl 1124.60034, Zbl 1155.60329, Zbl 1160.60315, Zbl 1164.60355, Zbl 1164.60307, Zbl pre05660763, Zbl 1181.60046).

To combine this material in a single volume is highly desirable; on the other hand it is clear that it has not yet reached its definitive form which would be required for a proper monograph. The present lecture note is a first step in this direction and, at least implicitly, the authors say so: The fourth chapter is practically a to-do list inviting further investigations; moreover, many theorems depend on a yet unproven conjecture (C), see p. 148. In this sense the text is in the great tradition of Springer's Lecture Notes Series which puts timeliness over form and which may be informal, preliminary and sometimes even tentative.

But it continues also another great tradition: with its abundance of explicit formulae the present volume is a perfect example of Paul Lévy's way to study Brownian motion.

René L. Schilling
Institut für Stochastik
TU Dresden
D-01062 Dresden, Germany
rene.schilling@tu-dresden.de