Review commissioned by: ZMath–Zentralblatt für Mathematik

Yoichi Oshima: *Semi-Dirichlet forms and Markov processes.* De Gruyter, Studies in Mathematics **48**, Berlin, 2013. x+284 pp. €74.95, £76.10, US-$ 126.00. ISBN: 978-3-11-030200-4.

Habent sua fata libelli—books have their own (hi-)stories; the story of this book begins in 1988. In this year, the author of the volume under review delivered a lecture course at the University of Erlangen on not necessarily symmetric Dirichlet forms. Oshima's lecture notes were the first expository presentation of non-symmetric Dirichlet forms. Even though they never made it past the preprint stage and could only be obtained by mail order from Erlangen's mathematics department, they were widely quoted. At that time Fukushima's seminal monograph *Dirichlet Forms and Markov Processes* (Kodansha 1975, Elsevier 1980, Zbl. 0422.31007) had been long out of print and the first book on non-symmetric Dirichlet forms, Ma & Röckner's *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms* (Springer 1992, Zbl. 0826.31001) was still in the making. Around the same time we see the publication of Bouleau & Hirsch *Dirichlet Forms and Analysis on Wiener Space* (de Gruyter 1991, Zbl. 0748.60046) and the second edition of *Dirichlet Forms and Symmetric Markov* Processes by Fukushima, Oshima and Takeda (de Gruyter 1994, Zbl. 0838.31001). Again at Erlangen University, Oshima lectured in 1994 on time-dependent Dirichlet forms and his lecture notes *A short introduction to the general theory of Dirichlet forms* was the first account on temporally inhomogeneous Dirichlet forms which was accessible to graduate students—and this has not changed much.

The present monograph is a synthesis of the 1988 and 1994 lecture notes. At first glance, the presentation of the first five chapters (pp. 1–215) follows the 1988 lecture notes while Chapter 6 covers the time-dependent case as in the 1994 lectures. A second look quickly reveals that this is only true for the structuring of the material. Other than in the lecture notes, the proofs are carefully worked-out with all details, and it is very clear that there is much more material than one could cover in a semester-long lecture course. Addressing an advanced readership—one should have a sound background in functional analysis and at least some ideas of Markov processes—the text is indeed self-contained. What is more, recent developments are taken into account and the material is consequently presented in the context of lower bounded semi-Dirichlet forms; throughout the monograph they are simply referred to as Dirichlet forms. Usually, the regularity of the forms is assumed, i.e. suitable subsets of continuous functions are assumed to be dense in the form-domain, both uniformly and in the form-sense.

Lower bounded semi-Dirichlet forms are generalizations of (non-symmetric) Dirichlet forms: These are closed bilinear forms defined on a subspace of $L^2(X, m)$, which are coercive (lower bounded), satisfy the (weak) sector condition and the semi-Dirichlet property which is, loosely speaking, 'half' the usual Markovian contraction property. The latter is equivalent to the Markov property of the associated operator semigroup $(T_t)_{t\geq 0}$ or the positivity and $L^1(X, m)$ contraction property of the formally adjoint semigroup $(\widehat{T}_t)_{t\geq 0}$. The basic (functional analytic) theory of semi-Dirichlet forms is developed in Chapter 1 and many properties (as well as their proofs) are readily transferred from the non-symmetric setting. A novel feature is the introduction of an auxiliary bilinear form which turns out to be useful for the probabilistic treatment of the duality theory. Chapter 2 treats (the analytical side of) potential theory. Here we find capacities, quasi-continuity, potentials, excessive functions etc., culminating in the important orthogonal decomposition of semi-Dirichlet forms which opens up the possibility to treat forms defined on subsets of the underlying space.

In Chapter 3 we learn how to associate a Markov (resp. Hunt) process to a (regular, lower-bounded) semi-Dirichlet form. The construction is classic, basically addressing the abundance of exceptional sets through capacities and quasi-continuity. Although the construction is (known to be rather) cumbersome, the presentation is very clear. For the novice it is probably better to jump directly to those parts of the chapter where the analytic notion of exceptional sets is characterized in probabilistic terms and where the decomposition of forms is discussed at the level of Hunt processes. This study naturally involves positive continuous additive functionals and time-changes. These concepts are covered in the fourth chapter; again most results and their proofs run parallel to the (non-)symmetric situation. Chapter 5 continues this theme where we find a carefully worked-out proof of the Fukushima decomposition which establishes the connection to stochastic calculus. Here a major modification is necessary: The notion of energy is not well-defined for semi-Dirichlet forms since constant functions are, in general, not co-excessive; instead, one has to use a weak formulation for the energy. From that point onward one can follow the 'standard' theory of Dirichlet forms, considering, for instance, transformations of semi-Dirichlet forms by multiplicative functionals and recurrence–transience questions.

The final chapter, Chapter 6, is devoted to time-dependent semi-Dirichlet forms and their parabolic potential theory. Once the general theory (Chapter

1–5) is developed, this is now a matter of following Oshima's construction from 1994 (including a small correction from 2004, Zbl. 1078.60060 *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** (2004) 281–316, based on Stannat's theory of generalized Dirichlet forms, *Mem. Am. Math. Soc.* **678** (1999) Zbl. 1230.60006.)

This new book is a most welcome addition to the existing literature on Dirichlet forms. It is a readily accessible, advanced graduate-level account of analytic and probabilistic potential theory of Hunt processes given by (lower bounded semi-) Dirichlet forms.

Dresden, February 2014
René L. Schilling