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Øksendal, Bernt: Stochastic Differential Equations. An Introduction with Applications. Springer-Verlag (Universitext), Berlin-Heidelberg-New York 2003 (6th ed.), xxiv, 360 pp., £27.00, ISBN 3-540-04758-1.

Our modern world—telecommunications, banking, finance innovations, insurance products,...—would be unthinkable without mathematical modelling. One of the central problems is the question how to model risks and uncertainties. The impact such models can have is best encapsulated in the celebrated Black-Scholes option pricing formula: it is present on the executive floor of banks, throughout the world's stock exchanges, in investors' minds and has, in the last 5-10 years, found its way into the curricula of maths and business degrees. To illustrate the basic problem of randomness let us begin with the classical (deterministic) equation of motion. In its simplest form it is an ordinary differential equation of the type

$$\dot{x}_t = a(t, x_t), \qquad x_t\Big|_{t=0} = x_0.$$

Such an equation is highly idealistic and does not take into account the various perturbations of the real world. If we want to model these we should add some perturbation, e.g.,

$$\dot{x}_t = a(t, x_t) + b(t, x_t)\dot{W}_t, \qquad x_t\Big|_{t=0} = x_0, \tag{$\star$}$$

in form of the derivative of some suitable random noise W_t. It is, of course, the actual physical phenomenon that determines the choice of the particular noise term. Such problems have been considered for at least 100 years. Already in 1900 L. Bachelier used a random motion which should later become known as *Brownian Motion* to model the fluctuations of the Paris stock market. This *Brownian Motion*, in its differential form also known as *White Noise*, is still one of the most popular and universal random processes. The name Brownian Motion is in honour of the Scottish botanist Robert Brown who observed in 1828 the seemingly erratic and incessant movement of pollen suspended in a watery solution. Only some 75 years later, A. Einstein (1905), M. Ritter von Smoluchowski (1906) and J. Perrin (1909) gave physically satisfactory explanations of this phenomenon; and it was some years before N. Wiener (1922-3) came up with the first mathematically rigourous construction. While these approaches concentrate on the fluctuations, the equation

($\star$) describes the infinitesimal change of our system x_t in terms of the infinitesimal change of the fluctuations. But not every noise has a derivative. Unfortunately that is what happens with Brownian Motion: it is with probability 1 nowhere differentiable (Paley-Wiener-Zygmund 1933)! Since we are only interested in x_t, one might try to integrate ($\star$) in order to get rid of derivatives. Indeed, the integral equation

$$x_t = x_0 + \int_0^t a(s, x_s)\, ds + \int_0^t b(s, x_s)\, dW_s \qquad (\star\star)$$

(in Stieltjes' sense) only requires that W_t is of bounded variation. But not even this is true. In 1942, there enters the Japanese mathematician K. Itô. Based on ($\star\star$) he gave a stochastic extension of the Riemann-Stieltjes integral which is general enough to accommodate the irregularities of Brownian Motion. In the definition of the Riemann-Stieltjes integral we encounter limits of the type

$$\int f\, dg = \lim_{\pi} \sum_j f(\tau_j)(g(t_{j+1}) - g(t_j))$$

where the limit must exist for *all* partitions $\pi = \{t_1 < t_2 < \ldots < t_n\}$ and *all* choices of $\tau_j \in [t_j, t_{j+1}]$. Take $\tau_j = t_j$, replace lim by a limit in probability and you get Itô's extension. For nice deterministic functions f, g (continuous and bounded variation will do) we recover the usual integral but for less smooth functions things are different. This is best seen in *Itô's formula* which says that for Brownian motion

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, \underbrace{(dW_s)^2}_{=ds} \qquad (\star\star\star)$$

holds. The second term on the right is completely unexpected in the classical theory since second variations of continuous functions of bounded variation do vanish. Needless to say that ($\star\star\star$) gives rise to a new *stochastic* calculus nowadays called *Itô calculus.*

This, and much more (for the specialists among the readers: s.d.e.s driven by continuous martingales, Itô calculus for continuous diffusions), is what Øksendal's book is all about. When the first edition came out in 1985 it was probably the one and only English-language textbook on the subject which would be accessible to a wide audience, ranging from mathematically erudite engineers to advanced undergraduates and postgraduates alike. Since then the book is in its 6th edition, has undergone many changes and additions and has evolved from a 200-page typewritten booklet to a modern classic. Part of its charm and success is the fact that the author does not bother too much with the (for the novice) cumbersome rigourous theory of stochastic processes from the outset—Brownian motion is almost axiomatically treated—but comes quickly to stochastic integrals and differential equations. This does not mean that the book is not rigourous, it is just the timing and dosage of mathematical rigour following a different rhythm which is palatable for undergraduates and interested scholars from subjects other than maths. This makes it possible to deal with the basics of stochastic integration and differential equations in the first quarter of the book, Chapters 1-5. Thereafter select topics are discussed: filtering (ch. 6), diffusions and boundary value problems (ch. 7, 8, 9), optimal stopping and stochastic control (ch. 10, 11) and mathematical finance (ch. 12). These topics can be read almost independently of each other which makes the book an ideal textbook for an introductory course of the subject with some applications. With none of the applications does the book aim to be encyclopedic, not even in-depth. The intention is rather to illustrate the use and power of Itô's calculus and to spark the readers' interest for the subject. This makes Øksendal's book so popular with students and academic teachers: it is a highly readable account, suitable for self-study and for use in the classroom.

University of Sussex @ Brighton René L. Schilling