Review commissioned by: Math Reviews

Nualart, David; Nualart, Eulalia: *Introduction to Malliavin Calculus*. Cambridge University Press, IMS Textbooks vol. 9, Cambridge 2018. xii + 236 pp., £27.99, € 28.99 (D), US-$ 38.99 (RRP). ISBN 978-1-107-61198-6.

The first paper of what we nowadays call *Malliavin's calculus* was P. Malliavin's seminal contribution on a *Stochastic Calculus of Variations and Hypoelliptic Operators* [MR0536013] which he presented at the 1976 Symposium on stochastic differential equations in Kyoto. Malliavin's aim was to create a differential calculus on Wiener space in order to show smoothness of rather general random functionals, e.g. solutions to Itô differential equations. One of the first applications was a probabilistic proof of Hörmander's theorem on the hypoellipticity of partial differential operators which are sums of squares of vector fields. In contrast to earlier attempts to create a differential calculus on infinite dimensional spaces, Malliavin did not use the 'usual' Fréchet derivative as his basic differential operator, but he based his calculus on the Malliavin derivative D which has the property that $-\delta D = L$; L is the Ornstein–Uhlenbeck operator and δ is Skorokhod's integral which is the adjoint of D. The field rapidly developed and in the hands of the japanese school around K. Itô and S. Watanabe soon more accessible formulations of Malliavin's ideas appeared, see e.g. Watanabe's Tata Lectures [MR0742628] or Chapters V.8 and V.9 in the second (1989) edition of Ikeda and Watanabe [MR1011252]. The name *Malliavin's calculus* was coined in 1982 by Stroock [MR0794531]. From 1985 Malliavin's ideas were extended to jump processes, leading to a stochastic calculus of variations on Poisson space.

Nowadays, Malliavin's calculus is a fundamental tool in mathematical research but it is still not regularly taught in Master courses. One reason certainly are the formidable technical entrance hurdles to the theory—a sound working knowledge of Hilbertian analysis as well as a good training in advanced probability, two side-coditions which are not always concomitant—and the scarcity of accessible texts on the subject matter. The already mentioned lectures by Watanabe and the monograph by Nualart [MR1344217] are still among the most reliable guides to the theory (in the Wiener case), the book by Ishikawa [MR3495001] treats both Wiener and Poisson space, making it a valuable reference on which one may base a lecture, although it is not primarily written for the classroom.

The authors of the present text were clearly aware of this situation and they wanted to write a text which is concise, comes straight to the heart of the matter and includes modern applications of Malliavin's calculus to Stein's method or fractional Brownian motion. Moreover, it is—apart from Ishikawa—the only text that develops Malliavin's calculus for both continuous and jump processes. In order to make the material easier to access, the first two chapters give a rapid introduction to Brownian motion and stochastic calculus. Remedial material from intermediary probability can be found in the appendix. The main matter starts from page 50 where Malliavin's deriviative D and its dual, the divergence (or Skorokhod integral) δ are introduced as well as the scale of Sobolev spaces. This part follows Watanabe's approach. Chapter 4 introduces iterated Wiener integrals and the Wiener chaos. Similar to D. Nualart's monograph, D and δ are discussed using chaos expansions. Chapter 5 treats the Ornstein–Uhlenbeck semigroup and its infinitesimal generator L which is identified with $-\delta D$. Here we find the extremely useful integration-by-parts formulae and Meyer's inequality which is important to treat L^p-Sobolev spaces. This chapter marks also the end of the theoretical development of the Malliavin calculus. The next three Chapters (Chapters 6–8, pp. 87–157) are devoted to various applications of Malliavin's calculus. These chapters may be read independently of each other. Chapter 6 begins with the Clark–Ocone formula (a representation of sufficiently smooth Wiener functionals) which is then applied to find the modulus of continuity of Brownian local times and the derivative of self-intersection local time; a short excursion to local times of fractional Brownian motions ends this chapter. Malliavin's original aim was to capture the smoothness of Wiener functionals; often this amounts to showing the existence of a probability density. This is the subject matter of Chapter 7 which discusses the existence of densities in several settings. From an application point of view the treatment of Malliavin differentiability of SDEs and the related Hörmander conditions for their infinitesimal generators are the highlights of this chapter. Recently, see Nourdin and Peccati [MR2962301], the connections of Malliavin's calculus to the normal approximation problem has led to a renewed interest in Malliavin's calculus. Some of these developments on Stein's method are discussed in Chapter 8, with focus on CLT- and non-CLT results.

The last 40 pages are devoted to Malliavin's calculus on Poisson space. The approach is in the spirit of J. Picard (see also Ishikawa's monograph) and the discussion starts with an introduction to Lévy processes via Poisson random measures. In this chapter also elements of stochastic calculus for Lévy processes (Itô's formula, SDEs, Girsanov) and Poisson–Wiener chaos are explained. In the following two chapters, Malliavin's calculus is first developed using the Wiener chaos and Picard's difference operator. The presentation follows closely the path laid out for the Wiener space and often the proofs refer to the corresponding steps of the first part. The last chapter gives an introduction to the Bichteler–Jacod approach to jump-type Malliavin calculus, finally focussing on the existence of densities of SDEs driven by a Lévy process (or a Poisson random measure).

The book is written as a first encounter with Malliavin's calculus. The choice of the material is well thought-out, guiding the reader from the basics to the current frontier of research. It is quite a feature that the authors manage to do this on barely 150 pages (for the Wiener space) and 50 more pages (for the Poisson space). But this comes at a price. Each chapter has a few exercises, quite often the reader is asked to fill in gaps or whole proofs of the more simple results. Unfortunately, there are no solutions, and this may be a problem for the novice—student or first-time lecturers alike. There is a lot of ‘remedial’ material on stochastic processes and stochastic calculus in Chapters 1,2 and the appendix, but for the intended audience—Master students which have been exposed to a second course in probability—some words and calculation techniques on Hilbert spaces might have been more appropriate. The style of the proofs is terse, at many places one would wish that there were more explanations. The overall presentation is elegant and streamlined which sometimes hides the motivation or the fact why things are as they are. In the hands of an experienced lecturer—such as the authors—this is a cleverly written and most useful text, the very beginner may need more details.

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Dresden, June 2019

MSC2010 Primary: 60-01; Secondary: 60H07; 60J65; 60G51; 60G55; 60H10.