

Review commissioned by: *Mathematical Reviews*

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**Peter Mörters and Yuval Peres: Brownian Motion.** Cambridge University Press, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge 2010, xii + 403 pp., £40.00, US-\$ 65.00, ISBN 978-0-521-76018-8.

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Brownian motion or the Wiener process is arguably the single most important example of a stochastic process. Over the last 100-or-so years it has been a mathematical success story, firmly establishing Brownian motion in ‘almost all’ graduate curricula. Already the birth as one of Einstein’s three miracles of the *annus mirabilis* 1905 had been glorious, its first mathematically rigorous construction by Norbert Wiener in 1923 introduced measure theory into probability theory (quite some years before Kolmogorov’s axioms) and the tender loving care of Paul Lévy and Kiyosi Itô paved the way which led Brownian motion to become one of the central objects of mathematics in the second half of the 20th century.

It is pretty much here where the storyline of the monograph by Mörters and Peres begins. The authors set out to capture the spirit of Paul Lévy’s great classic *Processus Stochastiques et Mouvement Brownian* [MR 0029120, MR 0190953] both in the presentation and the choice of the material. The focus is on sample path properties—which makes this book different from most of its predecessors focussing on Brownian motion as a diffusion process, on potential theory, on stochastic calculus or on white noise analysis—and the presentation is rather direct, coming quickly to the questions at hand. The proofs, again in the spirit of Lévy, are elegant and they work out the ideas very clearly. On the other hand, the already concise presentation relies often on a description how to prove things rather than explicit formulae, making the text not always easy for the beginner. The text is accompanied by numerous exercises which form an integral part of the book. Particularly important problems are flagged and hints or complete solutions are included in an appendix.

Not much time is spent on the construction of one-dimensional Brownian motion—which is Paul Lévy’s original interpolation construction from 1940 [MR 0002734]—and classical path properties, e.g. the smoothness of the sample paths. All this takes place in the opening chapter. Chapter 2 still contains basics, here we encounter  $d$ -dimensional Brownian motion and explore the (strong) Markov property of the Wiener process. Very naturally, the authors’ approach to the (strong) Markov property is the observation

that  $B = (B_t)_{t \leq \tau}$  and  $B^{(\tau)} = (B_{t+\tau} - B_\tau)_{t \geq 0}$  are, for every stopping time  $\tau$ , independent Brownian motions; this Brownian (strong) Markov property is stronger than the usual (strong) Markov property, but it makes it much easier to give a rigorous justification for the ‘naive’ approach to the reflection principle. The chapter ends with a short study of Brownian motion as a martingale. The relation of Brownian motion to harmonic analysis and PDEs is developed in Chapter 3. The presentation is probabilistic (much in the spirit of Kai Lai Chung), avoiding the technicalities of semigroup theory and infinitesimal generators and, at the same time, giving a clear probabilistic meaning to objects such as Green’s function or harmonic measure. This theme is continued in Chapter 8 where Dirichlet’s and Poisson’s problem are revisited and the connection between polar sets and capacity-zero sets is studied. Here we also find a complete proof of Wiener’s test of the regularity of boundary points.

The first three introductory chapters are followed by the presentation of some of the main tools of the book: fractal dimensions, covering techniques, capacities and Frostman’s lemma. The idea is that many properties of the Brownian paths are best understood within the framework of random fractals, trees and (fractal) geometric measure theory. Although the ideas are not new—they date back to McKean’s papers in the late 50s, Hawkes’ and Kahane’s contributions in 60s and 70s—it is surprisingly difficult to find an easily accessible presentation *ad usum delphini*. Here the authors do an excellent job and this short, technical chapter is eminently readable. As we will see, Hausdorff measure and Hausdorff dimensions are central ingredients for the remaining 200 pages of the book.

Chapter 5 returns to the proper topic of the tract: the Brownian paths, this time seen from a random walk perspective. We get a gentle introduction to (path) properties which have ‘random-walk character’, culminating in Donsker’s invariance principle and some fluctuation identities. The proof of Donsker’s theorem is based on Skorokhod’s embedding theorem which can be seen as the missing link for the duality between Brownian motion and classical random walks. The exposition of Brownian local time (Chapter 6) is best characterized as *Paul Lévy’s way to his local time* [MR 1118437] leading to an elegant existence proof for Brownian local times by random walk approximations. Other topics of this chapter are an introduction to Ray-Knight theorems and an interpretation of local time as the Hausdorff measure of the time spent in a point  $x$ . Stochastic calculus is not the main aim of the book and therefore its presentation is limited to the bare necessities needed later on. The readers get a brief introduction to Itô’s stochastic integral and the change-of-variable formula (Itô’s Lemma) which is generalized to non-smooth functions, i.e. Tanaka’s formula. Continuing the theme

from Chapter 3 and as a preparation of Chapter 8, the authors use stochastic integration to give a probabilistic proof of the Feynman-Kac formula. The highlight of this chapter, however, is the proof of the conformal invariance of Brownian motion and of harmonic measure. This may well be seen as *hors-d'œuvre* for Chapter Eleven treating stochastic Loewner evolution.

While most topics up to this point were in one way or another well known, Chapters 9 and 10 contain topics of the authors' own research. Chapter 9 focusses on self-intersections of Brownian paths, in particular, the existence of multiple points and the size—again measured in terms of Hausdorff measures—of the set of multiple points. The rather surprising fact (due to Dvoretzky-Erdős-Kakutani and Le Gall) that there is some  $x \in \mathbb{R}^2$  such that for a planar Brownian motion the set  $\{t : B_t = x\}$  is *not countable* is undoubtedly one of the highlights of the book; the new proof given here is easier than that by Le Gall. The closing chapter of the book is again devoted to exceptional sets of Brownian motion. Using methods from fractal geometry, fast points, slow points and cone points of planar Brownian motion, in particular their (packing) dimension, are studied.

In an appendix, a short non-technical and not completely rigorous survey of stochastic Loewner evolution is given; the focus is on how certain Loewner exponents are related to dimension numbers of some classes of exceptional points of (self-avoiding) planar Brownian motion. Originally, the authors asked Oded Schramm to write this chapter; after Oded's tragic death Wendelin Werner completed the task. The discussion is similar to Werner's survey talk in Perthame et al.: *Leçons de mathématiques d'aujourd'hui—III*, Cassini, Paris 2007 [MR2572391].

This is a lovely written book guiding the interested reader from the humble beginnings to the cutting edge of current research in Brownian motion. It excels in its careful selection of topics, the very clear presentation and, although quite advanced material is presented, never gives the reader the impression of being fraught with technicalities.

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