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**Mandrekar, Vidyadhar S. and Redett, David A.:** *Weakly stationary random fields, invariant subspaces and applications*. CRC Press, Chapman & Hall, Boca Raton 2018. x + 182 pp., EUR 127.99, £110.00, US-$ 139.95 (RRP). ISBN 978-1-138-56224-0.

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A weakly stationary process is a (real or complex-valued) random process $X = (X_t)_{t\in T}$ which has finite second moments $\mathbb{E}|X_t|^2 < \infty$ and whose covariance function $\operatorname{cov}(X_s, X_t)$ depends only on $t - s$. In the monograph under review the focus is on $T = \mathbb{Z}$ (weakly stationary random sequences) and $T = \mathbb{Z}^2$ (weakly stationary random fields) – the fact that the authors consider only $d = 2$ is not essential, most results carry over to higher-dimensional index sets. A word of caution seems to be in order: 'stationary' or 'strongly stationary' often refers to the situation where the finite-dimensional distributions of $X$ are invariant under shifts in the parameter set, i.e. $\operatorname{Law}(X_{t_1}, \dots, X_{t_n}) = \operatorname{Law}(X_{t_1+h}, \dots, X_{t_n+h})$ for all $n \in \mathbb{N}$ and $h \in T$. To distinguish this notion from the type of stationarity considered here, many authors use 'weakly stationary', 'stationary in the wide sense' or 'stationary in the sense of Khintchine'.

Along with Markov processes, strongly and weakly stationary processes belong to the most important stochastic processes and they have been studied early on. The relaxation of the notion of strong stationarity goes back to Khintchine [Korrelationstheorie der stationären stochastischen Prozesse. *Math. Ann.* 109 (1934), MR1512911] and H. Wold [*A Study in the Analysis of Stationary Time Series*, Almqvist & Wiksell, Uppsala 1938, 2nd edn. 1954], starting a vibrant field with seminal contributions by Kolomogorov on extrapolation (nowadays we would use the term 'prediction') and filtering in the 1940s [e.g. MR0009098, Stationary sequences in Hilbert's space. *Boll. Moskovskogo Gosudarstvenogo Universiteta. Matematika* 2 (1941)], M. Krein [On an extrapolation problem of A.N. Kolmogorov. *Dokl. Akad. Nauk SSSR (N.S.)* 46 (1945)], Wiener's influential *Extrapolation, Interpolation, and Smoothing of Stationary Time Series. With Engineering Applications* [MIT Press & John Wiley 1949, MR0031213] and culminating in the famous structure result by Karhunen [Über lineare Methoden in der Wahrscheinlichkeitsrechnung. *Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys.* (1947), MR0023013].

This part of the story is covered in the first Chapter (pp. 1–39) which gives a careful introduction to weakly stationary random sequences and their spectral and moving-average representations. The Wold decomposition which splits every weakly stationary process into two orthogonal parts – one with zero innovation (singular) and one with non-zero innovation (regular) – is done in a functional analytic way, and the exposition aims to characterize all regular sequences in terms of their spectral measure. This is finally achieved in the Szegö–Krein–Kolmogorov theorem.

In Chapter two (pp. 40–107) stationary random fields indexed by $\mathbb{Z}^2$ are discussed. This is also the main part of the monograph. The approach follows closely the lay-out of the first chapter. Since $\mathbb{Z}^2$ does not have a total ordering, the Wold decomposition, regularity and singularity are treated for the 'horizontal' and 'vertical' directions. This corresponds to the half-space ordering of $\mathbb{Z}^2$. Under the additional assumption of a commuting condition which relates the projections onto the closed liner subspaces generated by $\{X_{j,k}, j \leq m \,\&\, k \leq n\}$ with those of $\{X_{j,k}, j \leq m \,\&\, k < \infty\}$, $\{X_{j,k}, j < \infty \,\&\, k \leq n\}$, a fourfold Wold decomposition is established. Under this condition, there exists also a quarter-plane moving average representation. Replacing the half-space ordering of $\mathbb{Z}^2$ by a semigroup ordering (i.e. if $(S, +)$ is a semigroup in $(\mathbb{Z}^2, +)$ we say that $(m', n') \prec (m, n)$ iff $(m' - m, n' - n) \in S$) the analogue of the Szegö–Krein–Kolmogorov theorem (due to Helson and Lowdenslager [MR0097688, Prediction theory and Fourier series in several variables. *Acta Math.* 99 (1958)] is presented. In the following two sections – on the semigroup moving average representation and Wold-type decompositions – the connections of the Helson–Lowdenslager theory with weakly stationary random fields is fully developed; this part was merely sketched in [MR0097688].

Chapter three (pp. 109–133) contains a careful rendering of Halmos' results on decomposition of complex Hilbert spaces in the presence of isometries; with these methods, Beurling's theorem on shift-invariant subspaces of $H^2(\mathbb{T})$ (this is the Hardy space on the torus $\mathbb{T} \subset \mathbb{R}^2$) is derived. With weakly stationary random fields indexed by $\mathbb{Z}^2$ in mind, these concepts are then generalized to pairs of isometries and $\mathbb{T}^2$, leading to a fourfold Halmos decomposition and the characterization of invariant subspaces of $L^2(\mathbb{T}^2)$; it is not suprising that, again, a certain commutation property (of the isometries) plays a central role. The material in Chapter 3 follows an idea of Masani [MR0140930, Shift invariant spaces and prediction theory. *Acta Math.* 107 (1962)] who uses Halmos' decomposition to establish the close relation between the analysis of shift-invariant subspaces of $H^2(\mathbb{T})$ and the prediction theory of weakly stationary stochastic processes. In this sense, the material in Chapter 3 is supplementary, but it leads to an elegant unifying presentation and a deeper understanding of the material.

Tha last Chapter (pp. 135–169) contains a few applications, mainly on texture identification, and a brief outlook on non-square integrable sequences: (non-Gaussian) harmonizable symmetric $\alpha$-stable sequences. This is very much in the spirit of Samorodnitsky & Taqqu [MR1280932 (95f:60024) *Stable non-Gaussian random processes.* Chapman & Hall, New York, 1994] and the deep generalization by Rosiński [MR1349166 (96k:60091) On the structure of stationary stable processes. *Ann. Probab.* 23 (1995)].

Summing up, this is an interesting contribution to the literature where – for the first time in a monograph – the connections between weakly stationary random fields and invariant subspaces are presented. The material is very clearly presented and the first two chapters may well be used in an introductory course at graduate level.

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