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Serge Cohen, Alexey Kuznetsov, Andreas E. Kyprianou and Victor Rivero: *Lévy Matters II: Recent Progress in Theory and Applications: Fractional Lévy Fields, and Scale Functions.* Springer, Lecture Notes in Mathematics **2061**, Berlin 2012, xii + 186 pp., €37.40, £31.99, US-$ 49.95 ISBN 978-3-642-31406-3.

The series *Lévy Matters* is a subseries of the Springer Lecture Notes in Mathematics which is devoted to contributions around Lévy processes, their generalizations and related topics. The present booklet is the second volume of this subseries and, like its predecessor volume, comprises independent contributions

pp. 1–95 *Fractional Lévy Fields* by Serge Cohen [MR 3 014 146].

pp. 97–186 *The Theory of Scale Functions for Spectrally Negative Lévy Processes* by Alexey Kuznetsov, Andreas Kyprianou and Victor Rivero [MR 3 014 147].

Both papers are authoritative survey papers covering recent progress in their respective field.

Cohen's paper is a review of fractional Lévy fields, in particular random fields which are neither Gaussian nor stable. The following two models are considered:

(A) Moving-average random fields which are given by

$$X_H(t) \sim \int_{\mathbb{R}^d} \left(\|t-s\|^{H-d/2} - \|s\|^{H-d/2} \right) M(ds)$$

where $H \in (0,1)$, $H \neq d/2$ and $M(ds) = \int_{\mathbb{R}} x\tilde{N}(ds,dx)$ is a Lévy random measure; that is, $N(ds,dx)$ is a Poisson random measure on $\mathbb{R}^d \times \mathbb{R}$ whose control measure (compensator) is of the form $ds \times \mu(dx)$ for some Lévy measure μ on $\mathbb{R}$; in dimension $d=1$, such random measures represent the jump structure of Lévy processes. Throughout it is assumed that μ has finite pth moments of any order $p \geq 2$.

(B) Real harmonizable fractional Lévy fields which are of the form

$$X_H(x) \sim \int_{\mathbb{R}^d} \frac{e^{-ix\xi} - 1}{\|\xi\|^{H+d/2}} M(d\xi)$$

with a complex isotropic random Lévy measure given by

$$M(d\xi) = \int_{\mathbb{C}} zN(d\xi, dz) + \int_{\mathbb{R}^d} \bar{z}N(-d\xi, dz)$$

and $\tilde{N}(d\xi, dz)$ is a compensated Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$. The control measure μ is isotropic in $\mathbb{C}$ and admits again pth moments of any order $p \geq 2$.

The paper starts out with a brief review of random measures—in particular Poisson, Lévy and stable random measures—and stable random fields, both Gaussian and non-Gaussian. This part of the paper does not contain all proofs, but it is nevertheless very useful for the novice. The main body of the paper are Chapters 4–6 where Lévy fields are introduced (Chapter 4) with the primary focus on their construction, sample path regularity and statistical self-similarity properties. Although both models of fractional Lévy fields have the same covariance structure as a fractional Brownian field, their sample path properties differ significantly. This shows that the covariance alone is not enough to study such processes. Chapter 5 is devoted to statistical questions related to fractional Lévy fields. For both real harmonizable and moving average Lévy fields estimators for the index H are constructed, thus allowing to identify the underlying Lévy fields from real-world data and to calibrate the model. The rather lengthy and technical proofs are given in the appendix.

The concluding Chapter 6 covers simulation of various Lévy fields provided that their integral representation is known. It starts with a careful error analysis and discussion of the rate of convergence for the approximation by series expansions. Then the very useful 'normal approximation' method is discussed which essentially replaces the small jumps by a Gaussian process with the same covariance structure as that of the small-jump part. The chapter closes with a brief outlook on the simulation of harmonizable infinitely divisible random fields.

There is a partial overlap with Chapters 4 and 5.4 of the recent monograph by Serge Cohen and Jacques Istas: *Fractional Fields and Applications* (Springer, Berlin 2011 [MR 3088856]). Since that monograph concentrates mainly on Gaussian fields, the present survey paper is a complement rather than a competitor.

The paper by Kuznetsov *et al.* is a state-of-the art survey on scale functions for spectrally negative (i.e. having only negative jumps, one-dimensional) Lévy processes. The reader is assumed to have a basic working knowledge of Lévy processes, otherwise most techniques are explained (or referenced) in detail. A spectrally negative Lévy process X has a finite Laplace transform $\mathbb{E}e^{\lambda X_t} = e^{t\psi(\lambda)}$ where ψ is the characteristic (Laplace) exponent given by the Lévy-Khintchine formula. The q-scale function is the unique right-continuous function such that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\,dx = \frac{1}{\psi(\lambda)-q}, \quad \lambda \gg 1,\ q \geq 0.$$

Scale functions appear naturally in fluctuation theory, typically in exit and level-crossing problems:

$$\mathbb{E}^x\left(e^{-q\tau_a^+}1_{\{\tau_a^+<\tau_0^-\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \quad 0 < x < a,\ q \geq 0,$$

where τ_a^+ is the first time $X_t > a$ and τ_0^- is the first time $X_t < 0$. In the Russian literature $W^{(q)}$ is often called the resolvent.

The paper starts with a brief overview on applications of scale functions in various fields. Chapter 2 treats the existence of scale functions, their relation to excursions and their role in fluctuation theory of spectrally negative Lévy processes, in particular the Wiener-Hopf decomposition. In Chapter 3 various analytical properties of scale functions are discussed: Their asymptotic behaviour at 0 and $+\infty$, convexity, analyticity in the parameter q, and the smoothness of $x \mapsto W^{(q)}(x)$ in terms of the Lévy (jump) measure.

A major problem is to find explicit examples of scale functions. Using the relationship between scale functions and the potential measures of a subordinator, the authors give a recipe how to 'engineer' many examples of scale functions. The key ingredient here is the theory of Bernstein functions, in particular special and complete Bernstein functions, see Schilling, Song, Vondraček: *Bernstein Functions. Theory and Applications.* De Gruyter, Berlin 2010 [MR 2978140] and 2012 [MR 2978140]. The final Chapter 5 is devoted to several numerical methods to calculate scale functions for concrete spectrally negative Lévy processes.

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