
Jossi Klafter and Igor M. Sokolov: *First Steps in Random Walks. From Tools to Applications.* Oxford University Press, Oxford 2011, vi + 152 pp., €54.99, £39.95, US-\$ 72.95 ISBN 978-0-19-923486-8.

Anomalous diffusions are widely used in the sciences to describe ‘diffusion’ phenomena whose mean square displacement (variance) is not linear (as for Brownian motion, the archetype of all diffusion models) but sub- or super-linear. There are plenty of examples for such behaviour in nature, e.g. solute transport in heterogeneous media, foraging patterns of animals, mixing of the coins of different Euro countries to name but a few; the hallmark of such behaviour are non-Gaussian power-law tails of the probability distribution function. Among the most popular models for anomalous diffusions are continuous-time random walks (CTRWs) with not necessarily exponentially distributed waiting times between the steps, Lévy flights (which is physicists’ parlance for stable pure-*jump* Lévy processes) and fractional Brownian motions. Such phenomena were, for the first time, described during the decade 1965–1975 which is marked by two seminal papers by Montroll and Weiss [MR0172344 Random walks on lattices. II. *J. Mathematical Phys.* **6** (1965) 167–181] and Scher and Montroll [Anomalous transit-time dispersion in amorphous solids. *Phys. Rev. B.* **12** (1975) 2455–2477]. Initially, the idea that actual physical phenomena might have infinite spatial or temporal moments was not well-received by other scientists, and it took some time until these models became popular. One of the authors, Jossi Klafter, was at the forefront of this development; by now it is an accepted fact that anomalous diffusions are real.

Although there are many research papers on such phenomena, there has been no (physics) textbook on the mathematical tools of anomalous diffusion. This much needed monograph tries to fill the gap. Starting with ordinary random walks the reader is carefully guided through probability theory: random variables and simple random walks, the central limit theorem, generating functions, Tauberian theorems. In a next step the so-called continuous-time random walks (CTRWs) are introduced where a new operational time scale (‘subordination’) is used to measure the time between two steps of the random walk. The waiting times are themselves random, they may or may not

be independent of the random walk. Contrary to the Markovian world, the waiting times need not be exponential, i.e. here is the possibility to model memory and aging effects and to introduce ergodicity breaking—probably still an unpleasant surprise for the working scientist. Now the focus shifts to the (generalized) master equation which is, essentially, a Kolmogorov or Fokker–Planck equation governing the transition probabilities of the underlying random process. Since we have to deal with anomalous diffusions, the Fokker–Planck equation has to be non-local—i.e. it contains not a differential operator but an integro-differential or pseudo-differential operator—either in time (leading to limits of CTRWs) or space (leading to limits of Lévy walks and Lévy flights) or both. Fractional derivatives (Riemann–Liouville and Caputo fractional derivatives in time, fractional Laplacians in space) are discussed at length, along with the corresponding stochastic processes. The last theoretical chapter is devoted to coupled CTRWs and Lévy walks, where the temporal and spatial movements are not any longer stochastically independent. The book closes with two chapters on various applications and an outlook to random walks on percolation structures and random environments.

Despite of all what has been said so far, this is a difficult book to review. In the eyes of a physicist, this is a *mathematics book*, for a mathematician, it is most likely a *physics treatise*. Let me take the latter stance. Clearly, the presentation is far from Landau style, there are no formal definitions or theorems; proofs are usually reduced to calculations used to introduce and connect enunciations of facts, and quite often mathematical intricacies are gentleman-like swept under the carpet; the mathematics is, roughly, at advanced undergraduate level. What sounds like harsh criticism, isn't—after all as this is a *physics* textbook.

The actual concern is more subtle. The authors reduce random processes exclusively to their one-step distribution functions and the Kolmogorov (aka master) equations. All developments in mathematics after the appearance of (the third resp. second edition of) Feller's magnificent textbook [*An introduction to probability theory and its applications*. MR0270403 (vol II, 2nd edn 1971) and MR0228020 (vol I, 3rd edn)] are not even mentioned. Again, I do not want to blame the authors for this. When physics embraced random walks, mathematics was not interested—the lukewarm review MR0172344 of the important Montroll-Weiss paper on random walks on lattices from 1965 by Frank Spitzer, one of the grand-masters of random walk theory, is symptomatic—, and physics developed in parallel; not much interaction

took place. Let us take a different example: the MR database (accessed in August 2012) has 375 entries with the phrase ‘anomalous diffusion’ and 260 with ‘sub- or super-diffusion’; this compares badly with Google scholar. One of the most important survey papers in the field (183 quotations in MR, 2376 in Google scholar), [MR1809268 Metzler and Klafter: The random walk’s guide to anomalous diffusion: a fractional dynamics approach] has no proper review, and the second-most quoted paper [49 hits in MR, 616 in Google scholar—Metzler and Klafter: The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics.] is marked as *{There will be no review of this item}*.

I think that this is a timely and important book and mathematicians should read it; not so much for the mathematical contents which are present, but for those which are missing. It is about time that mathematicians and physicists start to talk again. Stochastic models in the sciences are a rich and interesting field and we mathematicians must not neglect it.

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