To appear in: *ZMath – Zentralblatt*

N. Jacob: Pseudo-differential operators and Markov processes Vol. III: Markov Processes and Applications. London: Imperial College Press, ISBN 1–86094–568–6. xxviii, 474 pp. £48.00; US-$ 78.00 (2005).

This is the third and final instalment of the treatise *Pseudo-Differential Operators and Markov Processes.* The first two volumes, vol. 1 *Fourier Analysis and Semigroups* and vol. 2 *Generators and Their Potential Theory* are reviewed in this journal, cf. [ZBl. 987.60003] and [ZBl. 1005.60004], respectively.

The overall theme of the trilogy is the connection between operator semigroups—both in L^p and C_∞—, infinitesimal generators and stochastic processes. Part III concentrates on stochastic processes and, in particular, their construction (starting from a given generator) and their probabilistic properties. As we know from the first two volumes, almost all Feller processes and many L^p sub-Markovian semigroups are generated by pseudo-differential operators. In the present volume it is shown that one can use a single deterministic function, the *symbol* of the pseudo-differential operator, to construct and analyse the corresponding stochastic process.

In texts like these the reader is confronted with advanced material from both analysis and probability theory. Similar to the first two volumes which require substantial analytic background and where the author took great care to introduce probabilists to higher analysis, it is now the analysts' turn to be initiated to probability theory. The exposition starts with a rapid review of the basics, e.g. measurability aspects, extensions of measure theory (which are needed in probability theory but not always covered in 'analytic' measure theory courses), conditional expectations and continuous-time martingales. The classical Kolmogorov construction for stochastic processes *of function space type* on the product space $E^{[0,\infty)}$ is presented in Chapter 3; building on this all the 'usual' and 'standard' topics of a course on stochastic processes are presented: regularization of sample paths, the Markov and strong Markov properties, stopping times, shift operators and a first review of the martingale problem which will take centre stage in the following chapter. As an illustration the author constructs Lévy processes from convolution semigroups. This class of processes is also chosen for a different reason: the symbol of a Lévy process is the characteristic exponent (i.e. the logarithm of the characteristic function) and it is well known that this is the very object to study path properties. A few properties (path decompositions, Hausdorff dimensions, variations ...) are briefly mentioned as examples. The chapter closes with the stochastic interpretation of the symbol of the generator of a general Feller process and provides examples and an explanation as to why the symbol should be (and indeed is) the analogue of the Lévy exponent when it comes to path properties.

Chapter 4 is devoted to the martingale problem in Skorokhod space. It starts with a few generalities and then presents the existence proof for a large class of Feller processes with prescribed generator using the martingale problem; well-posedness (hence, uniqueness) is treated in a separate section. This elegant and powerful approach is due to W. Hoh.

A different class of processes is discussed in Chapter 5: Hunt processes generated by L^p semigroups. This chapter is motivated by Fukushima's construction of Hunt processes from an L^2 semigroup using Dirichlet forms. The chapter continues with the potential theoretic considerations from volume 2 and completes them with elements from probabilistic potential theory for Hunt processes; these are needed in order to associate a stochastic process with L^p semigroups. The main result of this chapter is the theorem that (under some weak assumptions) it is indeed possible to construct a process from an L^p semigroup; this theorem of the author appears for the first time in a book.

Probabilistic potential theory for Markov processes is developed in Chapter 6; again the particularly nice case of Lévy processes—where many properties can be traced back and expressed in terms of the characteristic exponents—is thoroughly discussed. The chapter closes with a treatment of the balayage-Dirichlet problem for nonlocal pseudo-differential operators.

The last chapter of the monograph contains a selection of applications and extensions of the theory, e.g. fractional derivatives and boundary value problems (on the half-space) for certain classes of pseudo-differential operators. Another example which should be interesting for applications starts with a Lévy process whose transition probabilities are parameterized families of probability distributions (e.g., Meixner distributions, normal inverse Gaussian distributions etc ...). If one makes the parameters state-space dependent (this is a very realistic assumption for many models) one leaves the realm of Lévy processes and it is actually not clear at all if this will lead to a stochastic process. What is clear, however, is that the generator of such a process is necessarily a pseudo-differential operator and using the theory developed so far it is possible to find (reasonable!) conditions under which one can play with the parameters and still get a stochastic process.