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N. Jacob: Pseudo-differential operators and Markov processes Vol. II Generators and their potential theory. London: Imperial College Press, ISBN 1–86094–324–1. xxii, 453 p. £46.00; US-$ 68.00 (2002).

The present monograph is the second part of the three-volume treatise *Pseudo-Differential Operators and Markov Processes*: vol. 1 *Fourier Analysis and Semigroups*, see the review [ZBl. 987.60003], vol. 2 *Generators and Their Potential Theory* which is under review here and vol. 3 *Markov Processes and Applications* which is scheduled for autumn 2004. The overall theme of the treatise is the construction and study stochastic processes and sub-Markovian operator semigroups through their infinitesimal generators. The key observation is that almost all Feller processes and many L^p sub-Markovian semigroups are generated by pseudo-differential operators; this allows to reduce existence questions and even the study of stochastic properties of the associated processes to the knowledge of the *symbol* of the pseudo-differential operator, i.e., a single deterministic function.

This approach uses techniques from numerous mathematical disciplines, notably from harmonic analysis, PDEs and, of course, probability theory. This problem has been addressed in the first volume where the various basic analytical techniques are treated. Volume 2, although principally self-contained, builds on the material in the first part of the treatise and gives often very precise cross-references to its companion volume. The text is divided into three chapters, a 12 page *Introduction* (Chapter 1) which is a brief overview of the material in the following two main sections (Chapters 2 and 3) each of which comprises some 200 pages.

Chapter 2 on *Generators of Feller and Sub-Markovian Semigroups* starts with a look at elliptic as well as degenerate second-order differential operators $L = L(x, D)$. The main theorems are sufficient conditions as to when such operators generate sub-Markovian semigroups in $L^p(\mathbb{R}^n, dx)$ resp. $C_\infty(\mathbb{R}^n)$ (strongly elliptic case), resp., $L^2(\mathbb{R}^n, dx)$ in the degenerate case. The construction uses the Hille-Yosida (-Ray) Theorem where the hard part is to show that the range of $\lambda\mathrm{id} + L$ is dense in the Banach space under consideration. This amounts to solving $\lambda u + Lu = f$ in $\mathbb{R}^n$ for sufficiently many f. This problem is well understood and the author just collects (mainly without proofs) the necessary tools from classical sources. The degenerate case is treated using Dirichlet forms. The main thrust of the exposition, however, is aimed at non-local operators and here the monograph contains (for the first time in book form) the author's original contributions to the field. The rest of the chapter is devoted to the construction of Fellerian and sub-Markovian semigroups starting with (typically: non-local) pseudo-differential operators with negative definite symbols. As in the local case, the idea is to integrate the associated Kolmogorov equation using the Hille-Yosida (-Ray) Theorem which, again means solving the pseudo-differential equation $\lambda u + p(x, D)u = f$. The problem is that the symbols, negative definite functions in the sense of I.J. Schoenberg *Trans. Am. Math. Soc.* **44** (1938) 522–536 [ZBl. 0019.41502], are not smooth, neither homogeneous nor are they known to have a principal part. Consequently, many powerful techniques from the classical theory of pseudo-differential operators are not available at all or only in a very restricted way. The way around are either *ad hoc* methods or a more systematic approach (mainly due to W. Hoh *Osaka J. Math.* **35** (1998), 798–820 [ZBl. 0922.47045]) using a specially adapted first-order symbolic calculus (very much in the spirit of H. Kumano-go *Pseudo-Differential Operators*, MIT Press, Cambridge (Mass.) 1981 [ZBl. 0489.35003]) which can be used for a large class of negative definite symbols. This calculus yields Calderon-Vaillancourt-type boundedness estimates in L^2 as well as Gårding-type inequalities which are used to solve the above mentioned Ψdo equation. The chapter concludes with a brief review of other analytical constructions of Feller semigroups, some perturbation results which allow to relax some smoothness requirements for the symbols. As an application, operators with variable order of differentiation as well as generators of (Bochner-)subordinate semigroups are considered; this includes a detailed discussion of symbolic pseudo-differential operator vs. the functional calculus which is canonically associated with Bochner's subordination.

Chapter 3 contains elements of the *Potential Theory of Semigroups and Generators*. In a preparatory section results for capacities and abstract Bessel potential spaces associated with L^p sub-Markov semigroups are collected (with proofs). This is then implemented in the concrete situation where the semigroup is of convolution type, i.e., has a constant coefficient generator. The associated abstract function spaces can, in this case, be identified with a scale of anisotropic L^p Bessel potential spaces $H_p^{\psi,s}$ and various properties of these spaces, embeddings, interpolation results, capacities, quasi-continuous modifications etc., are treated in detail. This is followed by a digest of E. Stein's Littlewood-Paley Theory for L^p sub-Markovian semigroups, and one of the main applications are criteria guaranteeing the L^p boundedness of purely imaginary powers of infinitesimal generators. The chapter closes with *global properties* and *Nash and Sobolev type inequalities* of L^p semigroups. By global behaviour, one means recurrence, transience, irreducibility, comparison of capacities and various other "ergodic" properties of the semigroups along with on- and off-diagonal heat kernel estimates. This is, essentially, a non-symmetric L^p version of the global properties as they are being studied in the context of symmetric Dirichlet forms in the space L^2.