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N. Jacob: Pseudo differential operators and Markov processes. In 3 vol. Vol. 1: Fourier analysis and semigroups. London: Imperial College Press, (ISBN: 1-86094-293-8). xxii, 493 p, (2001).

This is the first of three volumes in a treatise on *Pseudo differential operators and Markov processes*. Under this common heading three parts are planned, *Fourier analysis and semigroups* (the volume under review), *Generators and potential theory* (vol. 2, to appear in winter 2002) and *Markov processes and applications* (vol. 3). The intention of these monographs is to give a complete and self-contained account of the theory of stochastic processes which are generated by pseudo differential operators, in particular on Lévy-type and Feller processes. Many contributions to the field were developed over the last ten years by the author and the treatise is a state-of-the art summary of the analytic side of the theory; various new and unpublished results are included. The present work elaborates a previous research note by the author which appeared some years earlier under the same title *Pseudo-differential operators and Markov processes* [Akademie Verlag, Berlin 1996, ZBl. 860.60002] and which was a non-technical survey of the topic, essentially without proofs.

The new monograph remedies this shortcoming. It is written in a style that is accessible to non-specialists and novices and strives to be as self-contained as possible. There are some good reasons for this: the subject under investigation requires tools from quite different disciplines, extending from modern methods of PDEs, harmonic analysis, semigroup theory, to potential theory, Dirichlet forms and, later, stochastic processes. Results which are needed as prerequisites are collected with careful references to accessible sources and more central results are developed in the text. This digest of material is not only an invaluable service for the reader but it serves also a further purpose: different theories have different notations and normalizations for the same objects. The present exposition gives a uniform and consistent approach aiming to close the (language-) gap between analysis and probability theory.

Volume 1 on *Fourier analysis and semigroups* contains mostly preparatory material which is, on the one hand, of ancillary nature to the main topic of the treatise but, on the other, of considerable interest in its own. There are four chapters, a brief *Introduction* and a terse 60-page review on *Essentials from analysis* where some basic material is collected, mainly for referencing purposes and notational convenience. Topics range from elements of calculus (inequalities, norms...), Schwartz distributions and basic functional analysis to more specialized aspects of measure theory (monotone class theorems, topological considerations, Bochner-integrals), convexity and interpolation theory.

The bulk of the material is in Chapters 3 and 4 on *Fourier analysis and convolution semigroups* (about 175 p.) and *One-parameter semigroups* (about 200 p.). Chapter three provides an introduction to the classical Fourier transform in the spaces $\mathcal{S}$ of Schwartz functions and L^p and to the Fourier transform of distributions $\mathcal{S}'$ and positive Borel measures. The presentation is already geared towards probability theory insofar as considerable weight is given to the notion of *positivity*; in particular, Bochner's Theorem and positive definite functions are discussed. The related notions of negative definite functions (in the sense of Schoenberg, central also to probability theory in the form of the Lévy-Khinchine formula) and their properties are treated in great detail and, similarly, the corresponding objects on the positive half-line with the one-sided Laplace-transform: completely monotone functions and Bernstein functions; a purely analytic proof for the Lévy-Khinchine formula in $\mathbb{R}^n$ is also included. Negative definite functions will play a major role in the study of generators of Feller processes. Therefore, a scale of anisotropic function spaces—modelled on the Hörmander $B_{p,k}$-class used to study regularity of partial differential operators—is introduced, and properties like embeddings and comparison results are proved. The chapter closes with a brief survey of various classical function spaces of Besov- and Triebel-Lizorkin type (without proofs) and a section where the Mikhlin-Hörmander Multiplier Theorem is proved.

Strongly continuous operator semigroups are treated in Chapter 4. The emphasis is not so much on the general theory, which is being developed up to and including the Hille-Yosida Theorem, elements of duality theory and some standard perturbation and approximation theory, but more on positivity preserving and sub-Markovian semigroups which crop up in probability theory. A careful proof of Stein's result, showing that symmetric sub-Markovian semigroups on L^2 (L^p) are analytic, is also included. *Subordination* (in the sense of Bochner) of semigroups and a related functional calculus (on the level of infinitesimal generators of semigroups) are presented in great detail. The discussion then moves on to Feller semigroups, i.e. Markov semigroups that leave the space of continuous functions vanishing at infinity invariant. The central result is the theorem of Courrège which states that the infinitesimal generators of Feller semigroups are necessarily pseudo-differential operators with negative definite symbols. The problem to find sufficient conditions on the symbol under which a pseudo-differential operator is a Feller generator, will be one of the main topics in volumes 2 and 3. Finally, sub-Markovian semigroups on L^p are discussed. Their treatment is in close analogy to L^2-sub-Markov semigroups appearing in connection with (non-symmetric) Dirichlet forms. The discussion focusses again on the generator; the notion of *L^p-Dirichlet operator* is introduced as a necessary and sufficient criterion for sub-Markovianity of the semigroup (close to the Beurling-Deny criteria). The case $p = 2$ and the theory of (non-symmetric) Dirichlet forms are outlined in a separate section; much emphasis is placed on non-local forms generated by pseudo-differential operators and concrete examples of Dirichlet forms. Worth mentioning is the somewhat unconventional treatment of *closability*: here the PDE viewpoint is dominant and closability is deduced from a lower inequality of Gårding-type and some boundedness estimates for the form. The chapter closes with a brief exposition how to extend Feller and sub-Markovian semigroups and their relation to the various possible definitions of the Feller property.