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Francis Hirsch, Christophe Profeta, Bernard Roynette, Marc Yor: *Peacocks and Associated Martingales, with Explicit Constructions*. Springer, Bocconi & Springer Series Mathematics, Statistics, Finance and Economics vol. **3**, Milan 2011, xxxii + 384 pp., €102.99, £81.00, US-$ 129.00 ISBN 978-88-470-1907-2.

This is a highly specialized monograph which explores the question how to construct (real-valued) martingales with prescribed one-dimensional marginals. At the origin is the following result due to H.G. Kellerer [Markov–Komposition und eine Anwendung auf Martingale, *Mathematische Annalen* **198** (1972) 99-122. MR0356250]: Let $(M_t)_{t\geq 0}$ be a real-valued martingale. Then $t \mapsto \mathbb{E}\psi(M_t) \in (-\infty, \infty]$ is, for every convex function ψ, an increasing function. Conversely, if $X = (X_t)_{t\geq 0}$ is a stochastic process such that $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$ and

$$t \mapsto \mathbb{E}\psi(X_t) \in (-\infty, \infty] \quad \text{is increasing for all convex } \psi, \tag{$*$}$$

then there is martingale $M = (M_t)_{t\geq 0}$ such that for every epoch $t \geq 0$ the random variables M_t and X_t have the same probability law; in general, X and M will be defined on a different probability spaces; one might call M a *martingale realization* of X.

Kellerer's results lay dormant for almost 30 years—the first citations are due to Müller and Rüschendorf [MR1860205] in 2001 and Madan and Yor [MR1914701] in 2002—which may well be due to the fact that the 1972 paper uses a slightly different stochastic ordering which does not yield an equivalence in terms of martingales (but rather sub-martingales—with only little effort one can show that it is, nevertheless, equivalent to the above statement).

The condition $(*)$ is known as the *convex order* and a process X satisfying $(*)$ is called (in French) a *Processus Croissant pour l'Ordre Convexe* (i.e. A stochastic process which increases with respect to the convex order); with a little imagination the acronym PCOC can be pronounced as *pea-cock*.

The difficult direction in Kellerer's theorem is the existence of the martingale M; unfortunately, the 1972 proof is not constructive. Madan and Yor addressed this problem in [MR1914701] in 2002, but the essential hint came from the paper P. Carr, C.O. Ewald and Y. Xiao: On the qualitative effect of volatility and duration on prices of Asian options, *Finance Research Letters* **5** (2008) 162–171 [not covered by MR] where, among other things, it is shown that for a standard Brownian motion $\left\{t^{-1}\int_0^t \exp(B_s - s/2)\,ds,\ t > 0\right\}$ is a peacock. The connection to mathematical finance is obvious: The payout profile of a (European) call is a convex function and it seems to be reasonable that the price of an option increases with the strike time. For an Asian call the option price is $\mathbb{E}\left[\left(t^{-1}\int_0^t M_s\,ds - K\right)^+\right]$ and if $\left\{t^{-1}\int_0^t M_s\,ds,\ t > 0\right\}$ were a peacock, the pricing of the Asian call could be reduced to a European call for the peacock's martingale realization.

With these motivations in mind, the authors explore in the first chapter various examples of peacocks. They identify two strains of peacocks: The F_1-strain of the form $\left\{\alpha(t)^{-1}\int_0^t M_s\,d\alpha(s),\ t > 0\right\}$ where M is a martingale and $\alpha : [0,\infty) \to [0,\infty)$ is deterministic, continuous and increasing function, and the F_2-strain $\left\{\int_0^t (M_s - M_0)\,d\alpha(s),\ t > 0\right\}$ with M, α as before; a typical F_2-animal would be $(tX)_{t\geq 0}$ where X is an integrable random variable. Further pheasants are obtained as temporal mixtures of certain (exponential) martingales where a further parameter, e.g. the volatility, takes over as the new operational time.

Over the next six chapters, Chapter 2 – Chapter 7, several methods to construct martingale realizations for a given peacock are discussed. Apart from Chapters 3 and 4 (time reversal and time inversion), these chapters do not depend on each other and can be read separately. Roughly speaking, each chapter has the same structure: Preliminaries and the general method are laid out in the first section, then things are discussed for (geometric) Brownian motion and the Brownian F_1- and F_2-animals and the subsequent sections extend this to more general processes, e.g. Gaussian or additive/Lévy processes. Given the whole henhouse of peacocks it is natural that many results are stated in the form of do-it-yourself exercises (without solutions, about half of them carry hints and instructions); although the core material comes with complete proofs, the huge number of open problems and the broad mixture of mathematical techniques (from analysis, PDEs, probability and stochastic analysis) require a certain sophistication on the side of the readers. The methods to construct martingale realizations range from embedding into two-dimensional processes (Chapter 2: The sheet method), reversal and projective inversion of the time (Chapters 3,4), embedding into self-similar (with index H) additive processes (so-called Hss additive or Sato processes; Chapter 5) over constructions as solutions of tailor-made SDEs (Chapter 6) or realizations as time-changed Brownian motions (Chapter 7: The Skorokhod Embedding Method). Of course, different types of peacocks (even in the same F_j-strain!) require different treatments and there are rare overlaps in the choice of the method. It seems to me that there cannot be a 'general' theory and that the approach must necessarily remain *ad hoc* on a case-to-case basis.

The last chapter is devoted to the question what can be said about higher-order marginals of peacocks and their martingale realizations. Here we encounter the uniqueness problem (can there be more than one martingale realization? — in general this is an open problem) and inequalities of mean values of convex functions involving more than one epoch. The monograph closes with a set of seven open problems.

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