Review commissioned by: Mathematical Reviews

Peter K. Friz, Nicolas B. Victoir: *Multidimensional stochastic processes as rough paths.* Cambridge University Press, Cambridge studies in advanced mathematics vol. **120**, Cambridge 2010, xiv + 656 pp., £50.00, US-$ 85.00 ISBN 978-0-521-87607-0.

The story of rough paths begins with a short paper by Lyons [MR1302388] where he discusses (deterministic, ordinary) differential equations of the form $dy_t = V(y)\,dx_t$ where the driving signal $x : [0,T] \to \mathbb{R}^d$ is not regular enough to admit the 'usual' differential dx_t. For probabilists this is a familiar situation: Brownian motion does not have finite variation, so one needs stochastic (Itô) differentials etc. If $t \mapsto x_t$ is of finite p-variation with $p \in (1,2)$, then—as Lyons points out—one can use Love-Young integrals to arrive at a satisfactory theory of ODEs driven by rough signals. Higher-order numerical approximation schemes for $dy_t = V(y)\,dx_t$ quickly lead to approximations of the form $y_t - y_s \approx \sum_n f_n(V) \int \cdots \int_{s \le t_1 < \ldots < t_n \le t} dx(t_1) \otimes \cdots \otimes dx(t_n)$ where the f_n are suitable linear operators. The sequence of iterated integrals $S(x) = (1, x^1, x^2, \ldots)$ where $x^n = \int \cdots \int_{t_1 < \ldots < t_n} dx(t_1) \otimes \cdots \otimes dx(t_n)$ is called the *signature* of the path x. If $p = 1$ or $p < 2$ we can work out x^n from the path using the Riemann-Stieltjes or the Love-Young integral, respectively.

A classical result by Chen tells us that we can reconstruct x if we know $S(x)$, cf. [MR0085251,MR0106258]. As soon as $p \ge 2$ it is no longer possible to use classical integration theory, and the (geometric) information contained in the iterated integrals must be *a priori* known; for paths arising from stochastic processes, e.g. Brownian paths, this can be done using Itô or Stratonovich integrals. The key observation is that for a path of finite p-variation the truncated signature $S_{[p]}(x) = (1, x^1, \ldots, x^{[p]})$ determines x^N for all $N > [p]$. This means that p is indeed an indicator for the 'roughness' of the paths; the higher p, the longer the signature $S_{[p]}$ has to be. This leads directly to the definition of a *rough path*: a (geometric p-) rough path is a continuous path x which is in the closure (w.r.t. the p-variation norm) of the group comprising elements of the form $S_{[p]}(f)$ where f is of bounded variation. A rough differential equation (RDE) is defined in a similar way as accumulation point (in a suitable Hölder/variational norm) of 'usual' ODEs

$dy^n_t = V(y^n_t)\, dx^n_t$ driven by signals x^n of bounded variation. Observe that the smoothness required of V is directly related to the roughness p of x: the larger p the more regular V has to be. Since the setup for RDEs is similar to that of ODEs, many advantages of the ODE theory can be relatively easily transferred to the rough-path setting, e.g. the dependence on the initial condition and the driving signal (i.e. flow properties), limit theorems (leading to pathwise approximations) or perturbation results. Moreover, the rough path integral complements Itô's theory since it is a powerful theory that covers non-semimartingale driving noises and even fully anticipative integrands.

The present monograph aims to give a self-contained introduction to the theory and applications of rough paths in four parts, 20 chapters and an 60-odd page appendix. Part 1 (pages 19–122) is a resumé of function spaces (in particular p-variation and Hölder spaces), ODEs (with a focus on the solution map and flow properties) and a short introduction to the Love-Young integral. Most of the material is classic, but the selection and the presentation is geared towards later applications to rough path theory. The *abstract theory of rough paths* is treated in the second part (pages 125–324). The presentation is deterministic and path-by-path (reminiscent of Föllmer's *Calcul d'Itô sans probabilités* [MR0622559]). It starts with some algebraic preliminaries on Lie groups, Lie algebras, and function spaces on free groups. This mixture of algebra, geometry, analysis and—later on probability theory—which makes rough path theory hard to access for the novice, and the new presentation does not really help to overcome the sorrows. The notion of a rough path and a rough differential equation is finally introduced from page 195 onwards based on the alternative approach of Davie [MR2387018] which is based on (modified) Euler approximations. Rough differential equations are discussed in great detail still following Davie's ideas. This approach gives natural existence, uniqueness and approximation results (the latter come almost for free due to the 'numerical' approach to RDEs) even under minimal regularity of the coefficients. The highlights of the theory are, however, the particularly nice smoothness properties of Itô-Lyons (solution) map and the flow property of the solution. The final chapter of part 2 is on RDEs with drift. Rather than considering a space-time path (t, x_t) and referring to the general theory, the authors decide to take advantage of the nicer character of dt allowing for (significantly) less smooth coefficients in t-direction. Part 3 (pp. 327–500) focusses on the main topic of the book: the combination of rough paths and stochastic processes. Although all definitions are given, the reader should have some working knowledge of stochastic processes and stochastic analysis. Brownian motion is briefly introduced and enhanced Brownian motion, that is the triplet $(1, B_t, A_t)$ comprising d-dimensional Brownian motion B_t and its Lévy area process A_t, is shown to be a rough path with index $p = 2$—as an aside: the Young integral approach has to fail since BM is almost surely of infinite strong 2-variation! The power and elegance of the rough path machinery is now illustrated using some classical theorems e.g. Wong-Zakai approximations, the Cameron-Martin theorem, large deviations or the support theorem. All of them are sketched for the classical case and worked out in the rough-path setting (i.e. including the Lévy area). In subsequent chapters similar rough path treatments are given to certain semi-martingales, Gaussian processes (including fractional Brownian motion) and Markov processes with continuous paths and generators of the type 'sum of squares of vector fields'. The last part (pages 503–567) is devoted to applications of rough paths to stochastic analysis. It starts with a *working summary on rough paths* and could be read almost as a stand-alone text. Topics include stochastic flows, SDEs driven by non-semimartingales, stochastic Taylor expansions, and the support theorem for RDEs. The focus is, however, on Malliavin's calculus for RDEs and heat kernel properties à la Hörmander. In the massive appendix the proof of the extremely useful Garsia-Rodemich-Rumsey estimate (with many applications to path regularity) is recorded [MR0267632], otherwise the readers are reminded of basics of infinite dimensional analysis, large deviations and local Dirichlet spaces.

The book under review is an interesting addition to the existing literature, notably the seminal paper by Lyons [MR1654527], the book by Lyons and Qian [MR2036784], the survey by Lejay [MR2053040] and the highly readable St.-Flour lecture notes [MR2314753] by Lyons. It is the first textbook-style treatment covering the whole breadth of the subject. Although it is sufficiently self-contained, it is not easy to read. Apart from the few motivating pages at the beginning it is a bit of a problem not to loose interest in the subject when plodding through the material in the middle until the reader is finally guided to the really interesting applications.

René L. Schilling
Institut für Stochastik
TU Dresden
D-01062 Dresden, Germany
rene.schilling@tu-dresden.de