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S.N. Cohen & R.J. Elliott: *Stochastic Calculus and Applications.* Second Edition, Birkhäuser, Probability and Its Applications, New York 2015. xxiii + 666 pp., € 74.89, £ 62.99, US-$ 89.99 (RRP). ISBN 978-1-4939-2866-8.

Elliott's 1982 monograph *Stochastic Calculus and Applications* [MR0678919] was the first comprehensive English-language text (indeed, it was the first non-French textbook treatment) covering P.A. Meyer's *general theory of processes* and the related stochastic calculus; as a matter of fact, this theory was still getting its definitive form laid out in the monumental five-volume treatise by Dellacherie, Maisonneuve (for vol. 5) and Meyer: *Probabilités et potentiel* (5 volumes, Ed. Hermann: Paris 1975, 1980, 1983, 1987, 1992. [MR0488194, MR0566768, MR0727641, MR0898005, vol. 5 not covered by MR]). We read in the preface of the 1982 edition that "it is surprising and unfortunate that, although this general theory is found so useful by theoretical engineers, it is not (with a few significant exceptions) widely taught or appreciated in the English-speaking world" and so the author set out to teach "such natural and basic concepts" along with a few applications (filtering and stochastic control) to a wider audience starting with first-year graduate students.

The wish "that still more applications and results might be found" came indeed true: Over the past three decades stochastic analysis flourished and the general theory advanced from an active field of research to a standard tool taught in every decent graduate programme. This development was facilitated by the emergence of quite a few monographs, for instance Rogers & Williams (1979 & 1987, MR0531031, MR0921238), Ikeda & Watanabe (1981, MR0637061), Jacod & Shiryayev (1987, MR0959133), Karatzas & Shreve (1988, MR0917065), Protter (1990, MR1037262), and Revuz & Yor (1991, MR1083357), just to mention the most influential ones. None of these texts, however, had engineering applications in mind as did Elliott's text.

These developments had to be taken into account in a second edition. There are a few obvious changes: First, the vastly increased size (it has now 666 pages, 22 chapters with subsections and 2 appendices compared with 302 pages, 18 chapters, no subsections and neither an appendix) and the addition of a new author, Sam Cohen, a former PhD student of Elliott. Second, the new role of the text, i.e. its morphogenesis from an avantgarde monograph to a standard textbook covering a big portion of the theory broadly and rigorously. At the same time, the original approach has been retained, starting out with the general setting (*théorie générale*) before the concrete problems are dealt with. What doubles the size of the text is a newly written introductory part (Chapters 1 & 2: on Measure and Integral & Probability and Expectation), the addition of illuminating comments, examples and problems, and a careful treatment of backward stochastic differential equations with applications in optimal control of jump processes.

Part 1 (2 chapters, approx. 70 pages) gives an overview on measure & integration and probability theory. In the first edition, the readers were assumed to be familiar with this part of the theory, and it seems doubtful whether a newcomer to measure-theoretic probability gets much out of the new presentation but an understanding for the notation. Part 2 (5 chapters, approx. 100 pages) is on stochastic processes, mainly martingales in continuous and discrete time and measurability issues. The presentation is geared towards stochastic calculus. Brownian motion and the Poisson process appear here as examples of continuous-time martingales, but without in-depth treatment. Both parts should be seen as preparatory for the main part of the monograph in the following chapters.

Part 3 (6 chapters, approx. 160 pages) deals with the stochastic integral for general semimartingales. Semimartingales are introduced in the standard way, as sum of processes of bounded variation (BV) and local martingales. Consequently, the part starts with BV processes, Stieltjes integrals, the central Doob–Meyer decomposition of square-integrable martingales, structural results for martingales and, most importantly, the scale of $\mathcal{H}^p$ spaces of martingales. Then the authors discuss quadratic variations of semimartingales, the Kunita–Watanabe and Burkholder–Davis–Gundy inequalities which allow a streamlined introduction of the stochastic integral. This is finally done in Chapter 12. The presentation is classic (Itô isometry, orthogonality, localization, ...) and contains now a discussion of Émery's semimartingale topology which is the 'correct' topology if one understands the stochastic integral as integral operator driven by a semimartingale. In the last chapter of this part (Chapter 13) these results are extended to general random measures (in the spirit of Jacod) with Lévy processes as the prime examples.

Part 4 (6 chapters, approx. 160 pages) is devoted to stochastic differential equations (SDEs), their properties and backwards SDEs. The part begins with a discussion of the Itô formula for jump processes, some extensions and classic applications thereof (Tanaka-Meyer-Itô formula, Lévy's characterization of Brownian motion, martingale representation) and Girsanov's theorem. For the latter the Doléans stochastic exponential and exponential integrability (Novikov–Kazamaki-type conditions) for jump processes are included. SDEs are first treated in a Lipschitz and strong-solution context, using extensively the scales of martingale spaces, and the Markov property of the solutions is discussed at length. Weak solutions are only briefly touched, the main focus being time changes à la Girsanov in order to cope with non-Lipschitz drift coefficients and jump compensators – and necessarily weak solutions. The chapter closes with a sketch of the connection between martingale problems and weak solutions. The closing chapter of this part is the newly added material on backward SDEs. It begins with a brief introduction to general BSDEs, then the comparison theorem is proved and the relation to (semilinear) partial integro-differential equations and viscosity solutions is derived.

The last Part 5 (3 chapters, approx. 170 pages) contains optimal control and filtering problems. The main new feature here is the approach of stochastic optimal control through BSDEs while the other sections are only slightly updated from the previous edition. A couple of appendices (ca. 75 pages) contain additional material, e.g. Carathéodory's extension theorem, Kolmogorov's existence theorem for stochastic processes, the existence of regular conditional probabilities, the Kolmogorov–Chentsov theorem, the role of semimartingales as stochastic integrators, a Novikov criterion for jump processes, the relation of martingales and spaces of bounded mean oscillation (BMO) and some material on non-Lipschitz BSDEs.

The authors say rightly that calling the resurrection of *Stochastic Calculus and Applications* a "second edition is an understatement". While the spirit of the book and the basic layout remain largely untouched, a lot of the material has been re-written, re-organized and newly added, the presentation is more chatty and textbook-like. On the other hand, the top-down, from general-to-particular approach, may not be too appealing for a first (graduate) course; the – rather terse – introductory Parts 1 & 2 preparing the reader for the main theme of the book are no easy read for the novice and hardly apt to replace a full course on advanced probability and a first encounter with stochastic processes. As supplementary reading for a second course or as a comprehensive (!) resource for the general theory of processes aimed at PhD students and scholars, this second edition will stay a valuable resource.

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