Commissioned for the *Looking Back* series of Zentralblatt.

Chung, Kai Lai; Zhao, Zhongxin: *From Brownian motion to Schrödinger's equation. 2nd corrected printing.* Springer, Grundlehren der Mathematischen Wissenschaften **312**, Berlin 2001, 287+xii pp., EUR 96,25, US-$ 129.00. ISBN 978-3-540-57030-1.

It is very hard to assess the impact of the œuvre, let alone a single monograph, of a scientist who is such a prolific writer as Kai Lai Chung. Kai Lai Chung (1917–2009) was an extraordinary researcher, a great expositor and a truly outspoken mind. All of these qualities show up in almost every single publication of Chung. Many of his most important books are both research monograph and textbook, some changed their character as time went on and once novel material became a 'standard' topic. This is true for the Grundlehren volumes *Markov chains with stationary transition probabilities* [Zbl. 0092.34304, 0146.38401] and *Lectures From Markov Processes to Brownian Motion* [Zbl. 0503.60073, 1082.60001], while the *Lectures on Boundary Theory for Markov Chains* [Zbl. 0204.51003] and the volume under review, *From Brownian Motion to Schrödinger's Equation* (with Z. Zhao) [Zbl. 0819.60068], are probably too specialized and (still) too advanced for course adoption. Brilliant examples of Chung's abilities as an academic teacher are his four textbooks, *A Course in Probability Theory* [Zbl. 0159.45701, 0345.60003, 0980.60001], *Elementary Probability Theory with Stochastic Processes* [Zbl. 0293.60001, 0328.60001, 0404.60002, 1019.60001], *Introduction to Stochastic Integration* [Zbl. 0527.60058, 0725.60050], *A New Introduction to Stochastic Processes (in Chinese)* [Zbl. 0917.60001] which are still in use after more than 40 years. So is his classic translation of Gnedenko-and-Kolmogorov's *Limit Distributions for Sums of Independent Random Variables* [Zbl. 0056.36001] which he extends and corrects.

Chung's exposition is very clear, easy to follow and always mathematically reliable. Chung has a very personal style of writing; not all theorems, techniques or colleagues are equally important in his opinion—and he is rather blunt about this! His views on mathematics, in general, and probability theory, in particular, are most poignantly presented in *Green, Brown and Probability* [Zbl. 0871.60001, 1014.60001] and *Introduction to Random Times and Quantum Randomness* [Zbl. 0989.81500, 1021.81500, 1041.81074].

Also in mathematics Chung held firm beliefs. That *a recent probabilistic proof* of almost anything *may be still further probabilisable* (p. 190, Notes on Chapter 6) has been his programme throughout his life. Kai Lai Chung will be remembered as one of the really influential probabilists in the second half of the twentieth century. His first publication appeared in 1936 when he was an undergraduate of Tsinghua University and he worked literally until he died in June 2009. Chung wrote 133 research and expository papers. A selection of his œuvre is contained in the two volumes *Chance and Choice: Memorabilia* (edited by K.L. Chung, World Scientific 2004 [Zbl. 1059.01017]) and *Selected Works of Kai Lai Chung* (with commentaries; edited by F. AitSahlia, E. Hsu, R. Williams, World Scientific 2008 [Zbl. 1165.60302]).

The volume under review is his last research monograph. Originally published in 1995 as volume 312 of Springer's *Grundlehren* series it is a compilation and refinement of the authors' results on the Schrödinger equation

$$\frac{1}{2}\,\Delta u(x) + q(x)u(x) = 0, \quad x \in \mathbb{R}^d, \tag{1}$$

and the time-dependent Schrödinger equation

$$\frac{\partial}{\partial t}\,w(t,x) = \frac{1}{2}\,\Delta w(t,x) + q(x)w(t,x), \quad x \in \mathbb{R}^d,\ t > 0, \tag{2}$$

($q : \mathbb{R}^d \to \mathbb{R}$ is a Borel-measurable function) up to 1995. Since the 2001 reprint contains only minor corrections, we refer to the original Zentralblatt review [ZBl. 0819.60068] for a detailed description of the contents. Chung initiated this line of research in the late

1970s. His 1980 Séminaire de Probabilités paper *On stopped Feynman-Kac functional* [ZBl. 0444.60061] started the interest in probabilistic solutions of (1) where q is not necessarily positive; Chung calls these solutions q-harmonic functions. These developments made it possible, in Chung's own words (Preface, p. vii), to tackle the *great problems of quantum potential theory: the representation of a q-harmonic function by its boundary values, the unique solvability of the Dirichlet boundary value problem, Poisson's equation, Harnack's inequality, ..., and all the lush Green landscape now tinted with the ubiquitious* q with the tools of probability theory. The central object is the *gauge*, i.e. the expression

$$\mathbb{E}^x \left(\mathbf{1}_{\{\tau<\infty\}} \int_0^\tau q(X_s)\, ds \right)$$

where τ is the first exit time of the stochastic process X_t associated with (1) from the domain D. The *gauge theorem* asserts that for bounded potentials q and for domains D with finite Lebesgue measure this expression is either everywhere infinite or bounded. The same conclusion remains true, if q is a *Kato class* function. If one uses conditioned Brownian motion (instead of Brownian motion) one can derive conditional gauge theorems, but in this case the boundary of the domain D needs to be sufficiently smooth, e.g. Lipschitz continuous. Considering different domains D with a fixed potential q, it is possible to study various objects such as the principal eigenvalue and eigenfunction of (1) or the boundary Harnack principle.

Chung states in the preface to *From Brownian Motion to Schrödinger's Equation* that *this book (...) is a new departure with a new bent. (...) Here the focus is on a few central themes and their (...) developments.* And what a departure this has been: since its publication, the notion of the Kato class is firmly established in the research literature and the use of stochastic methods in the study of the Harnack inequality, the boundary Harnack principle, eigenvalue estimates and subsequent developments originated from the ideas developed in this monograph.

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