Review commissioned by: Zentralblatt Math

Krzystof Burdzy: *Brownian Motion and its Applications to Mathematical Analysis.* Springer, Lecture Notes in Mathematics **2106**, Cham 2014, xii + 137 pp., €37.44, SFR 47-00, £31.99, US-$ 49.99 (RRP). ISBN 978-3-319-04393-7.

These are the notes of the lectures delivered by Chris Burdzy at the 43rd St. Flour 2013 École d'Été de Probabilités. The focus is on Brownian motion, in particular, on the links between Brownian motion and real analysis. The text starts with a brief overview on Brownian motion and its properties, but this is mainly intended to fix notation and for cross-referencing; any prospecitve reader should have sound knowledge of Brownian motion, stochastic calculus and probabilistic semigroup theory. Chapter 2 gives a flavour of the tings to come. It contains probabilistic proofs of a few classical results: The fundamental theorem of algebra (with three proofs), a probabilistic version of Privalov's theorem on nontangential limits and Plessner's theorem on angular limits. As a counterexample (to the belief that probabilistic proofs are always 'simpler' or 'more enlightening') the author presents the standard probability proof of Picard's theorem on the range of entire functions, and he shows the limitations of the stochastic approach in a short exposition on the (yet?) undiscovered stochastic proof of McMillan's twist point theorem.

The main part of the lecture notes is devoted to Jeffreys Rauch's *hot spot conjecture* of 1974; it appeared for the first time in print in the 1985 lecture notes *Rearrangements and Convexity of Level Sets in PDE* by B. Kawohl [Springer LNM 1150, Berlin 1985]): *The second eigenfunction of the Neumann Laplacian in a bounded domain of* $\mathbb{R}^d$*,* $d \geq 1$*, attains its maximum on the boundary.* Apart from Kawohl's observation that the hot spots conjecture is true in cylindrical domains of the type $D \times [0,1]$, $D \subset \mathbb{R}^{d-1}$, and that it should be true for general convex domains, there has been not much progress until 1999. In this year, the present author verified the hot spots conjecture for bounded symmetric (along one line) convex sets and for bounded lip-shaped domains (in $\mathbb{R}^2$, ∂D is made up of two Lipschitz curves with Lipschitz constant 1) [Bañuelos *&* Burdzy: *Journal of Functional Analysis* **164** (1999) 1–22, with additions by Pascu: *Transactions of the AMS*

354 (2002) 4681–4702, and Atar *&* Burdzy: *Electronic Communications in Probability* **7** (2004) 129–139]. As to-date, it is still unknown whether the conjecture holds for general convex domains or in simply connected domains. On the donwside, Werner *&* Burdzy [*The Annals of Mathematics* **149** (1999) 309–317] showed that the conjecture cannot be true in general.

The subsequent seven chapters (Chapter 4–10) discuss problems and probabilistic techniques related to the hot spot conjecture. Chapter 4 deals with properties of the Neumann eigenfunctions and eigenvalues. In Chapter 5 one of the principal tools of the 1999 paper is introduced: Various types of (nontrivial) coupling methods for Brownian motions. A short interlude, with topics being interesting in their own right, is contained in Chapter 6 where the parabolic Harnack principle is treated in relation to coupling. The coupling theme continues in Chapter 7 with Pascu's 2002 scaling coupling. The final Chapters 8–10 contain several applications: Nodal lines (Chapter 8), monotonicity of Neumann heat kernels (Chapter 9) and reflecting Brownian motion in time-dependent domains (Chapter 10), establishing the existence of heat atoms in rapidly contracting domains.

Given the vast range of possible applications of Brownian motion, the present selection is rather personal; quite often those parts are selected where the author has made substantial contributions himself. The style of the notes is expository, most results are quoted, sketched and/or intuitively explained. The best way do describe the approach is to quote the author *that these notes do not present material at the level of rigor that is expected from journal articles or textbooks* [...] *It would take several hundred pages to present rigorously all the mathematical tools needed in these notes* [Preface, p. viii]—suffice it to say that the notes comprise less than 140 pages.

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Dresden, January 2014