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Buldygin, Valeriĭ, Indlekover, Karl-Heinz, Klesov, Oleg I., Steinebach, Josef G.: *Pseudo-Regularly Varying Functions and Generalized Renewal Processes*. Springer, Probability Theory and Mathematical Modelling **91**, Cham 2018. xxii + 482 pp., €106.99 (D), US-$ 119.99, £89.99 (RRP). ISBN 978-3-319-99537-3.

A measurable function $f : \mathbb{R} \to (0, \infty)$ is said to be of regular variation (at infinity) if for all $\lambda > 0$ the limits $\lim_{x\to\infty} f(\lambda x)/f(x) = g(\lambda) \in (0.\infty)$ exists. One can show that in this case $g(\lambda)$ is of the form λ^ρ for some index $\rho \in \mathbb{R}$. This notion is due to Karamata in the 1930s – he applied regular variation originally to Tauberian theorems – and it was further developed by his students and collaborators. Regular variation appears in many contexts, e.g. in Abelian, Tauberian and Mercerian theorems, in complex analysis (representation of entire functions) and, very prominently, in probability theory: limit laws, domains of attraction, the central limit problem, renewal theory, extrem value theory, to mention but a few applications. The monograph by Bingham, Goldie and Teugels [*Regular Variation*, Encyclopaedia of Mathematics and Its Applications Vol. 27, Cambridge University Press, Cambridge 1987] is the still unsurpassed comprehensive tract on the topic.

The authors of the book under review are well-known specialists in limit theorems and renewal theory. Many of their results are collected in this tract for the first time as a monograph. Regular variation is extremely successful to describe limit theorems for classical renewal processes which are described by counting processes of (i) independent, (ii) identically distributed and (iii) non-negative inter-arrival times. As soon as one of the conditions (i)–(iii) is violated, one deals with a generalized renewal process and most of the classic arguments break down. As it turns out, the notion of pseudo-regular variation and certain variants thereof, can take the role which regular variation takes in the study of classic renewal problems.

A measurable function f which is eventually strictly positive is pseudo-regularly varying if $\limsup_{c\to 1} \limsup_{x\to\infty} f(cx)/f(x) = 1$. Notice that a regularly varying function satisfies this relation with $\lim_{c\to 1}$ instead of $\limsup_{c\to 1}$. The function $f(x) := x^\alpha \exp(\sin\log(x))$, $x > 0$, is an example of a pseudo-regularly varying function which fails to be of regular variation.

The present text gives a systematic and encylopaedic introduction into pseudo-regular variation and related function classes – I came across a zoo of more than 15 different types –, the relations between them and characterizations of these classes (Chapters 3–7, pp. 53–310). One should mention the rather extensive discussion on (generalized) inverses which is rarely found in the literature. In Chapter 8 we find generalized renewal processes and Chapter 9 treat limit problems connected with ordinary and stochastic differential equations (of diffusion type). Renewal sequences constructed from random walks indexed by a multi-dimensional set are studied in Chapter 10 and the last chapter treats precise asymptotics for the complete convergence of stable random variables. There is a short appendix on various (technical) results from probability and analysis which the authors could not find in the literature.

The style of the book is encyclopaedic, but the lack of a list of symbols and notations and the relatively short index makes the text not easy to use as a reference text.

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