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V. I. Bogachev: Measure Theory (2 vols.). Springer-Verlag, Berlin 2007, xviii+500 pp. (vol. 1) and xiv+575 pp. (vol. 2), ISBN: 3-540-34513-2, EUR 128.35, US-$ 159.00.

Every now and then it is important to have an up-to-date, comprehensive view of a mathematical discipline. If we exclude the collection of articles edited by G. Pap *Handbook of Measure Theory (2. vols.)*, North-Holland, ISBN 0444502637, no such attempt has recently been made for the subject of measure theory. The present 2-volume treatise by Vladimir Bogachev sets out to do just this. The author says very clearly that the times of a Dunford-and-Schwartz-like treatment containing everything of a subject are over (and they probably had been over at their time) and he tries to find the right balance between in-depth treatment and survey style.

Volume one contains the modern foundations of measure and integration theory, starting with the concept of a measure rather than the functional approach due to Daniell. The exposition goes straight for (finite, positive) measures and cuts short lengthy discussions about pre-measures, contents and the corresponding systems of sets. Sidelining this anyway cumbersome material makes it easier to come straight to the notion of (the abstract) Lebesgue integral. The construction of the integral and its basic properties, up to and including convergence theorems and elementary inequalities, is dealt with in Chapter 2. The treatment is influenced by functional analysis which shows, for example in the way the convergence theorems are treated: rather than the 'classical' approach Beppo-Levi, then Fatou, then dominated convergence, the author uses Egorov's theorem and the completeness of L^1. Decomposition theorems, the Radon-Nikodým theorem, product measures, including infinite products, Fubini-type theorems and a simple version of the general transformation theorem are treated in Chapter 3. The basic course on measure and integration is completed in the fourth chapter, where L^p-spaces are introduced and, again, treated with a functional analytic slant. In this chapter we encounter also uniform integrability and various notions of convergence of measures – weak convergence will be treated in the second volume. The fifth and last chapter of volume 1 is devoted to the interplay of integration and differentiation, covering theorems and a concise introduction to the Kurzweil-Henstock integral.

This is the core material of volume 1, covered in the first 50 or so pages of each chapter. Things get really interesting in the 'Supplements and exercises' sections of each chapter; these comprise 40-50 pages each, and about half the number of pages is devoted to problems. The problems themselves are not just drill problems but are often guided exercises illustrating or extending the core material. I should mention that the problems are accessible to students – some of the exercises are especially recommended for students and marked as such – and that in many places hints or references to solutions are given. The supplements contain a wealth of material which is not usually covered in many books on measure theory. Let me, exemplarily, mention the nice treatment Borel vs. Baire measurability, the Brunn-Minkowski inequality, the Hellinger distance or the space BMO – unfortunately without duality theory. If short proofs were available, they are usually given, technically more demanding material is surveyed with precise references. These appendices are not necessarily meant for linear reading but rather for leafing them through whenever time permits.

Volume 2 is organised in a similar way – 5 chapters of approximately 50 pages core material and 50 pages of supplements and exercises – but the material covered there is of a different quality and *is to a large extent the result of the development of ideas generated in 1930-1960 by a number of mathematicians, among which primarily one should mention A. N. Kolmogorov, J. von Neumann, and A. D. Alexandroff* (vol. 1, p. vii). The main thrust goes into the direction of the interplay of measure and topology and the reader should now have profound knowledge of (graduate-level) mathematical analysis and some interest in probability theory. While volume 1 had a more functional-analytic slant, volume 2 is sprinkled with probability theory and represents to some extent the research interests of the author. In contrast to the first part of the monograph, volume 2 is not intended for linear reading but more for a 'pick-and-mix' type strategy; accordingly, the chapters are mostly independent or at least very well cross-referenced. The opening chapter 6 is on Borel, Baire and Souslin sets and can safely be used as a reference text. Chapter 7 treats measures on topological spaces, in particular Radon measures, projective families of measures and Kolmogorov's theorem as well as the Daniell approach to measure and integration and a short introduction to Fourier transforms of measures. Weak convergence of measures and its ramifications, Prokhorov's theorem, Lévy's continuity theorem etc., is treated in-depth in Chapter 8; since the author has infinite-dimensional situations in mind, the important concept of vague convergence of measures does not appear at all. In Chapter 9 (non-linear) transformations of measures, images and pre-images of measures, and isomorphisms of measure spaces are treated. Related is the last chapter which is on conditional measures and conditional expectations. At the centre of the exposition are regular conditional probabilities, liftings and disintegration theorems for measures. The chapter closes in a natural way with Birkhoff's pointwise ergodic theorem.

Again a lot of unusual or hard-to-find material is in the supplements and exercises. At this point I only mention Blackwell (and a whole zoo of other) spaces which are used in (infinite-dimensional) probability theory but which are usually not very well documented otherwise.

The monograph excels by its clear, scholarly style and the wealth of (unobtrusive!) historical comments and references. Of the 2038 bibliographic entries approximately 750 are to textbooks and monographs on or containing a substantial amount of measure theory, and this collection of books should be pretty exhaustive. It is good to see that the important Russian school is equally represented alongside the western contributions to measure and integration theory – a feature which is not found in too many books.

The text grew out of the author's lectures at Moscow's Lomonossov University, but it is hardly a textbook – although Bogachev singles out some 100 pages of core text in volume 1 which may serve as a first course on measure and integration. Nevertheless, it is written in a friendly way and has often the reader or the student in mind. The true value, however, of this monograph is that it presents many known, many not-so-well-known and *folklore* results on measure and integration theory side by side and in a unified manner. The numerous references to the existing research and textbook literature add to this impression. For any library and researchers in mathematical analysis or probability theory these two volumes are a must-have – for the mathematical literature this is a wonderful addition.

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