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Solution Manual

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1 Prologue.
Solutions to Problems 1.1–1.5

Problem 1.1 Solution: We have to calculate the area of an isosceles triangle of side-length r, base b, height h and opening angle $\phi := 2\pi/2^j$. From elementary geometry we know that

$$\cos\tfrac{\phi}{2} = \tfrac{h}{r} \quad \text{and} \quad \sin\tfrac{\phi}{2} = \tfrac{b}{2r}$$

so that

$$\text{area (triangle)} = \frac{1}{2}\,hb = r^2\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2} = \frac{r^2}{2}\,\sin\phi.$$

Since we have $\lim_{\phi\to 0}\frac{\sin\phi}{\phi} = 1$ we find

$$\begin{aligned}\text{area (circle)} &= \lim_{j\to\infty} 2^j\,\frac{r^2}{2}\,\sin\tfrac{2\pi}{2^j}\\ &= r^2\pi\lim_{j\to\infty}\frac{\sin\frac{2\pi}{2^j}}{\frac{2\pi}{2^j}}\\ &= r^2\pi\end{aligned}$$

just as we had expected.

Problem 1.2 Solution: By construction,

$$C_{n+1} = [0,1]\setminus\left(\bigcup_{i=1}^{n+1}\ \bigcup_{t_1,\dots,t_i\in\{0,2\}} I_{t_1,\dots,t_i}\right)$$

and each interval $I_{t_1,\dots,t_i}$ has length 2^{-i}. We have used this when calculating $\ell(C_{n+1})$

$$\ell(C_{n+1}) = \ell[0,1] - 2^0\times\frac{1}{3^1} - 2^1\times\frac{1}{3^2} - \cdots - 2^n\times\frac{1}{3^{n+1}}$$

(note that we have removed 2^n intervals of length 3^{-n-1}). If we let $n\to\infty$, we get for all removed intervals

$$\ell\left(\bigcup_{i=1}^{\infty}\ \bigcup_{t_1,\dots,t_i\in\{0,2\}} I_{t_1,\dots,t_i}\right) = \sum_{i=1}^{\infty} 2^{i-1}\times\frac{1}{3^i} = 1.$$

The last line requires σ-additivity. (Just in case: you will see in the next chapter that the number of removed intervals is indeed countable).

Problem 1.3 Solution: We record the lenghts of the removed pieces in each step

1. In Step 1 we remove one ($= 2^0$) piece of length $\frac{1}{2}r$;
2. In Step 2 we remove two ($= 2^1$) pieces, each of length $\frac{1}{8}r$;
3. In Step 3 we remove four ($= 2^2$) pieces, each of length $\frac{1}{32}r$;

n. In Step n we remove 2^n pieces, each of length $\frac{1}{2^{2n-1}}r$;

In each step we remove $2^n \times 2^{-2n+1} \times r = 2^{-n+1}r$ units of length, i.e. we remove

$$\sum_{n=1}^{\infty} \frac{r}{2^{n-1}} = r.$$

Thus, $\ell(I) = \ell[0,1] - r = 1 - r$.

This means that the modified Cantor set does have a length! Consequently it cannot be empty.

■■

Problem 1.4 Solution: In each step the total length is increased by the factor $4/3$, since we remove the middle interval (relative length $1/3$) and replace it by two copies constituting the sides of an equilateral triangle (relative length $2/3$). Thus,

$$\ell(K_n) = \frac{4}{3} \times \ell(K_{n-1}) = \cdots = \left(\frac{4}{3}\right)^n \ell(K_0) = \left(\frac{4}{3}\right)^n.$$

In particular, $\lim_{n\to\infty} \ell(K_n) = \infty$.

Again σ-additivity comes in in the form of a limit (compare with Problem 1.2).

■■

Problem 1.5 Solution: In each step the total area is decreased by the factor $3/4$, since we remove the middle triangle (relative area $1/4$). Thus,

$$\text{area}(S_n) = \frac{3}{4} \times \text{area}(S_{n-1}) = \cdots = \left(\frac{3}{4}\right)^n \text{area}(S_0) = \left(\frac{3}{4}\right)^n \frac{\sqrt{3}}{4}.$$

In particular, $\text{area}(S) = \lim_{n\to\infty} \text{area}(S_n) = 0$.

Again σ-additivity comes in in the form of a limit (compare with Problem 1.2). Notice that S is not empty as it contains the vertices of all black triangles (see figure) of each stage.

■■

2 The pleasures of counting. Solutions to Problems 2.1–2.22

Problem 2.1 Solution:

(i) We have

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \in B^c \\ &\iff x \in A \cap B^c. \end{aligned}$$

(ii) Using (i) and de Morgan's laws (*) yields

$$\begin{aligned} (A \setminus B) \setminus C &\overset{\text{(i)}}{=} (A \cap B^c) \cap C^c = A \cap B^c \cap C^c \\ &= A \cap (B^c \cap C^c) \overset{(*)}{=} A \cap (B \cup C)^c = A \setminus (B \cup C). \end{aligned}$$

(iii) Using (i), de Morgan's laws (*) and the fact that $(C^c)^c = C$ gives

$$\begin{aligned} A \setminus (B \setminus C) &\overset{\text{(i)}}{=} A \cap (B \cap C^c)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C) \\ &= (A \cap B^c) \cup (A \cap C) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \cap C). \end{aligned}$$

(iv) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cap C) &\overset{\text{(i)}}{=} A \cap (B \cap C)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C^c) \\ &= (A \cap B^c) \cup (A \cap C^c) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \setminus C) \end{aligned}$$

(v) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cup C) &\overset{\text{(i)}}{=} A \cap (B \cup C)^c \\ &\overset{(*)}{=} A \cap (B^c \cap C^c) \\ &= A \cap B^c \cap C^c \end{aligned}$$

$$
\begin{aligned}
&= A \cap B^c \cap A \cap C^c \\
&\overset{\text{(i)}}{=} (A \setminus B) \cap (A \setminus C)
\end{aligned}
$$

(vi) By definition and the distributive laws for sets we find

$$
\begin{aligned}
(A \cup B) \setminus C &= (A \cup B) \cap C^c \\
&= (A \cap C^c) \cup (B \cap C^c) \\
&= (A \setminus C) \cup (B \setminus C).
\end{aligned}
$$

Problem 2.2 Solution: Observe, first of all, that

$$
A \setminus C \subset (A \setminus B) \cup (B \setminus C). \tag{*}
$$

This follows easily from

$$
\begin{aligned}
A \setminus C &= (A \setminus C) \cap X \\
&= (A \cap C^c) \cap (B \cup B^c) \\
&= (A \cap C^c \cap B) \cup (A \cap C^c \cap B^c) \\
&\subset (B \cap C^c) \cup (A \cap B^c) \\
&= (B \setminus C) \cup (A \setminus B).
\end{aligned}
$$

Using this and the analogous formula for $C \setminus A$ then gives

$$
\begin{aligned}
&(A \cup B \cup C) \setminus (A \cap B \cap C) \\
&\quad= (A \cup B \cup C) \cap (A \cap B \cap C)^c \\
&\quad= [A \cap (A \cap B \cap C)^c] \cup [B \cap (A \cap B \cap C)^c] \cup [C \cap (A \cap B \cap C)^c] \\
&\quad= [A \setminus (A \cap B \cap C)] \cup [B \setminus (A \cap B \cap C)] \cup [C \setminus (A \cap B \cap C)] \\
&\quad= [A \setminus (B \cap C)] \cup [B \setminus (A \cap C)] \cup [C \setminus (A \cap B)] \\
&\quad\overset{\text{2.1(iv)}}{=} (A \setminus B) \cup (A \setminus C) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus A) \cup (C \setminus B) \\
&\quad\overset{(*)}{=} (A \setminus B) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus B) \\
&\quad= (A \triangle B) \cup (B \triangle C)
\end{aligned}
$$

Problem 2.3 Solution: It is clearly enough to prove (2.3) as (2.2) follows if I contains 2 points. De Morgan's identities state that for any index set I (finite, countable or not countable) and any collection of subsets $A_i \subset X$, $i \in I$, we have

$$
\text{(a)} \quad \Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c \qquad \text{and} \qquad \text{(b)} \quad \Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c.
$$

In order to see (a) we note that

$$\begin{aligned} a \in \left(\bigcup_{i \in I} A_i \right)^c &\iff a \notin \bigcup_{i \in I} A_i \\ &\iff \forall\, i \in I \,:\, a \notin A_i \\ &\iff \forall\, i \in I \,:\, a \in A_i^c \\ &\iff a \in \bigcap_{i \in I} A_i^c, \end{aligned}$$

and (b) follows from

$$\begin{aligned} a \in \left(\bigcap_{i \in I} A_i \right)^c &\iff a \notin \bigcap_{i \in I} A_i \\ &\iff \exists\, i_0 \in I \,:\, a \notin A_{i_0} \\ &\iff \exists\, i_0 \in I \,:\, a \in A_{i_0}^c \\ &\iff a \in \bigcup_{i \in I} A_i^c. \end{aligned}$$

Problem 2.4 Solution:

(i) The inclusion $f(A \cap B) \subset f(A) \cap f(B)$ is *always* true since $A \cap B \subset A$ and $A \cap B \subset B$ imply that $f(A \cap B) \subset f(A)$ and $f(A \cap B) \subset f(B)$, respectively. Thus, $f(A \cap B) \subset f(A) \cap f(B)$.

Furthermore, $y \in f(A) \setminus f(B)$ means that there is some $x \in A$ but $x \notin B$ such that $y = f(x)$, that is: $y \in f(A \setminus B)$. Thus, $f(A) \setminus f(B) \subset f(A \setminus B)$.

To see that the converse inclusions cannot hold we consider some *non injective* f. Take $X = [0,2]$, $A = (0,1)$, $B = (1,2)$, and $f : [0,2] \to \mathbb{R}$ with $x \mapsto f(x) = c$ (c is some constant). Then f is not injective and

$$\emptyset = f(\emptyset) = f((0,1) \cap (1,2)) \neq f((0,1)) \cup f((1,2)) = \{c\}.$$

Moreover, $f(X) = f(B) = \{c\} = f(X \setminus B)$ but $f(X) \setminus f(B) = \emptyset$.

(ii) Recall, first of all, the definition of f^{-1} for a map $f : X \to Y$ and $B \subset Y$

$$f^{-1}(B) := \{x \in X \,:\, f(x) \in B\}.$$

Observe that

$$\begin{aligned} x \in f^{-1}(\cup_{i \in I} C_i) &\iff f(x) \in \cup_{i \in I} C_i \\ &\iff \exists\, i_0 \in I \,:\, f(x) \in C_{i_0} \\ &\iff \exists\, i_0 \in I \,:\, x \in f^{-1}(C_{i_0}) \\ &\iff x \in \cup_{i \in I} f^{-1}(C_i), \end{aligned}$$

and

$$\begin{aligned} x \in f^{-1}(\cap_{i\in I} C_i) &\iff f(x) \in \cap_{i\in I} C_i \\ &\iff \forall\, i \in I \,:\, f(x) \in C_i \\ &\iff \forall\, i \in I \,:\, x \in f^{-1}(C_i) \\ &\iff x \in \cap_{i\in I} f^{-1}(C_i), \end{aligned}$$

and, finally,

$$\begin{aligned} x \in f^{-1}(C \setminus D) &\iff f(x) \in C \setminus D \\ &\iff f(x) \in C \quad\text{and}\quad f(x) \notin D \\ &\iff x \in f^{-1}(C) \quad\text{and}\quad x \notin f^{-1}(D) \\ &\iff x \in f^{-1}(C) \setminus f^{-1}(D). \end{aligned}$$

Problem 2.5 Solution:

(i), (vi) For every x we have

$$\begin{aligned} \mathbb{1}_{A\cap B}(x) = 1 &\iff x \in A \cap B \\ &\iff x \in A,\ x \in B \\ &\iff \mathbb{1}_A(x) = 1 = \mathbb{1}_B(x) \\ &\iff \begin{cases} \mathbb{1}_A(x) \cdot \mathbb{1}_B(x) = 1 \\ \min\{\mathbb{1}_A(x), \mathbb{1}_B(x)\} = 1 \end{cases} \end{aligned}$$

(ii), (v) For every x we have

$$\begin{aligned} \mathbb{1}_{A\cup B}(x) = 1 &\iff x \in A \cup B \\ &\iff x \in A \text{ or } x \in B \\ &\iff \mathbb{1}_A(x) + \mathbb{1}_B(x) \geqslant 1 \\ &\iff \begin{cases} \min\{\mathbb{1}_A(x) + \mathbb{1}_B(x), 1\} = 1 \\ \max\{\mathbb{1}_A(x), \mathbb{1}_B(x)\} = 1 \end{cases} \end{aligned}$$

(iii) Since $A = (A \cap B) \cup (A \setminus B)$ we see that $\mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x)$ can never have the value 2, thus part (ii) implies

$$\begin{aligned} \mathbb{1}_A(x) = \mathbb{1}_{(A\cap B)\cup(A\setminus B)}(x) &= \min\{\mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x), 1\} \\ &= \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x) \end{aligned}$$

and all we have to do is to subtract $\mathbb{1}_{A\cap B}(x)$ on both sides of the equation.

(iv) With the same argument that we use in (iii) and with the result of (iii) we get

$$\begin{aligned}\mathbb{1}_{A\cup B}(x) &= \mathbb{1}_{(A\setminus B)\uplus(A\cap B)\uplus(B\setminus A)}(x)\\ &= \mathbb{1}_{A\setminus B}(x) + \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{B\setminus A}(x)\\ &= \mathbb{1}_{A}(x) - \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{B}(x) - \mathbb{1}_{A\cap B}(x)\\ &= \mathbb{1}_{A}(x) + \mathbb{1}_{B}(x) - \mathbb{1}_{A\cap B}(x).\end{aligned}$$

(vii) We have

$$\forall i : \mathbb{1}_{A_i} \leqslant \mathbb{1}_{\bigcup_i A_i} \implies \sup_i \mathbb{1}_{A_i} \leqslant \mathbb{1}_{\bigcup_i A_i}.$$

On the other hand,

$$x_0 \in \bigcup_i A_i \implies \exists i_0 : x \in A_{i_0}.$$

Thus,

$$\mathbb{1}_{\bigcup_i A_i}(x_0) = 1 \implies \mathbb{1}_{A_{i_0}}(x_0) = 1 \implies \sup_i \mathbb{1}_{A_i}(x_0) = 1$$

and we get $\sup_i \mathbb{1}_{A_i} \geqslant \mathbb{1}_{\bigcup_i A_i}$.

(viii) One possibility is to mimic the proof of (vii). We prefer to argue like this: using (iii) and de Morgan's identities we get

$$\mathbb{1}_{\bigcap_i A_i} \underset{\text{de Morgan}}{\overset{\text{(iii)}}{=}} \mathbb{1}_X - \mathbb{1}_{\bigcup_i A_i^c} \overset{\text{(vii)}}{=} 1 - \sup_i \mathbb{1}_{A_i^c} = \inf_i (1 - \mathbb{1}_{A_i^c}) \overset{\text{(iii)}}{=} \inf_i \mathbb{1}_{A_i}.$$

Problem 2.6 Solution:

(i) Using 2.5(iii), (iv) we see that

$$\begin{aligned}\mathbb{1}_{A\,\triangle\,B}(x) &= \mathbb{1}_{(A\setminus B)\uplus(B\setminus A)}(x)\\ &= \mathbb{1}_{A\setminus B}(x) + \mathbb{1}_{B\setminus A}(x)\\ &= \mathbb{1}_{A}(x) - \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{B}(x) - \mathbb{1}_{A\cap B}(x)\\ &= \mathbb{1}_{A}(x) + \mathbb{1}_{B}(x) - 2\mathbb{1}_{A\cap B}(x)\end{aligned}$$

and this expression is 1 if, and only if, x is either in A or B but not in both sets. Thus

$$\mathbb{1}_{A\,\triangle\,B}(x) \iff \mathbb{1}_{A}(x) + \mathbb{1}_{B}(x) = 1 \iff \mathbb{1}_{A}(x) + \mathbb{1}_{B}(x) \mathrm{mod} 2 = 1.$$

It is also possible to show that

$$\mathbb{1}_{A\,\triangle\,B} = |\mathbb{1}_{A} - \mathbb{1}_{B}|.$$

This follows from

$$\mathbb{1}_A(x) - \mathbb{1}_B(x) = \begin{cases} 0, & \text{if } x \in A \cap B; \\ 0, & \text{if } x \in A^c \cap B^c; \\ +1, & \text{if } x \in A \setminus B; \\ -1, & \text{if } x \in B \setminus A. \end{cases}$$

Thus,

$$|\mathbb{1}_A(x) - \mathbb{1}_B(x)| = 1 \iff x \in (A \setminus B) \cup (B \setminus A) = A \vartriangle B.$$

(ii) From part (i) we see that

$$\begin{aligned} \mathbb{1}_{A \vartriangle (B \vartriangle C)} &= \mathbb{1}_A + \mathbb{1}_{B \vartriangle C} - 2\mathbb{1}_A \mathbb{1}_{B \vartriangle C} \\ &= \mathbb{1}_A + \mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C - 2\mathbb{1}_A\big(\mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C\big) \\ &= \mathbb{1}_A + \mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C - 2\mathbb{1}_A\mathbb{1}_B - 2\mathbb{1}_A\mathbb{1}_C + 4\mathbb{1}_A\mathbb{1}_B\mathbb{1}_C \end{aligned}$$

and this expression treats A, B, C in a completely symmetric way, i.e.

$$\mathbb{1}_{A \vartriangle (B \vartriangle C)} = \mathbb{1}_{(A \vartriangle B) \vartriangle C}.$$

(iii) ***Step 1:*** $(\mathscr{P}(X), \vartriangle, \emptyset)$ is an abelian group.

Neutral element: $A \vartriangle \emptyset = \emptyset \vartriangle A = A$;

Inverse element: $A \vartriangle A = (A \setminus A) \cup (A \setminus A) = \emptyset$, i.e. each element is its own inverse.

Associativity: see part (ii);

Commutativity: $A \vartriangle B = B \vartriangle A$.

Step 2: For the multiplication $\cap$ we have

Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$;

Commutativity: $A \cap B = B \cap A$;

One-element: $A \cap X = X \cap A = A$.

Step 3: Distributive law:

$$A \cap (B \vartriangle C) = (A \cap B) \vartriangle (A \cap C).$$

For this we use again indicator functions and the rules from (i) and Problem 2.5:

$$\begin{aligned} \mathbb{1}_{A \cap (B \vartriangle C)} = \mathbb{1}_A \mathbb{1}_{B \vartriangle C} &= \mathbb{1}_A(\mathbb{1}_B + \mathbb{1}_C \mod 2) \\ &= \big[\mathbb{1}_A(\mathbb{1}_B + \mathbb{1}_C)\big] \mod 2 \\ &= \big[\mathbb{1}_A\mathbb{1}_B + \mathbb{1}_A\mathbb{1}_C\big] \mod 2 \\ &= \big[\mathbb{1}_{A\cap B} + \mathbb{1}_{A\cap C}\big] \mod 2 \\ &= \mathbb{1}_{(A\cap B) \vartriangle (A\cap C)}. \end{aligned}$$

Problem 2.7 Solution: Let $f : X \to Y$. One has

$$\begin{aligned} f \text{ surjective} &\iff \forall\, B \subset Y : f \circ f^{-1}(B) = B \\ &\iff \forall\, B \subset Y : f \circ f^{-1}(B) \supset B. \end{aligned}$$

This can be seen as follows: by definition $f^{-1}(B) = \{x : f(x) \in B\}$ so that

$$f \circ f^{-1}(B) = f\big(\{x : f(x) \in B\}\big) = \{f(x) : f(x) \in B\} \subset \{y : y \in B\}$$

and we have equality in the last step if, and only if, we can guarantee that every $y \in B$ is of the form $y = f(x)$ for some x. Since this must hold for all sets B, this amounts to saying that $f(X) = Y$, i.e. that f is surjective. The second equivalence is clear since our argument shows that the inclusion '$\subset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^2$ and $B = [-1, 1]$. Then

$$f^{-1}([-1, 1]) = [0, 1] \text{ and } f \circ f^{-1}([-1, 1]) = f([0, 1]) = [0, 1] \subsetneq [-1, 1].$$

On the other hand

$$\begin{aligned} f \text{ injective} &\iff \forall\, A \subset X : f^{-1} \circ f(A) = A \\ &\iff \forall\, A \subset X : f^{-1} \circ f(A) \subset A. \end{aligned}$$

To see this we observe that because of the definition of f^{-1}

$$f^{-1} \circ f(A) = \{x : f(x) \in f(A)\} \supset \{x : x \in A\} = A \tag{*}$$

since $x \in A$ always entails $f(x) \in f(A)$. The reverse is, for non-injective f, wrong since then there might be some $x_0 \notin A$ but with $f(x_0) = f(x) \in f(A)$ i.e. $x_0 \in f^{-1} \circ f(A) \setminus A$. This means that we have equality in ($*$) if, and only if, f is injective. The second equivalence is clear since our argument shows that the inclusion '$\supset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f \equiv 1$. Then

$$f([0, 1]) = \{1\} \text{ and } f^{-1} \circ f([0, 1]) = f^{-1}(\{1\}) = \mathbb{R} \supsetneq [0, 1].$$

Problem 2.8 Solution: Assume that for x, y we have $f \circ g(x) = f \circ g(y)$. Since f is injective, we conclude that

$$f(g(x)) = f(g(y)) \implies g(x) = g(y),$$

and, since g is also injective,

$$g(x) = g(y) \implies x = y$$

showing that $f \circ g$ is injective.

Problem 2.9 Solution:

- Call the set of odd numbers $\mathcal{O}$. Every odd number is of the form $2k-1$ where $k \in \mathbb{N}$. We are done, if we can show that the map $f : \mathbb{N} \to \mathcal{O}$, $k \mapsto 2k-1$ is bijective. Surjectivity is clear as $f(\mathbb{N}) = \mathcal{O}$. For injectivity we take $i, j \in \mathbb{N}$ such that $f(i) = f(j)$. The latter means that $2i - 1 = 2j - 1$, so $i = j$, i.e. injectivity.

- The quickest solution is to observe that $\mathbb{N} \times \mathbb{Z} = \mathbb{N} \times \mathbb{N} \cup \mathbb{N} \times \{0\} \cup \mathbb{N} \times (-\mathbb{N})$ where $-\mathbb{N} := \{-n : n \in \mathbb{N}\}$ are the strictly negative integers. We know from Example 2.5(iv) that $\mathbb{N}\times\mathbb{N}$ is countable. Moreover, the map $\beta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times(-\mathbb{N})$, $\beta((i,k)) = (i,-k)$ is bijective, thus $\#\mathbb{N} \times (-\mathbb{N}) = \#\mathbb{N} \times \mathbb{N}$ is also countable and so is $\mathbb{N} \times \{0\}$ since $\gamma : \mathbb{N} \to \mathbb{N} \times \{0\}$, $\gamma(n) := (n, 0)$ is also bijective.

  Therefore, $\mathbb{N} \times \mathbb{Z}$ is a union of three countable sets, hence countable.

  An *alternative approach* would be to write out $\mathbb{Z} \times \mathbb{N}$ (the swap of $\mathbb{Z}$ and $\mathbb{N}$ is for notational reasons—since the map $\beta((j,k)) := (k,j)$ from $\mathbb{Z} \times \mathbb{N}$ to $\mathbb{N} \times \mathbb{Z}$ is bijective, the cardinality does not change) in the following form

$$\begin{array}{ccccccccc}
\dots & (-3,1) & (-2,1) & (-1,1) & (0,1) & (1,1) & (2,1) & (3,1) & \dots \\
\dots & (-3,2) & (-2,2) & (-1,2) & (0,2) & (1,2) & (2,2) & (3,2) & \dots \\
\dots & (-3,3) & (-2,3) & (-1,3) & (0,3) & (1,3) & (2,3) & (3,3) & \dots \\
\dots & (-3,4) & (-2,4) & (-1,4) & (0,4) & (1,4) & (2,4) & (3,4) & \dots \\
\dots & (-3,5) & (-2,5) & (-1,5) & (0,5) & (1,5) & (2,5) & (3,5) & \dots \\
\dots & (-3,6) & (-2,6) & (-1,6) & (0,6) & (1,6) & (2,6) & (3,6) & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

  and going through the array, starting with $(0,1)$, then $(1,1) \to (1,2) \to (0,2) \to (-1,2) \to (-1,1)$, then $(2,1) \to (2,2) \to (2,3) \to (1,3) \to ...$ in clockwise oriented $\sqcup$-shapes down, left, up.

- In Example 2.5(iv) we have shown that $\#\mathbb{Q} \leqslant \#\mathbb{N}$. Since $\mathbb{N} \subset \mathbb{Q}$, we have a canonical injection $\jmath : \mathbb{N} \to \mathbb{Q}, i \mapsto i$ so that $\#\mathbb{N} \leqslant \#\mathbb{Q}$. Using Theorem 2.7 we conclude that $\#\mathbb{Q} = \#\mathbb{N}$.

  The proof of $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$ can be easily adapted—using some pretty obvious notational changes—to show that the Cartesian product of any two countable sets of cardinality $\#\mathbb{N}$ has again cardinality $\#\mathbb{N}$. Applying this $m-1$ times we see that $\#\mathbb{Q}^m = \#\mathbb{N}$.

- $\bigcup_{m\in\mathbb{N}} \mathbb{Q}^m$ is a countable union of countable sets, hence countable, cf. Theorem 2.6.

Problem 2.10 Solution: Following the hint it is clear that $\beta : \mathbb{N} \to \mathbb{N} \times \{1\}, i \mapsto (i,1)$ is a bijection and that $\jmath : \mathbb{N} \times \{1\} \to \mathbb{N} \times \mathbb{N}, (i,1) \mapsto (i,1)$ is an injection. Thus, $\#\mathbb{N} \leqslant \#(\mathbb{N} \times \mathbb{N})$.

On the other hand, $\mathbb{N} \times \mathbb{N} = \bigcup_{j\in\mathbb{N}} \mathbb{N} \times \{j\}$ which is a countable union of countable sets, thus $\#(\mathbb{N} \times \mathbb{N}) \leqslant \#\mathbb{N}$.

Applying Theorem 2.7 finally gives $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$.

Problem 2.11 Solution: Since $E \subset F$ the map $\jmath : E \to F, e \mapsto e$ is an injection, thus $\#E \leqslant \#F$.

Problem 2.12 Solution: Assume that the set $\{0,1\}^{\mathbb{N}}$ were indeed countable and that $\{s_j\}_{j\in\mathbb{N}}$ was an enumeration: each s_j would be a sequence of the form $(d_1^j, d_2^j, d_3^j, ..., d_k^j, ...)$ with $d_k^j \in \{0,1\}$. We could write these sequences in an infinite list of the form:

$$
\begin{array}{ccccccccc}
s_1 & = & d_1^1 & d_2^1 & d_3^1 & d_4^1 & \dots & d_k^1 & \dots \\
s_2 & = & d_1^2 & d_2^2 & d_3^2 & d_4^2 & \dots & d_k^2 & \dots \\
s_3 & = & d_1^3 & d_2^3 & d_3^3 & d_4^3 & \dots & d_k^3 & \dots \\
s_4 & = & d_1^4 & d_2^4 & d_3^4 & d_4^4 & \dots & d_k^4 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
s_k & = & d_1^k & d_2^k & d_3^k & d_4^k & \dots & d_k^k & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}
$$

and produce a new 0-1-sequence $S = (e_1, e_2, e_3, \dots)$ by setting

$$
e_m := \begin{cases} 0, & \text{if} \quad d_m^m = 1 \\ 1, & \text{if} \quad d_m^m = 0 \end{cases} .
$$

Since S differs from s_ℓ exactly at position ℓ, S cannot be in the above list, thus, the above list did not contain all 0-1-sequences, hence a contradiction.

Problem 2.13 Solution: Consider the function $f : (0,1) \to \mathbb{R}$ given by

$$
f(x) := \frac{1}{1-x} - \frac{1}{x}.
$$

This function is obviously continuous and we have $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = +\infty$. By the intermediate value theorem we have therefore $f((0,1)) = \mathbb{R}$, i.e. surjectivity.

Since f is also differentiable and $f'(x) = \dfrac{1}{(1-x)^2} + \dfrac{1}{x^2} > 0$, we see that f is strictly increasing, hence injective, hence bijective.

Problem 2.14 Solution: Since $A_1 \subset \bigcup_{i\in\mathbb{N}} A_i$ it is clear that $\mathfrak{c} = \#A_1 \leqslant \#\bigcup_{i\in\mathbb{N}} A_i$. On the other hand, $\#A_i = \mathfrak{c}$ means that we can map A_i bijectively onto $\mathbb{R}$ and, using Problem 2.13, we map $\mathbb{R}$ bijectively onto $(0,1)$ or $(i-1,i)$. This shows that $\#\bigcup_{i\in\mathbb{N}} A_i \leqslant \#\bigcup_{i\in\mathbb{N}}(i-1,i) \leqslant \#\mathbb{R} = \mathfrak{c}$. Using Theorem 2.7 finishes the proof.

Problem 2.15 Solution: Since we can write each $x \in (0,1)$ as an infinite dyadic fraction (o.k. if it is finite, fill it up with an infinite tail of zeroes !), the proof of Theorem 2.8 shows that $\#(0,1) \leqslant \#\{0,1\}^{\mathbb{N}}$.

On the other hand, thinking in base-4 expansions, each element of $\{1,2\}^{\mathbb{N}}$ can be interpreted as a unique base-4 fraction (having no 0 or 3 in its expansion) of some number in $(0,1)$. Thus, $\#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}$.

But $\#\{1,2\}^{\mathbb{N}} = \#\{0,1\}^{\mathbb{N}}$ and we conclude with Theorem 2.7 that $\#(0,1) = \#\{0,1\}^{\mathbb{N}}$.

Problem 2.16 Solution: Just as before, expand $x \in (0,1)$ as an n-adic fraction, then interpret each element of $\{1,2,\dots,n+1\}^{\mathbb{N}}$ as a unique $(n+1)$-adic expansion of a number in $(0,1)$ and observe that $\#\{1,2,\dots,n+1\}^{\mathbb{N}} = \{0,1,\dots,n\}^{\mathbb{N}}$.

Problem 2.17 Solution: Take a vector $(x,y) \in (0,1)\times(0,1)$ and expand its coordinate entries x, y as dyadic numbers:

$$x = 0.x_1x_2x_3\dots, \qquad y = 0.y_1y_2y_3\dots.$$

Then $z := 0.x_1y_1x_2y_2x_3y_3\dots$ is a number in $(0,1)$. Conversely, we can 'zip' each $z = 0.z_1z_2z_3z_4\dots \in (0,1)$ into two numbers $x, y \in (0,1)$ by setting

$$x := 0.z_2z_4z_6z_8\dots, \qquad y := 0.z_1z_3z_5z_7\dots$$

This is obviously a bijective operation.

Since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ it is clear that we have also a bijection between $(0,1)\times(0,1) \leftrightarrow \mathbb{R}\times\mathbb{R}$.

Problem 2.18 Solution: We have seen in Problem 2.18 that $\#\{0,1\}^{\mathbb{N}} = \#\{1,2\}^{\mathbb{N}} = \mathfrak{c}$. Obviously, $\{1,2\}^{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ and since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ one extends this (using coordinates) to a bijection between $(0,1)^{\mathbb{N}} \leftrightarrow \mathbb{R}^{\mathbb{N}}$. Using Theorem 2.9 we get

$$\mathfrak{c} = \#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}^{\mathbb{N}} \leqslant \#\mathbb{R}^{\mathbb{N}} = \mathfrak{c},$$

and, because of Theorem 2.7 we have equality in the above formula.

Problem 2.19 Solution: Let $F \in \mathscr{F}$ with $\#F = n$ Then we can write F as a tuple of length n (having n pairwise different entries...) and therefore we can interpret F as an element of $\bigcup_{m\in\mathbb{N}} \mathbb{N}^m$. In this sense, $\mathscr{F} \hookrightarrow \bigcup_{m\in\mathbb{N}} \mathbb{N}^m$ and $\#\mathscr{F} \leqslant \bigcup_{m\in\mathbb{N}} \mathbb{N}^m = \#\mathbb{N}$ since countably many countable sets are again countable. Since $\mathbb{N} \subset \mathscr{F}$ we get $\#\mathscr{F} = \#\mathbb{N}$ by Theorem 2.7.

Alternative: Define a map $\phi : \mathscr{F} \to \mathbb{N}$ by

$$\mathscr{F} \ni A \mapsto \phi(A) := \sum_{a\in A} 2^a$$

. It is clear that ϕ increases if A gets bigger: $A \subset B \implies \phi(A) \leqslant \phi(B)$. Let $A, B \in \mathscr{F}$ be two finite sets, say $A = \{a_1, a_2, \dots, a_M\}$ and $\{b_1, b_2, \dots, b_N\}$ (ordered according to size with a_1, b_1 being the smallest and a_M, b_N the biggest) such that $\phi(A) = \phi(B)$. Assume, to the contrary, that $A \neq B$. If $a_M \neq b_N$, say $a_M > b_N$, then

$$\begin{aligned}
\phi(A) \geqslant \phi(\{a_M\}) \geqslant 2^{a_M} > \frac{2^{a_M} - 1}{2 - 1} &= \sum_{j=1}^{a_M - 1} 2^j \\
&= \phi(\{1, 2, 3, \dots a_M - 1\}) \\
&\geqslant \phi(B),
\end{aligned}$$

which cannot be the case since we assumed $\phi(A) = \phi(B)$. Thus, $a_M = b_N$. Now consider recursively the next elements, a_{M-1} and b_{N-1} and the same conclusion yields their equality etc. The process stops after $\min\{M, N\}$ steps. But if $M \neq N$, say $M > N$, then A would contain at least one more element than B, hence $\phi(A) > \phi(B)$, which is also a contradiction. This, finally shows that $A = B$, hence that ϕ is injective.

On the other hand, each natural number can be expressed in terms of finite sums of powers of base-2, so that ϕ is also surjective.

Thus, $\#\mathscr{F} = \#\mathbb{N}$.

■■

Problem 2.20 Solution: (Let $\mathscr{F}$ be as in the previous exercise.) Observe that the infinite sets from $\mathscr{P}(\mathbb{N})$, $\mathscr{I} := \mathscr{P}(\mathbb{N}) \setminus \mathscr{F}$ can be surjectively mapped onto $\{0,1\}^{\mathbb{N}}$: if $\{a_1, a_2, a_3, \dots\} = A \subset \mathbb{N}$, then define an infinite 0-1-sequence $(b_1, b_2, b_3, \dots)$ by setting $b_j = 0$ or $b_j = 1$ according to whether a_j is even or odd. This is a surjection of $\mathscr{P}(\mathbb{N})$ onto $\{0,1\}^{\mathbb{N}}$ and so $\#\mathscr{P}(\mathbb{N}) \geqslant \#\{0,1\}^{\mathbb{N}}$. Call this map γ and consider the family $\gamma^{-1}(s)$, $s \in \{0,1\}^{\mathbb{N}}$ in $\mathscr{I}$, consisting of obviously disjoint infinite subsets of $\mathbb{N}$ which lead to the same 0-1-sequence s. Now choose from each family $\gamma^{-1}(s)$ a representative, call it $r(s) \in \mathscr{I}$. Then the map $s \mapsto r(s)$ is a bijection between $\{0,1\}^{\mathbb{N}}$ and a subset of $\mathscr{I}$, the set of all representatives. Hence, $\mathscr{I}$ has at least the same cardinality as $\{0,1\}^{\mathbb{N}}$ and as such a bigger cardinality than $\mathbb{N}$.

■■

Problem 2.21 Solution: Denote by Θ the map $\mathscr{P}(\mathbb{N}) \ni A \mapsto \mathbb{1}_A \in \{0,1\}^{\mathbb{N}}$. Let $\delta = (d_1, d_2, d_3, \dots) \in \{0,1\}^{\mathbb{N}}$ and define $A(\delta) := \{j \in \mathbb{N} : d_j = 1\}$. Then $\delta = (\mathbb{1}_{A(\delta)}(j))_{j\in\mathbb{N}}$ showing that Θ is surjective.

On the other hand,

$$\mathbb{1}_A = \mathbb{1}_B \iff \mathbb{1}_A(j) = \mathbb{1}_B(j) \ \ \forall j \in \mathbb{N} \iff A = B.$$

This shows the injectivity of Θ, and $\#\mathscr{P}(\mathbb{N}) = \#\{0,1\}^{\mathbb{N}}$ follows.

Problem 2.22 Solution: Since for $A, A', B, B' \subset X$ we have the 'multiplication rule'

$$(A \cap B) \cup (A' \cap B') = (A \cup A') \cap (A \cup B') \cap (B \cup A') \cap (B \cup B')$$

and since this rule carries over to the infinite case, we get the formula from the problem by 'multiplying out' the countable union

$$(A_1^0 \cap A_1^1) \cup (A_2^0 \cap A_2^1) \cup (A_3^0 \cap A_3^1) \cup (A_4^0 \cap A_4^1) \cup \cdots .$$

More formally, one argues as follows:

$$x \in \bigcup_{n\in\mathbb{N}} (A_n^0 \cap A_n^1) \iff \exists\, n_0 : x \in A_{n_0}^0 \cap A_{n_0}^1 \tag{*}$$

while

$$\begin{aligned} x \in \bigcap_{i=(i(k))_{k\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}} \ \bigcup_{k\in\mathbb{N}} A_k^{i(k)} \\ &\iff \forall i = (i(k))_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} : x \in \bigcup_{k\in\mathbb{N}} A_k^{i(k)} \\ &\iff \forall i = (i(k))_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\ \exists\, k_0 \in \mathbb{N} : x \in A_{k_0}^{i(k_0)} \end{aligned} \tag{**}$$

Clearly, (*) implies (**). On the other hand, assume that (**) holds but that (*) is wrong, i.e. suppose that for every n we have that either $x \in A_n^0$ or $x \in A_n^1$ or x is in neither of A_n^0, A_n^1. Thus we can construct a uniquely defined sequence $i(n) \in \{0,1\}$, $n \in \mathbb{N}$, by setting

$$i(n) = \begin{cases} 0 & \text{if } x \in A_n^0; \\ 1 & \text{if } x \in A_n^1; \\ 0 & \text{if } x \notin A_n^0 \text{ and } x \notin A_n^1. \end{cases}$$

Define by $i'(n) := 1 - i(n)$ the 'complementary' 0-1-sequence. Then

$$x \in \bigcup_n A_n^{i(n)} \ \text{ but } \ x \notin \bigcup_n A_n^{i'(n)}$$

contradicting our assumption (**).

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3 σ-Algebras.
Solutions to Problems 3.1–3.16

Problem 3.1 Solution:

(i) It is clearly enough to show that $A, B \in \mathscr{A} \implies A \cap B \in \mathscr{A}$, because the case of N sets follows from this by induction, the induction step being

$$\underbrace{A_1 \cap \ldots \cap A_N}_{=: B \in \mathscr{A}} \cap A_{N+1} = B \cap A_{N+1} \in \mathscr{A}.$$

Let $A, B \in \mathscr{A}$. Then, by (Σ_2) also $A^c, B^c \in \mathscr{A}$ and, by (Σ_3) and (Σ_2)

$$A \cap B = (A^c \cup B^c)^c = (A^c \cup B^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathscr{A}.$$

Alternative: Of course, the last argument also goes through for N sets:

$$\begin{aligned} A_1 \cap A_2 \cap \ldots \cap A_N &= (A_1^c \cup A_2^c \cup \ldots \cup A_N^c)^c \\ &= (A_1^c \cup \ldots \cup A_N^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathscr{A}. \end{aligned}$$

(ii) By (Σ_2) we have $A \in \mathscr{A} \implies A^c \in \mathscr{A}$. Use A^c instead of A and observe that $(A^c)^c = A$ to see the claim.

(iii) Clearly $A^c, B^c \in \mathscr{A}$ and so, by part (i), $A \setminus B = A \cap B^c \in \mathscr{A}$ as well as $A \vartriangle B = (A \setminus B) \cup (B \setminus A) \in \mathscr{A}$.

■■

Problem 3.2 Solution:

(iv) Let us assume that $B \neq \emptyset$ and $B \neq X$. Then $B^c \notin \{\emptyset, B, X\}$. Since with B also B^c must be contained in a σ-algebra, the family $\{\emptyset, B, X\}$ cannot be one.

(vi) Set $\mathscr{A}_E := \{E \cap A : A \in \mathscr{A}\}$. The key observation is that all set operations in $\mathscr{A}_E$ are now relative to E and not to X. This concerns mainly the complementation of sets! Let us check (Σ_1)–(Σ_3).

Clearly $\emptyset = E \cap \emptyset \in \mathscr{A}_E$. If $B \in \mathscr{A}_E$, then $B = E \cap A$ for some $A \in \mathscr{A}$ and the complement of B relative to E is $E \setminus B = E \cap B^c = E \cap (E \cap A)^c = E \cap (E^c \cup A^c) = E \cap A^c \in \mathscr{A}_E$ as $A^c \in \mathscr{A}$. Finally, let $(B_j)_{j \in \mathbb{N}} \subset \mathscr{A}_E$. Then there are $(A_j)_{j \in \mathbb{N}} \subset \mathscr{A}$ such that $B_j = E \cap A_j$. Since $A = \bigcup_{j \in \mathbb{N}} A_j \in \mathscr{A}$ we get $\bigcup_{j \in \mathbb{N}} B_j = \bigcup_{j \in \mathbb{N}} (E \cap A_j) = E \cap \bigcup_{j \in \mathbb{N}} A_j = E \cap A \in \mathscr{A}_E$.

(vii) Note that f^{-1} interchanges with all set operations. Let $A, A_j, j \in \mathbb{N}$ be sets in $\mathscr{A}$. We know that then $A = f^{-1}(A'), A_j = f^{-1}(A'_j)$ for suitable $A, A'_j \in \mathscr{A}'$. Since $\mathscr{A}'$ is, by assumption a σ-algebra, we have

$$\emptyset = f^{-1}(\emptyset) \in \mathscr{A} \qquad \text{as} \quad \emptyset \in \mathscr{A}'$$

$$A^c = \left(f^{-1}(A')\right)^c = f^{-1}(A'^c) \in \mathscr{A} \qquad \text{as} \quad A'^c \in \mathscr{A}'$$

$$\bigcup_{j\in\mathbb{N}} A_j = \bigcup_{j\in\mathbb{N}} f^{-1}(A'_j) = f^{-1}\left(\bigcup_{j\in\mathbb{N}} A'_j\right) \in \mathscr{A} \qquad \text{as} \quad \bigcup_{j\in\mathbb{N}} A'_j \in \mathscr{A}'$$

which proves (Σ_1)–(Σ_3) for $\mathscr{A}$.

Problem 3.3 Solution: Denote by $\Sigma = \sigma(\{x\}, x \in \mathbb{R})$. Let $\mathscr{A}$ be the σ-algebra defined in Example 3.3(v). It is clear that $\{x\} \in \mathscr{A}$, and so $\Sigma \subset \mathscr{A}$. On the other hand, if $A \in \mathscr{A}$, then either A or A^c is countable. Wlog assume that A is countable. Then A is a countable union of singletons, as such $A \in \Sigma$ as well as $A^c \in \Sigma$. This means $\mathscr{A} \subset \Sigma$.

Problem 3.4 Solution:

(i) Since $\mathscr{G}$ is a σ-algebra, $\mathscr{G}$ 'competes' in the intersection of all σ-algebras $\mathscr{C} \supset \mathscr{G}$ appearing in the definition of $\mathscr{A}$ in the proof of Theorem 3.4(ii). Thus, $\mathscr{G} \supset \sigma(\mathscr{G})$ while $\mathscr{G} \subset \sigma(\mathscr{G})$ is always true.

(ii) Without loss of generality we can assume that $\emptyset \neq A \neq X$ since this would simplify the problem. Clearly $\{\emptyset, A, A^c, X\}$ is a σ-algebra containing A and no element can be removed without losing this property. Thus $\{\emptyset, A, A^c, X\}$ is minimal and, therefore, $= \sigma(\{A\})$.

(iii) Assume that $\mathscr{F} \subset \mathscr{G}$. Then we have $\mathscr{F} \subset \mathscr{G} \subset \sigma(\mathscr{G})$. Now $\mathscr{C} := \sigma(\mathscr{G})$ is a potential 'competitor' in the intersection appearing in the proof of Theorem 3.4(ii), and as such $\mathscr{C} \supset \sigma(\mathscr{F})$, i.e. $\sigma(\mathscr{G}) \supset \sigma(\mathscr{F})$.

Problem 3.5 Solution:

(i) $\{\emptyset, (0, \frac{1}{2}), \{0\} \cup [\frac{1}{2}, 1], [0, 1]\}$.

We have 2 *atoms* (see the explanations below): $(0, \frac{1}{2}), (0, \frac{1}{2})^c$.

(ii) $\{\emptyset, [0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1], [0, \frac{3}{4}], [\frac{1}{4}, 1], [0, \frac{1}{4}) \cup (\frac{3}{4}, 1], [0, 1]\}$.

We have 3 *atoms* (see below): $[0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1]$.

(iii) —same solution as (ii)—

Parts (ii) and (iii) are quite tedious to do and they illustrate how difficult it can be to find a σ-algebra containing two distinct sets.... imagine how to deal with something that is generated by 10, 20, or infinitely many sets. Instead of giving a particular answer, let us describe the method to find $\sigma(\{A, B\})$ practically, and then we are going to prove it.

1. Start with trivial sets and given sets: $\emptyset, X, A, B$.
2. now add their complements: A^c, B^c
3. now add their unions and intersections and differences: $A \cup B, A \cap B, A \setminus B, B \setminus A$
4. now add the complements of the sets in 3.: $A^c \cap B^c, A^c \cup B^c, (A \setminus B)^c, (B \setminus A)^c$
5. finally, add unions of differences and their complements: $(A \setminus B) \cup (B \setminus A), (A \setminus B)^c \cap (B \setminus A)^c$.

All in all one should have 16 sets (some of them could be empty or X or appear several times, depending on how much A differs from B). That's it, but the trouble is: is this construction correct? Here is a somewhat more systematic procedure:

Definition: An *atom* of a σ-algebra $\mathscr{A}$ is a non-void set $\emptyset \neq A \in \mathscr{A}$ that contains no other set of $\mathscr{A}$.

Since $\mathscr{A}$ is stable under intersections, it is also clear that all atoms are disjoint sets! Now we can make up every set from $\mathscr{A}$ as union (finite or countable) of such atoms. The task at hand is to find atoms if A, B are given. This is easy: the atoms of our future σ-algebra must be: $A \setminus B$, $B \setminus A$, $A \cap B$, $(A \cup B)^c$. (Test it: if you make a picture, this is a tesselation of our space X using disjoint sets and we can get back A, B as union! It is also minimal, since these sets must appear in $\sigma(\{A, B\})$ anyway.) The crucial point is now:

Theorem. *If $\mathscr{A}$ is a σ-algebra with N atoms (finitely many!), then $\mathscr{A}$ consists of exactly 2^N elements.*

Proof. The question is how many different unions we can make out of N sets. Simple answer: we find $\binom{N}{j}$, $0 \leqslant j \leqslant N$ different unions involving exactly j sets ($j = 0$ will, of course, produce the empty set) and they are all different as the atoms were disjoint. Thus, we get $\sum_{j=0}^{N} \binom{N}{j} = (1+1)^N = 2^N$ different sets.

It is clear that they constitute a σ-algebra. □

This answers the above question. The number of atoms depends obviously on the relative position of A, B: do they intersect, are they disjoint etc. Have fun with the exercises and do not try to find σ-algebras generated by three or more sets..... (By the way: can you think of a situation in $[0, 1]$ with two subsets given and exactly *four* atoms? Can there be more?)

■■

Problem 3.6 Solution:

(i) See the solution to Problem 3.5.

(ii) If $A_1, \dots, A_N \subset X$ are given, there are at most 2^N atoms. This can be seen by induction. If $N = 1$, then there are $\#\{A, A^c\} = 2$ atoms at most (usually exactly 2 but there is always the case $A = X$). If we add a further set A_{N+1}, then the worst case would be that both A_{N+1} and

A_{N+1}^c intersects with each of the at most 2^N atoms in a non-trivial way, thus splitting each atom into two sets, which amounts to saying that there are $2 \cdot 2^N = 2^{N+1}$ atoms.

Another way of seeing things is as follows: Denote by B_i either the set A_i or the set A_i^c, $i = 1, \dots, N$ and form all possible intersections $P_k := B_1 \cap B_2 \cap \dots \cap B_N$. There are 2^N possibilities since each term in the intersection has 2 possible values. Some of the sets P_k may be void, but it is clear that the P_K's are mutually disjoint (including the degenerate case where one or two of them are empty). These intersections are a "disjointification" of the original sets $A_1, \dots, A_N$ extended by $A_0 := (A_1 \cup \dots \cup A_N)^c$, and they are the atoms. It is clear that

$$\sigma(A_1, \dots, A_N) = \sigma(P_k, k = 1, \dots, 2^N).$$[1]

Indeed, $P_k \in \sigma(A_1, \dots, A_N)$ since the $B_i \in \sigma(A_1, \dots, A_N)$, so we get "$\supset$". Since we can write each A_k as a union of $P_k's$ (think!), the inclusion "$\subset$" is also clear.

It is best to visualize this with $N = 3$ and a Venn diagram where A_1, A_2, A_3 intersect all in a non-tivial way. This gives a Picture with 7 different non-overlapping regions. If you add the complement $(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$ you get region number 8 and we have in total 2^3 atoms. Clearly, less atoms are possible, depending on the way, how the original sets intersect.

■■

Problem 3.7 Solution: We follow the hint. Since $\#\mathscr{A} = \#\mathbb{N}$, the following set is a countable intersection of measurable sets, hence itself in $\mathscr{A}$:

$$\forall x \in X : A(x) := \bigcap_{A \in \mathscr{A}, A \ni x} A \in \mathscr{A}. \tag{*}$$

Write $\mathscr{A}_0$ for the atoms of $\mathscr{A}$. Then

- $A(x) \in \mathscr{A}$ is an atom which contains x.

  Indeed: Otherwise, there is some $B \subset A(x)$ such that $B \in \mathscr{A}$, $B \neq \emptyset$, $B \neq A(x)$. We can assume that $x \in B$, or we would take $B' := A(x) \setminus B$ instead of B. Since $x \in B$, B is part of the intersection appearing in (*) so that $B \supset A(x)$, hence $B = A(x)$, which is impossible.

- Every atom $A \neq \emptyset$ of $\mathscr{A}$ is of the form (*).

  Indeed: By assumption, $x_0 \in A$ so that $A = A(x_0)$.

- $\mathscr{A}$ has $\#\mathbb{N}$ many atoms.

  Indeed: Since $\#\mathscr{A} = \#\mathbb{N}$, there are countably infinitely many disjoint sets in $\mathscr{A}$, thus the procedure (*) yields at least $\#\mathbb{N}$ many atoms. On the other hand, there cannot be more atoms than members of $\mathscr{A}$, and the claim follows.

[1] A word on notation: $\sigma(\dots)$ is defined for families of sets, so one should write $\sigma(\{A_1, \dots, A_n\})$ or $\sigma(\mathscr{G})$ with $\mathscr{G} := \{A_1, A_2, \dots, A_N\}$, but this is rarely done and the slightly iffy (but easier on the eye) notation $\sigma(A_1, \dots, A_n)$ is used.

Since $\mathscr{A}$ contains all countable unions of sets from $\mathscr{A}_0$, and since there are more than countably many such unions, it is clear that $\#\mathscr{A} > \#\mathbb{N}$.

Remark: A σ-algebra may have no non-empty atoms at all! Here is an example (which I learned from Julian Hollender). Let I be an uncountable set, e.g. $I = [0,1]$, and consider $\Omega = \{0,1\}^I$. We can construct a σ-algebra on Ω in the following way: Let $K \subset I$ and define $P_K : \{0,1\}^I \to \{0,1\}^K$ the coordinate projection. A *cylinder set* or *finitely based set* with basis $K \subset I$ is a set of the form $P_K^{-1}(B)$ where $\#K < \infty$ and $B \subset \{0,1\}^K$. Now consider the σ-algebra $\mathscr{A} := \sigma(\{\text{cylinder sets}\})$ on $\{0,1\}^I$. Intuitively, $A \in \mathscr{A}$ is of the form $P_L^{-1}(B)$ where L is countable. (The proof as such is not obvious, a possible source is Lemma 4.5 in Schilling & Partzsch: *Brownian Motion*. De Gruyter, Berlin 2012.) Assume that $A_0 \in \mathscr{A}$ were an atom. Then A_0 has the basis L. Take $i \in I \setminus L$, consider $L' = L \cup \{i\}$ and construct a set $P_{L'}^{-1}(B')$ where $B' = B \times \{0\}$, say. Then $P_{L'}^{-1}(B') \subset A_0$ and $P_{L'}^{-1}(B') \in \mathscr{A}$.

■■

Problem 3.8 Solution: We begin with an **example**: Let $X = (0,1]$ and $\mathscr{A} = \mathscr{B}(0,1]$ be the Borel sets. Define

$$\mathscr{A}_n := \sigma\big(((j-1)2^{-n}, j2^{-n}],\ j = 1,2,\dots,2^n\big)$$

the dyadic σ-algebra of step 2^{-n}. Clearly, $\#\mathscr{A}_n = 2^n$. Moreover,

$$\mathscr{A}_n \subsetneq \mathscr{A}_{n+1} \quad \text{and} \quad \mathscr{A}_\infty := \bigcup_n \mathscr{A}_n.$$

However, $\mathscr{A}_\infty$ is NOT a σ-algebra.

Argument 1: $I \in \mathscr{A}_\infty \iff I \in \mathscr{A}_n$ for some n, i.e. I is a finite union of intervals with dyadic end-points. (More precisely: the topological boundary $\overline{I} \setminus I^\circ$ consists of dyadic points).

On the other hand, every open set $(a,b) \subset [0,1]$ is a countable union of sets from $\mathscr{A}_\infty$:

$$(a,b) = \bigcup_{I \in \mathscr{A}_\infty, I \subset (a,b)} I$$

which follows from the fact that the dyadic numbers are dense in $(0,1]$. (If you want it more elementary, then approximate a and b from the right and left, respectively, by dyadic numbers and construct the approximating intervals by hand....). If, for example, a and b are irrational, then $(a,b) \notin \mathscr{A}_\infty$. This shows that $\mathscr{A}_\infty$ cannot be a σ-algebra.

In fact, our argument shows that $\sigma(\mathscr{A}_\infty) = \mathscr{B}(0,1]$.

Argument 2: Since $\#\mathscr{A}_n = 2^n$ we see that $\#\mathscr{A}_\infty = \#\mathbb{N}$. But Problem 3.7 tells us that $\mathscr{A}_\infty$ can't be a σ-algebra.

Let us now turn to the general case. We follow the note by

A. Broughton and B.W. Huff: A comment on unions of sigma-fields. *Am. Math. Monthly* **84** (1977) 553–554.

Since the $\mathscr{A}_n$ are strictly increasing, we may assume that $\mathscr{A}_1 \neq \{\emptyset, X\}$. Recall also the notion of a trace σ-Algebra

$$B \cap \mathscr{A}_n := \{B \cap A : A \in \mathscr{A}_n\}.$$

Step 1. Claim: There exists a set $E \in \mathscr{A}_1$ such that $(E \cap \mathscr{A}_{n+1}) \setminus (E \cap \mathscr{A}_n) \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

To see this, assume – to the contrary – that for some n and some $B \in \mathscr{A}_1$ we have

$$B \cap \mathscr{A}_n = B \cap \mathscr{A}_{n+1} \quad \text{and} \quad B^c \cap \mathscr{A}_n = B^c \cap \mathscr{A}_{n+1}.$$

If $U \in \mathscr{A}_{n+1} \setminus \mathscr{A}_n$, then

$$U = \underbrace{(B \cap U)}_{\in B\cap\mathscr{A}_{n+1}=B\cap\mathscr{A}_n\subset\mathscr{A}_n} \cup \underbrace{(B^c \cap U)}_{\in B^c\cap\mathscr{A}_{n+1}=B^c\cap\mathscr{A}_n\subset\mathscr{A}_n}$$

leading to the contradiction $U \in \mathscr{A}_n$. Thus the claim holds with either $E = B$ or $E = B^c$.

Step 2. Let E be the set from Step 1 and denote by $n_1, n_2, \dots$ a sequence for which the assertion in Step 1 holds. Then

$$\mathscr{F}_k := E \cap \mathscr{A}_{n_k}, \quad k \in \mathbb{N}$$

is a strictly increasing sequence of σ-Algebras over the set E. Again we may assume that $\mathscr{F}_1 \neq \{\emptyset, E\}$ As in Step 1, we find some $E_1 \in \mathscr{F}_1$ such that E_1 is not trivial (i.e. $E_1 \neq \emptyset$ and $E_1 \neq E$) and $(E_1 \cap \mathscr{F}_{k+1}) \setminus (E_1 \cap \mathscr{F}_k) \neq \emptyset$ holds for infinitely many k.

Step 3. Now we repeat Step 2 and construct recursively a sequence of σ-algebras $\mathscr{A}_{i_1} \subset \mathscr{A}_{i_2} \subset \mathscr{A}_{i_3} \dots$ and a sequence of sets $E_1 \supset E_2 \supset E_3 \dots$ such that

$$E_k \in \mathscr{A}_{i_k} \quad \text{and} \quad E_{k+1} \in (E_k \cap \mathscr{A}_{i_{k+1}}) \setminus (E_k \cap \mathscr{A}_{i_k}).$$

Step 4. The sets $F_k := E_k \setminus E_{k+1}$ have the property that they are disjoint and $F_k \in \mathscr{A}_{i_{k+1}} \setminus \mathscr{A}_{i_k}$. Since the σ-algebras are increasing, we have

$$\bigcup_{n\in\mathbb{N}} \mathscr{A}_n = \bigcup_{k\in\mathbb{N}} \mathscr{A}_{i_k}$$

which means that we can restrict ourselves to a subsequence. This means that we can assume that $i_k = k$.

Step 5. Without loss of generality we can identify F_k with $\{k\}$ and assume that the $\mathscr{A}_n$ are σ-algebras on $\mathbb{N}$ such that $\{k\} \in \mathscr{A}_{k+1} \setminus \mathscr{A}_k$. Let B_n the smallest set in $\mathscr{A}_n$ such that $n \in B_n$. Then $n \in B_n \subset \{n, n+1, n+2, \dots\}$ and $B_n \neq \{n\}$. Moreover

$$m \in B_n \implies B_m \subset B_n \quad \text{since} \quad m \in B_n \cap B_m \in \mathscr{A}_m.$$

Now define $n_1 = 1$ and pick n_{k+1} recursively: $n_{k+1} \in B_{n_k}$ such that $n_{k+1} \neq n_k$. Then $B_{n_1} \supset B_{n_2} \supset \dots$. Set $E = \{n_2, n_4, n_6, \dots\}$. If $\mathscr{A}_\infty$ were a σ-algebra, then $E \in \mathscr{A}_n$ for some n, thus $E \in \mathscr{A}_{n_{2k}}$ for some k. Then $\{n_{2k}, n_{2k+2}, \dots\} \in \mathscr{A}_{n_{2k}}$ and thus $B_{n_{2k}} \subset \{n_{2k}, n_{2k+2}, \dots\}$. This contradicts the fact $n_{2k+1} \in B_{n_{2k}}$.

■■

Problem 3.9 Solution:

$\mathscr{O}_1$ Since $\emptyset$ contains no element, every element $x \in \emptyset$ admits certainly some neighbourhood $B_\delta(x)$ and so $\emptyset \in \mathscr{O}$. Since for all $x \in \mathbb{R}^n$ also $B_\delta(x) \subset \mathbb{R}^n$, $\mathbb{R}^n$ is clearly open.

$\mathscr{O}_2$ Let $U, V \in \mathscr{O}$. If $U \cap V = \emptyset$, we are done. Else, we find some $x \in U \cap V$. Since U, V are open, we find some $\delta_1, \delta_2 > 0$ such that $B_{\delta_1}(x) \subset U$ and $B_{\delta_2}(x) \subset V$. But then we can take $h := \min\{\delta_1, \delta_2\} > 0$ and find

$$B_h(x) \subset B_{\delta_1}(x) \cap B_{\delta_2}(x) \subset U \cap V,$$

i.e. $U \cap V \in \mathscr{O}$. For finitely many, say N, sets, the same argument works. Notice that already for countably many sets we will get a problem as the radius $h := \min\{\delta_j : j \in \mathbb{N}\}$ is not necessarily any longer > 0.

$\mathscr{O}_2$ Let I be any (finite, countable, not countable) index set and $(U_i)_{i \in I} \subset \mathscr{O}$ be a family of open sets. Set $U := \bigcup_{i \in I} U_i$. For $x \in U$ we find some $j \in I$ with $x \in U_j$, and since U_j was open, we find some $\delta_j > 0$ such that $B_{\delta_j}(x) \subset U_j$. But then, trivially, $B_{\delta_j}(x) \subset U_j \subset \bigcup_{i \in I} U_i = U$ proving that U is open.

The family $\mathscr{O}^n$ cannot be a σ-algebra since the complement of an open set $U \neq \emptyset, \neq \mathbb{R}^n$ is closed.

■■

Problem 3.10 Solution: Let $X = \mathbb{R}$ and set $U_k := (-\frac{1}{k}, \frac{1}{k})$ which is an open set. Then $\bigcap_{k \in \mathbb{N}} U_k = \{0\}$ but a singleton like $\{0\}$ is closed and not open.

■■

Problem 3.11 Solution: We know already that the Borel sets $\mathscr{B} = \mathscr{B}(\mathbb{R})$ are generated by any of the following systems:

$$\{[a, b) : a, b \in \mathbb{Q}\}, \quad \{[a, b) : a, b \in \mathbb{R}\},$$
$$\{(a, b) : a, b \in \mathbb{Q}\}, \quad \{(a, b) : a, b \in \mathbb{R}\}, \quad \mathscr{O}^1, \text{ or } \mathscr{C}^1$$

Here is just an example (with the dense set $D = \mathbb{Q}$) how to solve the problem. Let $b > a$. Since $(-\infty, b) \setminus (-\infty, a) = [a, b)$ we get that

$$\{[a, b) : a, b \in \mathbb{Q}\} \subset \sigma(\{(-\infty, c) : c \in \mathbb{Q}\})$$
$$\implies \mathscr{B} = \sigma(\{[a, b) : a, b \in \mathbb{Q}\}) \subset \sigma(\{(-\infty, c) : c \in \mathbb{Q}\}).$$

On the other hand we find that $(-\infty, a) = \bigcup_{k\in\mathbb{N}}[-k, a)$ proving that

$$\{(-\infty, a) : a \in \mathbb{Q}\} \subset \sigma(\{[c, d) : c, d \in \mathbb{Q}\}) = \mathscr{B}$$
$$\implies \sigma(\{(-\infty, a) : a \in \mathbb{Q}\}) \subset \mathscr{B}$$

and we get equality.

The other cases are similar.

Problem 3.12 Solution: Let $\mathbb{B} := \{B_r(x) : x \in \mathbb{R}^n,\ r > 0\}$ and let $\mathbb{B}' := \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$. Clearly,

$$\mathbb{B}' \subset \mathbb{B} \subset \mathscr{O}^n$$
$$\implies \sigma(\mathbb{B}') \subset \sigma(\mathbb{B}) \subset \sigma(\mathscr{O}^n) = \mathscr{B}(\mathbb{R}^n).$$

On the other hand, any open set $U \in \mathscr{O}^n$ can be represented by

$$U = \bigcup_{B\in\mathbb{B}',\ B\subset U} B. \tag{*}$$

Indeed, $U \supset \bigcup_{B\in\mathbb{B}',\ B\subset U} B$ follows by the very definition of the union. Conversely, if $x \in U$ we use the fact that U is open, i.e. there is some $B_\epsilon(x) \subset U$. Without loss of generality we can assume that ϵ is rational, otherwise we replace it by some smaller rational ϵ. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can find some $q \in \mathbb{Q}^n$ with $|x - q| < \epsilon/3$ and it is clear that $B_{\epsilon/3}(q) \subset B_\epsilon(x) \subset U$. This shows that $U \subset \bigcup_{B\in\mathbb{B}',\ B\subset U} B$.

Since $\#\mathbb{B}' = \#(\mathbb{Q}^n \times \mathbb{Q}) = \#\mathbb{N}$, formula $(*)$ entails that

$$\mathscr{O}^n \subset \sigma(\mathbb{B}') \implies \sigma(\mathscr{O}^n) = \sigma(\mathbb{B}') \quad \text{and, therefore,} \quad \sigma(\mathscr{O}^n) = \sigma(\mathbb{B})$$

and we are done.

Problem 3.13 Solution:

(i) $\mathscr{O}_1$: We have $\emptyset = \emptyset \cap A \in \mathscr{O}_A$, $A = X \cap A \in \mathscr{O}_A$.

$\mathscr{O}_1$: Let $U' = U \cap A \in \mathscr{O}_A$, $V' = V \cap A \in \mathscr{O}_A$ with $U, V \in \mathscr{O}$. Then $U' \cap V' = (U \cap V) \cap A \in \mathscr{O}_A$ since $U \cap V \in \mathscr{O}$.

$\mathscr{O}_2$: Let $U_i' = U_i \cap A \in \mathscr{O}_A$ with $U_i \in \mathscr{O}$. Then $\bigcup_i U_i' = \left(\bigcup_i U_i\right) \cap A \in \mathscr{O}_A$ since $\bigcup_i U_i \in \mathscr{O}$.

(ii) We use for a set A and a family $\mathscr{F} \subset \mathscr{P}(X)$ the shorthand $A \cap \mathscr{F} := \{A \cap F : F \in \mathscr{F}\}$.

Clearly, $A \cap \mathscr{O} \subset A \cap \sigma(\mathscr{O}) = A \cap \mathscr{B}(X)$. Since the latter is a σ-algebra, we have

$$\sigma(A \cap \mathscr{O}) \subset A \cap \mathscr{B}(X) \ \text{ i.e. } \ \mathscr{B}(A) \subset A \cap \mathscr{B}(X).$$

For the converse inclusion we define the family

$$\Sigma := \{B \subset X : A \cap B \in \sigma(A \cap \mathscr{O})\}.$$

It is not hard to see that Σ is a σ-algebra and that $\mathscr{O} \subset \Sigma$. Thus $\mathscr{B}(X) = \sigma(\mathscr{O}) \subset \Sigma$ which means that

$$A \cap \mathscr{B}(X) \subset \sigma(A \cap \mathscr{O}).$$

Notice that this argument does not really need that $A \in \mathscr{B}(X)$. If, however, $A \in \mathscr{B}(X)$ we have in addition to $A \cap \mathscr{B}(X) = \mathscr{B}(A)$ that

$$\mathscr{B}(A) = \{B \subset A : B \in \mathscr{B}(X)\}$$

Problem 3.14 Solution:

(i) We see, as in the proof of Theorem 3.4, that the intersection of arbitrarily many monotone classes (MC, for short) is again a MC. Thus,

$$\mathfrak{m}(\mathscr{F}) := \bigcap_{\substack{\mathscr{F} \subset \mathscr{G} \\ \mathscr{G}\ \mathrm{MC}}} \mathscr{G},$$

is itself a MC. Note, that the intersection is non-void as the power set $\mathscr{P}(X)$ is (trivially) a MC which contains $\mathscr{F}$. By construction, see also the argument of Theorem 3.4, $\mathfrak{m}(\mathscr{F})$ is a minimal MC containing $\mathscr{F}$.

(ii) Define

$$\mathscr{D} := \{F \in \mathfrak{m}(\mathscr{F}) : F^c \in \mathfrak{m}(\mathscr{F})\}.$$

By assumption, $\mathscr{F} \subset \mathscr{D}$. We are done, if we can show that $\mathscr{D}$ is a MC.

(MC_1) Let $(M_n)_{n\in\mathbb{N}} \subset \mathscr{D}$ be an increasing family $M_n \uparrow M = \bigcup_{n\in\mathbb{N}} M_n$. Since $\mathfrak{m}(\mathscr{F})$ is a MC, $M \in \mathfrak{m}(\mathscr{F})$ and

$$M^c = \left(\bigcup_{n\in\mathbb{N}} M_n\right)^c = \bigcap_{n\in\mathbb{N}} \underbrace{M_n^c}_{\in\mathfrak{m}(\mathscr{F})} \in \mathfrak{m}(\mathscr{F}).$$

Here we use that $M_n \uparrow \implies M_n^c \downarrow$ and so $\bigcap_{n\in\mathbb{N}} M_n^c \in \mathfrak{m}(\mathscr{F})$ because of (MC_2) for the system $\mathfrak{m}(\mathscr{F})$. This proves $M \in \mathscr{D}$.

(MC_2) Let $(N_n)_{n\in\mathbb{N}} \subset \mathscr{D}$ be a decreasing family $N_n \downarrow N = \bigcap_{n\in\mathbb{N}} N_n$. As in the first part we get from $N \in \mathfrak{m}(\mathscr{F})$ and $N_n^c \uparrow N^c$ that $N^c \in \mathfrak{m}(\mathscr{F})$ due to (MC_1) for the family $\mathfrak{m}(\mathscr{F})$. Consequently, $N \in \mathscr{D}$.

(iii) We follow the hint. Because of the $\cap$-stability of $\mathscr{F}$ we get $\mathscr{F} \subset \Sigma$. Let us check that Σ is a MC:

(MC$_1$) Let $(M_n)_{n\in\mathbb{N}} \subset \Sigma$ be an increasing sequence $M_n \uparrow M$ and $F \in \mathscr{F}$. Then $M \in \mathfrak{m}(\mathscr{F})$ and from $\mathfrak{m}(\mathscr{F}) \ni M_n \cap F \uparrow M \cap F$ we get (using (MC$_1$) for the system $\mathfrak{m}(\mathscr{F})$), that $M \cap F \in \mathfrak{m}(\mathscr{F})$, hence, $M \in \Sigma$.

(MC$_2$) This is similar to (MC$_1$).

Therefore, Σ is a MC and $\mathscr{F} \subset \Sigma$. This proves $\mathfrak{m}(\mathscr{F}) \subset \Sigma$ and $\mathscr{F} \subset \Sigma'$. Since Σ' is also a MC (the proof is very similar to the one for Σ; just replace "$F \in \mathscr{F}$" with "$F \in \mathfrak{m}(\mathscr{F})$") we get $\mathfrak{m}(\mathscr{F}) \subset \Sigma'$, too. This proves our claim.

(iv) Since $\mathscr{M} \supset \mathscr{F}$, we get
$$\mathscr{M} = \mathfrak{m}(\mathscr{M}) \supset \mathfrak{m}(\mathscr{F});$$
so it is enough to show that $\mathfrak{m}(\mathscr{F})$ is a σ-algebra containing $\mathscr{F}$. Clearly, $\mathscr{F} \subset \mathfrak{m}(\mathscr{F})$.

(Σ_1) By assumption, $X \in \mathscr{F} \subset \mathfrak{m}(\mathscr{F})$.

(Σ_2) This follows immediately from (ii).

(Σ_3) First we show that $\mathfrak{m}(\mathscr{F})$ is $\cup$-stable: since $\mathfrak{m}(\mathscr{F})$ is $\cap$-stable – by (iii) – we get
$$C, D \in \mathfrak{m}(\mathscr{F}) \implies C \setminus D = C \cap D^c \in \mathfrak{m}(\mathscr{F})$$
and so
$$C, D \in \mathfrak{m}(\mathscr{F}) \underset{(\Sigma_1)}{\overset{(\Sigma_2)}{\Longrightarrow}} C \cup D = X \setminus \big[(X \setminus C) \setminus D\big] \in \mathfrak{m}(\mathscr{F}).$$
If $(A_n)_{n\in\mathbb{N}} \subset \mathfrak{m}(\mathscr{F})$ is **any** sequence, the new sequence $B_n := A_1 \cup \cdots \cup A_n$ is increasing and $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$. Thus, ($\Sigma_3$) follows from (MC$_1$).

■■

Problem 3.15 Solution: By definition, $\mathscr{M} = \mathfrak{m}(\mathscr{O})$ is the monotone class generated by the open sets. Note that
$$\begin{aligned}\forall U \in \mathscr{O} \ : \ U &= \bigcup \big\{ B_r(x) \ : \ r \in \mathbb{Q}^+, x \in \mathbb{Q}^d, B_r(x) \subset U \big\} \\ &= \bigcup_{n\in\mathbb{N}} \bigcup \Big\{ B_r(x) \ : \ r \in \mathbb{Q}^+, x \in \mathbb{Q}^d, \overline{B_{r-1/n}(x)} \subset U \Big\}\end{aligned}$$
which means that we can write every $U \in \mathscr{O}$ as a union of countably many closed sets (i.e. it is a so-called F_σ-set). Since unions of finitely many closed sets are still closed, we can arrange the latter union to be an increasing union. Using the de-Morgan laws, this means that every closed set $C \in \mathscr{C} \iff C^c \in \mathscr{O}$ can be written as a countable intersection of (decreasing) open sets.

Since $\mathfrak{m}(\mathscr{O})$ is stable under countable intersections of its members, we get $\mathscr{O} \cup \mathscr{C} \subset \mathfrak{m}(\mathscr{O}) \subset \mathfrak{m}(\mathscr{O} \cup \mathscr{C})$, hence $\mathfrak{m}(\mathscr{O}) = \mathfrak{m}(\mathscr{O} \cup \mathscr{C})$. Please note that $\mathscr{O} \cup \mathscr{C} = \{A \ : \ A \in \mathscr{O} \text{ or } A \in \mathscr{C}\}$.

Since $\mathscr{O} \cup \mathscr{C}$ is stable under the formation of complements, the monotone class $\mathfrak{m}(\mathscr{O} \cup \mathscr{C})$ is stable under the formation of complements (see Problem 3.14 (ii)), thus it is a σ-algebra containing $\mathscr{O}$ and $\mathscr{C}$.

On the other hand, $\mathfrak{m}(\mathscr{O}) \subset \sigma(\mathscr{O})$ is trivial, so we get $\mathfrak{m}(\mathscr{O}) = \sigma(\mathscr{O})$.

The answer to the additional question is: yes, we can omit the monotonicity in the countable intersection and union. The argument is as follows: Problem 3.14 still works without the monotonicity (giving a slightly different notion of monotone class), and so the above proof goes through!

Problem 3.16 Solution: Write $\Sigma := \bigcup\{\sigma(\mathscr{C}) : \mathscr{C} \subset \mathscr{F},\ \mathscr{C} \text{ is a countable sub-family}\}$.

If $\mathscr{C} \subset \mathscr{F}$ we get $\sigma(\mathscr{C}) \subset \sigma(\mathscr{F})$, and so $\Sigma \subset \sigma(\mathscr{F})$.

Conversely, it is clear that $\mathscr{F} \subset \Sigma$, just take $\mathscr{C} := \mathscr{C}_F := \{F\}$ for each $F \in \mathscr{F}$. If we can show that Σ is a σ-algebra we get $\sigma(\mathscr{F}) \subset \sigma(\Sigma) = \Sigma$ and equality follows.

- Clearly, $\emptyset \in \Sigma$.
- If $S \in \Sigma$, then $S \in \sigma(\mathscr{C}_S)$ for some countable $\mathscr{C}_S \subset \mathscr{F}$. Moreover, $S^c \in \sigma(\mathscr{C}_S)$, i.e. $S^c \in \Sigma$.
- If $(S_n)_{n\geqslant 0} \subset \Sigma$ are countably many sets, then $S_n \in \sigma(\mathscr{C}_n)$ for some countable $\mathscr{C}_n \subset \mathscr{F}$ and each $n \geqslant 0$. Set $\mathscr{C} := \bigcup_n \mathscr{C}_n$. This is again countable and we get $S_n \in \sigma(\mathscr{C})$ for all n, hence $\bigcup_n S_n \in \sigma(\mathscr{C})$ and so $\bigcup_n S_n \in \Sigma$.

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4 Measures. Solutions to Problems 4.1–4.22

Problem 4.1 Solution:

(i) We have to show that for a measure μ and finitely many, pairwise disjoint sets $A_1, A_2, \dots, A_N \in \mathscr{A}$ we have

$$\mu(A_1 \uplus A_2 \uplus \dots \uplus A_N) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(i). Induction step: take $N+1$ disjoint sets $A_1, \dots, A_{N+1} \in \mathscr{A}$, set $B := A_1 \uplus \dots \uplus A_N \in \mathscr{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}
\mu(A_1 \uplus \dots \uplus A_N \uplus A_{N+1}) &= \mu(B \uplus A_{N+1}) \\
&= \mu(B) + \mu(A_{N+1}) \\
&= \mu(A_1) + \dots + \mu(A_N) + \mu(A_{N+1}).
\end{aligned}$$

(iv) To get an idea what is going on we consider first the case of three sets A, B, C. Applying the formula for strong additivity thrice we get

$$\begin{aligned}
\mu(A \cup B \cup C) &= \mu(A \cup (B \cup C)) \\
&= \mu(A) + \mu(B \cup C) - \mu(\underbrace{A \cap (B \cup C)}_{=(A\cap B)\cup(A\cap C)}) \\
&= \mu(A) + \mu(B) + \mu(C) - \mu(B \cap C) - \mu(A \cap B) \\
&\quad - \mu(A \cap C) + \mu(A \cap B \cap C).
\end{aligned}$$

As an educated guess it seems reasonable to suggest that

$$\mu(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\sigma \subset \{1,\dots,n\} \\ \#\sigma = k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big).$$

We prove this formula by induction. The induction start is just the formula from Proposition 4.3(iv), the hypothesis is given above. For the induction step we observe that

$$\begin{aligned}
\sum_{\substack{\sigma \subset \{1,\dots,n+1\} \\ \#\sigma = k}} &= \sum_{\substack{\sigma \subset \{1,\dots,n,n+1\} \\ \#\sigma = k,\ n+1 \notin \sigma}} + \sum_{\substack{\sigma \subset \{1,\dots,n,n+1\} \\ \#\sigma = k,\ n+1 \in \sigma}} \\
&= \sum_{\substack{\sigma \subset \{1,\dots,n\} \\ \#\sigma = k}} + \sum_{\substack{\sigma' \subset \{1,\dots,n\} \\ \#\sigma' = k-1,\ \sigma := \sigma' \cup \{n+1\}}}
\end{aligned} \tag{$*$}$$

Having this in mind we get for $B := A_1 \cup \ldots \cup A_n$ and A_{n+1} using strong additivity and the induction hypothesis (for $A_1, \ldots, A_n$ resp. $A_1 \cap A_{n+1}, \ldots, A_n \cap A_{n+1}$)

$$\begin{aligned}
\mu(B \cup A_{n+1}) &= \mu(B) + \mu(A_{n+1}) - \mu(B \cap A_{n+1}) \\
&= \mu(B) + \mu(A_{n+1}) - \mu\Big(\bigcup_{j=1}^{n}(A_j \cap A_{n+1})\Big) \\
&= \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big) + \mu(A_{n+1}) \\
&\qquad + \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\Big(A_{n+1} \bigcap_{j\in\sigma} A_j\Big).
\end{aligned}$$

Because of $(*)$ the last line coincides with

$$\sum_{k=1}^{n+1}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n,n+1\}\\ \#\sigma=k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big)$$

and the induction is complete.

(v) We have to show that for a measure μ and finitely many sets $B_1, B_2, \ldots, B_N \in \mathscr{A}$ we have

$$\mu(B_1 \cup B_2 \cup \ldots \cup B_N) \leqslant \mu(B_1) + \mu(B_2) + \ldots + \mu(B_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(v). Induction step: take $N+1$ sets $B_1, \ldots, B_{N+1} \in \mathscr{A}$, set $C := B_1 \cup \ldots \cup B_N \in \mathscr{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}
\mu(B_1 \cup \ldots \cup B_N \cup B_{N+1}) &= \mu(C \cup B_{N+1}) \\
&\leqslant \mu(C) + \mu(B_{N+1}) \\
&\leqslant \mu(B_1) + \ldots + \mu(B_N) + \mu(B_{N+1}).
\end{aligned}$$

■■

Problem 4.2 Solution:

(i) The Dirac measure is defined on an arbitrary measurable space $(X, \mathscr{A})$ by

$$\delta_x(A) := \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases},$$

where $A \in \mathscr{A}$ and $x \in X$ is a fixed point.

$\boxed{(\mathrm{M}_1)}$ Since $\emptyset$ contains no points, $x \notin \emptyset$ and so $\delta_x(\emptyset) = 0$.

$\boxed{(\mathrm{M}_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ a sequence of pairwise disjoint measurable sets. If $x \in \bigcup_{j\in\mathbb{N}} A_j$, there is exactly one j_0 with $x \in A_{j_0}$, hence

$$\delta_x\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = 1 = 1 + 0 + 0 + \ldots$$

$$= \delta_x(A_{j_0}) + \sum_{j \neq j_0} \delta_x(A_j)$$
$$= \sum_{j\in\mathbb{N}} \delta_x(A_j).$$

If $x \notin \bigcup_{j\in\mathbb{N}} A_j$, then $x \notin A_j$ for every $j \in \mathbb{N}$, hence

$$\delta_x\left(\bigcup_{j\in\mathbb{N}} A_j\right) = 0 = 0 + 0 + 0 + \ldots = \sum_{j\in\mathbb{N}} \delta_x(A_j).$$

(ii) The measure γ is defined on $(\mathbb{R}, \mathscr{A})$ by $\gamma(A) := \begin{cases} 0, \text{ if } \#A \leqslant \#\mathbb{N} \\ 1, \text{ if } \#A^c \leqslant \#\mathbb{N} \end{cases}$ where $\mathscr{A} := \{A \subset \mathbb{R} : \#A \leqslant \#\mathbb{N} \text{ or } \#A^c \leqslant \#\mathbb{N}\}$. (Note that $\#A \leqslant \#\mathbb{N}$ if, and only if, $\#A^c = \#\mathbb{R} \setminus A > \#\mathbb{N}$.)

$\boxed{(\mathrm{M}_1)}$ Since $\emptyset$ contains no elements, it is certainly countable and so $\gamma(\emptyset) = 0$.

$\boxed{(\mathrm{M}_2)}$ Let $(A_j)_{j\in\mathbb{N}}$ be pairwise disjoint $\mathscr{A}$-sets. If all of them are countable, then $A := \bigcup_{j\in\mathbb{N}} A_j$ is countable and we get

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 0 = \sum_{j\in\mathbb{N}} \gamma(A_j).$$

If at least one A_j is not countable, say for $j = j_0$, then $A \supset A_{j_0}$ is not countable and therefore $\gamma(A) = \gamma(A_{j_0}) = 1$. Assume we could find some other $j_1 \neq j_0$ such that A_{j_0}, A_{j_1} are not countable. Since $A_{j_0}, A_{j_1} \in \mathscr{A}$ we know that their complements $A^c_{j_0}, A^c_{j_1}$ are countable, hence $A^c_{j_0} \cup A^c_{j_1}$ is countable and, at the same time, $\in \mathscr{A}$. Because of this, $(A^c_{j_0} \cup A^c_{j_1})^c = A_{j_0} \cap A_{j_1} = \emptyset$ cannot be countable, which is absurd! Therefore there is at most one index $j_0 \in \mathbb{N}$ such that A_{j_0} is uncountable and we get then

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 1 = 1 + 0 + 0 + \ldots = \gamma(A_{j_0}) + \sum_{j\neq j_0} \gamma(A_j).$$

(iii) We have an arbitrary measurable space $(X, \mathscr{A})$ and the measure $|A| = \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{else} \end{cases}$.

$\boxed{(\mathrm{M}_1)}$ Since $\emptyset$ contains no elements, $\#\emptyset = 0$ and $|\emptyset| = 0$.

$\boxed{(\mathrm{M}_2)}$ Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathscr{A}$. Case 1: All A_j are finite and only finitely many, say the first k, are non-empty, then $A = \bigcup_{j\in\mathbb{N}} A_j$ is effectively a finite union of k finite sets and it is clear that

$$|A| = |A_1| + \ldots + |A_k| + |\emptyset| + |\emptyset| + \ldots = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 2: All A_j are finite and infinitely many are non-void. Then their union $A = \bigcup_{j\in\mathbb{N}} A_j$ is an infinite set and we get

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 3: At least one A_j is infinite, and so is then the union $A = \bigcup_{j\in\mathbb{N}} A_j$. Thus,

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

(iv) On a countable set $\Omega = \{\omega_1, \omega_2, \dots\}$ we define for a sequence $(p_j)_{j\in\mathbb{N}} \subset [0,1]$ with $\sum_{j\in\mathbb{N}} p_j = 1$ the set function

$$P(A) = \sum_{j:\omega_j\in A} p_j = \sum_{j\in\mathbb{N}} p_j\, \delta_{\omega_j}(A), \qquad A \subset \Omega.$$

$\boxed{(\mathrm{M}_1)}$ $P(\emptyset) = 0$ is obvious.

$\boxed{(\mathrm{M}_2)}$ Let $(A_k)_{k\in\mathbb{N}}$ be pairwise disjoint subsets of Ω. Then

$$\begin{aligned}
\sum_{k\in\mathbb{N}} P(A_k) &= \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} p_j\, \delta_{\omega_j}(A_k) \\
&= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} p_j\, \delta_{\omega_j}(A_k) \\
&= \sum_{j\in\mathbb{N}} p_j \Big(\sum_{k\in\mathbb{N}} \delta_{\omega_j}(A_k) \Big) \\
&= \sum_{j\in\mathbb{N}} p_j \delta_{\omega_j}\big(\textstyle\bigcup_k A_k \big) \\
&= P\big(\textstyle\bigcup_k A_k \big).
\end{aligned}$$

The change in the order of summation needs justification; one possibility is the argument used in the solution of Problem 4.7(ii). (Note that the reordering theorem for absolutely convergent series is not immediately applicable since we deal with a double series!)

(v) This is obvious.

Problem 4.3 Solution:

- On $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ the function γ is not be a measure, since we can take the sets $A = (1, \infty)$, $B = (-\infty, -1)$ which are disjoint, not countable and both have non-countable complements. Hence, $\gamma(A) = \gamma(B) = 1$. On the other hand, $A \uplus B$ is non-countable and has non-countable complement, $[-1, 1]$. So, $\gamma(A \uplus B) = 1$. This contradicts the additivity: $\gamma(A \uplus B) = 1 \neq 2 = \gamma(A) + \gamma(B)$. *Notice that the choice of the σ-algebra $\mathscr{A}$ avoids exactly this situation. $\mathscr{B}$ is the wrong σ-algebra for γ.*

- On $\mathbb{Q}$ (and, actually, any possible σ-algebra thereon) the problem is totally different: if A is countable, then $A^c = \mathbb{Q} \setminus A$ is also countable and vice versa. This means that $\gamma(A)$ is, according to the definition, both 1 and 0 which is, of course, impossible. This is to say: γ is not well-defined. *γ makes only sense on a non-countable set X.*

Problem 4.4 Solution:

(i) If $\mathscr{A} = \{\emptyset, \mathbb{R}\}$, then μ is a measure.

But as soon as $\mathscr{A}$ contains one set A which is non-trivial (trivial means either $\emptyset$ or X), we have actually $A^c \in \mathscr{A}$ which is also non-trivial. Thus,

$$1 = \mu(X) = \mu(A \uplus A^c) \neq \mu(A) + \mu(A^c) = 1 + 1 = 2$$

and μ cannot be a measure.

(ii) If we equip $\mathbb{R}$ with a σ-algebra which contains sets such that both A and A^c can be infinite (the Borel σ-algebra would be such an example: $A = (-\infty, 0) \implies A^c = [0, \infty)$), then ν is not well-defined. The only type of sets where ν is well-defined is, thus,

$$\mathscr{A} := \{A \subset \mathbb{R} : \#A < \infty \text{ or } \#A^c < \infty\}.$$

But this is no σ-algebra as the following example shows: $A_j := \{j\} \in \mathscr{A}$, $j \in \mathbb{N}$, are pairwise disjoint sets but $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{N}$ is not finite and its complement is $\mathbb{R} \setminus \mathbb{N}$ not finite either! Thus, $\mathbb{N} \notin \mathscr{A}$, showing that $\mathscr{A}$ cannot be a σ-algebra. We conclude that ν can never be a measure if the σ-algebra contains infinitely many sets. If we are happy with finitely many sets only, then here is an example that makes ν into a measure $\mathscr{A} = \{\emptyset, \{69\}, \mathbb{R} \setminus \{69\}, \mathbb{R}\}$ and similar families are possible, but the point is that they all contain only finitely many members.

Problem 4.5 Solution: Denote by λ one-dimensional Lebesgue measure and consider the Borel sets $B_k := (k, \infty)$. Clearly $\bigcap_k B_k = \emptyset$, $k \in \mathbb{N}$, so that $B_k \downarrow \emptyset$. On the other hand,

$$\lambda(B_k) = \infty \implies \inf_k \lambda(B_k) = \infty \neq 0 = \lambda(\emptyset)$$

which shows that the finiteness condition is indeed essential.

Problem 4.6 Solution: Mind the typo in the problem: it should read "infinite mass" – otherwise the problem is pointless.

Solution 1: Define a measure μ which assigns every point $n - \frac{1}{2k}$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$ the mass $\frac{1}{2k}$:

$$\mu = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \frac{1}{2k}\, \delta_{n-\frac{1}{2k}}.$$

(Since $\mathbb{Z}\times\mathbb{N}$ is countable, Problem 4.7 shows that this object is indeed a measure!) Obviously, any interval $[a, b)$ of length $b-a > 2$ contains some integer, say $m \in [a, b)$ so that $[m-1/2, m) \subset [a, b)$, thus

$$\mu[a, b) \geqslant \mu[m - 1/2, m) = \sum_{k\in\mathbb{N}} \frac{1}{2k} = \infty.$$

On the other hand, the sequence of sets

$$B_n := \bigcup_{k=-n}^{n} \left[k - 1, k - \tfrac{1}{2n}\right)$$

satisfies $\mu(B_n) < \infty$ and $\bigcup_n B_n = \mathbb{R}$.

Solution 2: Set $\mu(B) := \#(B \cap \mathbb{Q})$, $B \in \mathscr{B}(\mathbb{R})$, i.e. the counting measure of the rationals in $\mathbb{R}$. Clearly, $\mu[a, b) = \infty$ **for every (non-empty) interval with** $a < b$. On the other hand, if $(q_k)_{k\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$, the sets $B_n := (\mathbb{R} \setminus \mathbb{Q}) \cup \{q_1, \dots, q_n\}$ satisfy

$$B_n \uparrow \mathbb{R} \quad \text{and} \quad \mu(B_n) = n,$$

i.e. μ is σ-finite.

Problem 4.7 Solution:

(i) Clearly, $\rho := a\mu + b\nu : \mathscr{A} \to [0, \infty]$ (since $a, b \geqslant 0$!). We check (M_1), (M_2).

$\boxed{(\mathrm{M}_1)}$ Clearly, $\rho(\emptyset) = a\mu(\emptyset) + b\nu(\emptyset) = a \cdot 0 + b \cdot 0 = 0$.

$\boxed{(\mathrm{M}_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ be mutually disjoint sets. Then we can use the σ-additivity of μ, ν to get

$$\begin{aligned}\rho\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) &= a\mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) + b\nu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big)\\ &= a\sum_{j\in\mathbb{N}} \mu(A_j) + b\sum_{j\in\mathbb{N}} \nu(A_j)\\ &= \sum_{j\in\mathbb{N}} \big(a\mu(A_j) + b\mu(A_j)\big)\\ &= \sum_{j\in\mathbb{N}} \rho(A_j).\end{aligned}$$

Since all quantities involved are positive and since we allow the value $+\infty$ to be attained, there are no convergence problems.

(ii) Since all α_j are positive, the sum $\sum_{j\in\mathbb{N}} \alpha_j\mu_j(A)$ is a sum of positive quantities and, allowing the value $+\infty$ to be attained, there is no convergence problem. Thus, $\mu : \mathscr{A} \to [0, \infty]$ is well-defined. Before we check (M_1), (M_2) we prove the following

Lemma. *Let β_{ij}, $i, j \in \mathbb{N}$, be real numbers. Then*

$$\sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}} \beta_{ij} = \sup_{j\in\mathbb{N}}\sup_{i\in\mathbb{N}} \beta_{ij}.$$

Proof. Observe that we have $\beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij}$ for all $m, n \in \mathbb{N}$. The right-hand side is independent of m and n and we may take the *sup* over all n

$$\sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}$$

and then, with the same argument, take the sup over all m

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}.$$

The opposite inequality, '$\geqslant$', follows from the same argument with i and j interchanged. ☐

$\boxed{(\mathrm{M}_1)}$ We have $\mu(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \mu_j(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \cdot 0 = 0$.

$\boxed{(\mathrm{M}_2)}$ Take pairwise disjoint sets $(A_i)_{i\in\mathbb{N}} \subset \mathscr{A}$. Then we can use the σ-additivity of each of the μ_j's to get

$$\begin{aligned}
\mu\Bigg(\bigcup_{i\in\mathbb{N}} A_i\Bigg) &= \sum_{j\in\mathbb{N}} \alpha_j \mu_j \Bigg(\bigcup_{i\in\mathbb{N}} A_i\Bigg) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \sum_{i\in\mathbb{N}} \mu_j\left(A_i\right) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \lim_{M\to\infty} \sum_{i=1}^{M} \mu_j\left(A_i\right) \\
&= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j\left(A_i\right) \\
&= \sup_{N\in\mathbb{N}} \sup_{M\in\mathbb{N}} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j\left(A_i\right)
\end{aligned}$$

where we use that the limits are increasing limits, hence suprema. By our lemma:

$$\begin{aligned}
\mu\Bigg(\bigcup_{i\in\mathbb{N}} A_i\Bigg) &= \sup_{M\in\mathbb{N}} \sup_{N\in\mathbb{N}} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j\left(A_i\right) \\
&= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j\left(A_i\right) \\
&= \lim_{M\to\infty} \sum_{i=1}^{M} \sum_{j\in\mathbb{N}} \alpha_j \mu_j\left(A_i\right) \\
&= \lim_{M\to\infty} \sum_{i=1}^{M} \mu\left(A_i\right) \\
&= \sum_{i\in\mathbb{N}} \mu\left(A_i\right).
\end{aligned}$$

■■

Problem 4.8 Solution: Finite additivity implies monotonicity: $A \subset B \implies B = A \mathbin{\dot\cup} (B \setminus A)$ and so

$$\mu(B) = \mu(A \mathbin{\dot\cup} (B \setminus A)) = \mu(A) + \mu(B \setminus A) \geqslant \mu(A).$$

Let $B_n \uparrow B$ and $D_n := B_n \setminus B_{n-1}$ with $B_0 := \emptyset$. This gives

$$\begin{aligned}\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) \geqslant \sup_{n\in\mathbb{N}} \mu(B_n) &= \sup_{n\in\mathbb{N}} \mu\Big(\dot{\bigcup_{i=1}^{n}} D_i\Big) \\ &\overset{(1)}{=} \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \mu(D_i) = \sum_{i=1}^{\infty} \mu(D_i) \\ &\overset{(2)}{\geqslant} \mu\Big(\bigcup_{i=1}^{\infty} D_i\Big) = \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big).\end{aligned}$$

where we use finite additivity for (1) and σ-subaddtitivity for (2).

Problem 4.9 Solution: Set $\nu(A) := \mu(A \cap F)$. We know, by assumption, that μ is a measure on $(X, \mathscr{A})$. We have to show that ν is a measure on $(X, \mathscr{A})$. Since $F \in \mathscr{A}$, we have $F \cap A \in \mathscr{A}$ for all $A \in \mathscr{A}$, so ν is well-defined. Moreover, it is clear that $\nu(A) \in [0, \infty]$. Thus, we only have to check

$\boxed{(M_1)}$ $\nu(\emptyset) = \mu(\emptyset \cap F) = \mu(\emptyset) = 0.$

$\boxed{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ be a sequence of pairwise disjoint sets. Then also $(A_j \cap F)_{j\in\mathbb{N}} \subset \mathscr{A}$ are pairwise disjoint and we can use the σ-additivity of μ to get

$$\begin{aligned}\nu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \mu\Big(F \cap \bigcup_{j\in\mathbb{N}} A_j\Big) &= \mu\Big(\bigcup_{j\in\mathbb{N}} (F \cap A_j)\Big) \\ &= \sum_{j\in\mathbb{N}} \mu(F \cap A_j) \\ &= \sum_{j\in\mathbb{N}} \nu(A_j).\end{aligned}$$

Problem 4.10 Solution: Since P is a probability measure, $P(A_j^c) = 1 - P(A_j) = 0$. By σ-subadditivity,

$$P\Big(\bigcup_{j\in\mathbb{N}} A_j^c\Big) \leqslant \sum_{j\in\mathbb{N}} P(A_j^c) = 0$$

and we conclude that

$$P\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = 1 - P\Big(\Big[\bigcap_{j\in\mathbb{N}} A_j\Big]^c\Big) = 1 - P\Big(\bigcup_{j\in\mathbb{N}} A_j^c\Big) = 1 - 0 = 0.$$

Problem 4.11 Solution: Note that

$$\bigcup_j A_j \setminus \bigcup_k B_k = \bigcup_j \Big(A_j \setminus \underbrace{\bigcup_k B_k}_{\supset B_j \ \forall j} \Big) \subset \bigcup_j \big(A_j \setminus B_j \big)$$

Since $\bigcup_j B_j \subset \bigcup_j A_j$ we get from σ-subadditivity

$$\begin{aligned}
\mu\Big(\bigcup_j A_j\Big) - \mu\Big(\bigcup_j B_j\Big) &= \mu\Big(\bigcup_j A_j \setminus \bigcup_k B_k\Big) \\
&\leqslant \mu\Big(\bigcup_j \big(A_j \setminus B_j\big)\Big) \\
&\leqslant \sum_j \mu(A_j \setminus B_j).
\end{aligned}$$

Problem 4.12 Solution:

(i) We have $\emptyset \in \mathscr{A}$ and $\mu(\emptyset) = 0$, thus $\emptyset \in \mathscr{N}_\mu$.

(ii) Since $M \in \mathscr{A}$ (this is essential in order to apply μ to M!) we can use the monotonicity of measures to get $0 \leqslant \mu(M) \leqslant \mu(N) = 0$, i.e. $\mu(M) = 0$ and $M \in \mathscr{N}_\mu$ follows.

(iii) Since all $N_j \in \mathscr{A}$, we get $N := \bigcup_{j\in\mathbb{N}} N_j \in \mathscr{A}$. By the σ-subadditivity of a measure we find

$$0 \leqslant \mu(N) = \mu\Big(\bigcup_{j\in\mathbb{N}} N_j\Big) \leqslant \sum_{j\in\mathbb{N}} \mu(N_j) = 0,$$

hence $\mu(N) = 0$ and so $N \in \mathscr{N}_\mu$.

Problem 4.13 Solution:

(i) The one-dimensional Borel sets $\mathscr{B} := \mathscr{B}(\mathbb{R})$ are defined as the smallest σ-algebra containing the open sets. Pick $x \in \mathbb{R}$ and observe that the open intervals $(x - \frac{1}{k}, x + \frac{1}{k})$, $k \in \mathbb{N}$, are all open sets and therefore $(x - \frac{1}{k}, x + \frac{1}{k}) \in \mathscr{B}$. Since a σ-algebra is stable under countable intersections we get $\{x\} = \bigcap_{k\in\mathbb{N}} (x - \frac{1}{k}, x + \frac{1}{k}) \in \mathscr{B}$.

Using the monotonicity of measures and the definition of Lebesgue measure we find

$$0 \leqslant \lambda(\{x\}) \leqslant \lambda((x - \tfrac{1}{k}, x + \tfrac{1}{k})) = (x + \tfrac{1}{k}) - (x - \tfrac{1}{k}) = \tfrac{2}{k} \xrightarrow[k\to\infty]{} 0.$$

[Following the hint leads to a similar proof with $[x - \frac{1}{k}, x + \frac{1}{k})$ instead of $(x - \frac{1}{k}, x + \frac{1}{k})$.]

(ii) a) Since $\mathbb{Q}$ is countable, we find an enumeration $\{q_1, q_2, q_3, \dots\}$ and we get trivially $\mathbb{Q} = \bigcup_{j\in\mathbb{N}} \{q_j\}$ which is a disjoint union. (This shows, by the way, that $\mathbb{Q} \in \mathscr{B}$ as $\{q_j\} \in \mathscr{B}$.) Therefore, using part (i) of the problem and the σ-additivity of measures,

$$\lambda(\mathbb{Q}) = \lambda\Big(\bigcup_{j\in\mathbb{N}} \{q_j\}\Big) = \sum_{j\in\mathbb{N}} \lambda(\{q_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

b) Take again an enumeration $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$, fix $\epsilon > 0$ and define $C(\epsilon)$ as stated in the problem. Then we have $C(\epsilon) \in \mathscr{B}$ and $\mathbb{Q} \subset C(\epsilon)$. Using the monotonicity and σ-subadditivity of λ we get

$$\begin{aligned}
0 \leqslant \lambda(\mathbb{Q}) \leqslant \lambda\big(C(\epsilon)\big)& \\
&= \lambda\Big(\bigcup_{k\in\mathbb{N}} [q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\Big) \\
&\leqslant \sum_{k\in\mathbb{N}} \lambda\big([q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\big) \\
&= \sum_{k\in\mathbb{N}} 2\cdot\epsilon\cdot 2^{-k} \\
&= 2\epsilon \frac{\frac{1}{2}}{1-\frac{1}{2}} = 2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we can make $\epsilon \to 0$ and the claim follows.

(iii) Since $\bigcup_{0\leqslant x\leqslant 1}\{x\}$ is a disjoint union, only the countability assumption is violated. Let's see what happens if we could use 'σ-additivity' for such non-countable unions:

$$0 = \sum_{0\leqslant x\leqslant 1} 0 = \sum_{0\leqslant x\leqslant 1} \lambda(\{x\}) = \lambda\Big(\bigcup_{0\leqslant x\leqslant 1}\{x\}\Big) = \lambda([0,1]) = 1$$

which is impossible.

Problem 4.14 Solution: Without loss of generality we may assume that $a \neq b$; set $\mu := \delta_a + \delta_b$. Then $\mu(B) = 0$ if, and only if, $a \notin B$ and $b \notin B$. Since $\{a\}, \{b\}$ and $\{a, b\}$ are Borel sets, all null sets of μ are given by

$$\mathscr{N}_\mu = \big\{B \setminus \{a,b\} : B \in \mathscr{B}(\mathbb{R})\big\}.$$

(This shows that, in some sense, null sets can be fairly large!).

Problem 4.15 Solution: Let us write $\mathfrak{N}$ for the family of all (proper and improper) subsets of μ null sets. We note that sets in $\mathfrak{N}$ can be measurable (that is: $N \in \mathscr{A}$) but need not be measurable.

(i) Since $\emptyset \in \mathfrak{N}$, we find that $A = A \cup \emptyset \in \overline{\mathscr{A}}$ for every $A \in \mathscr{A}$; thus, $\mathscr{A} \subset \overline{\mathscr{A}}$. Let us check that $\overline{\mathscr{A}}$ is a σ-algebra.

(Σ_1) Since $\emptyset \in \mathscr{A} \subset \overline{\mathscr{A}}$, we have $\emptyset \in \overline{\mathscr{A}}$.

(Σ_2) Let $A^* \in \overline{\mathscr{A}}$. Then $A^* = A \cup N$ for $A \in \mathscr{A}$ and $N \in \mathfrak{N}$. By definition, $N \subset M \in \mathscr{A}$ where $\mu(M) = 0$. Now

$$\begin{aligned}
A^{*c} = (A \cup N)^c = A^c \cap N^c& \\
&= A^c \cap N^c \cap (M^c \cup M)
\end{aligned}$$

$$= (A^c \cap N^c \cap M^c) \cup (A^c \cap N^c \cap M)$$
$$= (A^c \cap M^c) \cup (A^c \cap N^c \cap M)$$

where we use that $N \subset M$, hence $M^c \subset N^c$, hence $M^c \cap N^c = M^c$. But now we see that $A^c \cap M^c \in \mathscr{A}$ and $A^c \cap N^c \cap M \in \mathfrak{N}$ since $A^c \cap N^c \cap M \subset M$ and $M \in \mathscr{A}$ is a μ null set: $\mu(M) = 0$.

(Σ_3) Let $(A_j^*)_{j\in\mathbb{N}}$ be a sequence of $\overline{\mathscr{A}}$-sets. From its very definition we know that each $A_j^* = A_j \cup N_j$ for some (not necessarily unique!) $A_j \in \mathscr{A}$ and $N_j \in \mathfrak{N}$. So,

$$\bigcup_{j\in\mathbb{N}} A_j^* = \bigcup_{j\in\mathbb{N}} (A_j \cup N_j) = \Big(\bigcup_{j\in\mathbb{N}} A_j\Big) \cup \Big(\bigcup_{j\in\mathbb{N}} N_j\Big) =: A \cup N.$$

Since $\mathscr{A}$ is a σ-algebra, $A \in \mathscr{A}$. All we have to show is that N_j is in $\mathfrak{N}$. Since each N_j is a subset of a (measurable!) null set, say, $M_j \in \mathscr{A}$, we find that $N = \bigcup_{j\in\mathbb{N}} N_j \subset \bigcup_{j\in\mathbb{N}} M_j = M \in \mathscr{A}$ and all we have to show is that $\mu(M) = 0$. But this follows from σ-subadditivity,

$$0 \leqslant \mu(M) = \mu\Big(\bigcup_{j\in\mathbb{N}} M_j\Big) \leqslant \sum_{j\in\mathbb{N}} \mu(M_j) = 0.$$

Thus, $A \cup N \in \overline{\mathscr{A}}$.

(ii) As already mentioned in part (i), $A^* \in \overline{\mathscr{A}}$ could have more than one representation, e.g. $A \cup N = A^* = B \cup M$ with $A, B \in \mathscr{A}$ and $N, M \in \mathfrak{N}$. If we can show that $\mu(A) = \mu(B)$ then the definition of $\bar\mu$ is independent of the representation of A^*. Since M, N are not necessarily measurable but, by definition, subsets of (measurable) null sets $M', N' \in \mathscr{A}$ we find

$$A \subset A \cup N = B \cup M \subset B \cup M',$$
$$B \subset B \cup M = A \cup N \subset A \cup N'$$

and since $A, B, B\cup M', A\cup N' \in \mathscr{A}$, we get from monotonicity and subadditivity of measures

$$\mu(A) \leqslant \mu(B \cup M') \leqslant \mu(B) + \mu(M') = \mu(B),$$
$$\mu(B) \leqslant \mu(A \cup N') \leqslant \mu(A) + \mu(N') = \mu(A)$$

which shows $\mu(A) = \mu(B)$.

(iii) We check (M_1) and (M_2)

(M_1) Since $\emptyset = \emptyset \cup \emptyset \in \overline{\mathscr{A}}, \emptyset \in \mathscr{A}, \emptyset \in \mathfrak{N}$, we have $\bar\mu(\emptyset) = \mu(\emptyset) = 0$.

(M_2) Let $(A_j^*)_{j\in\mathbb{N}} \subset \overline{\mathscr{A}}$ be a sequence of pairwise disjoint sets. Then $A_j^* = A_j \cup N_j$ for some $A_j \in \mathscr{A}$ and $N_j \in \mathfrak{N}$. These sets are also mutually disjoint, and with the arguments in (i) we see that $A^* = A \cup N$ where $A^* \in \overline{\mathscr{A}}, A \in \mathscr{A}, N \in \mathfrak{N}$ stand for the unions of

A_j^*, A_j and N_j, respectively. Since $\bar\mu$ does not depend on the special representation of $\overline{\mathscr{A}}$-sets, we get

$$\begin{aligned}\bar\mu\Big(\bigcup_{j\in\mathbb{N}} A_j^*\Big) = \bar\mu(A^*) = \mu(A) = \mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big)\\ = \sum_{j\in\mathbb{N}} \mu(A_j)\\ = \sum_{j\in\mathbb{N}} \bar\mu(A_j^*)\end{aligned}$$

showing that $\bar\mu$ is σ-additive.

(iv) Let M^* be a $\bar\mu$ null set, i.e. $M^* \in \overline{\mathscr{A}}$ and $\bar\mu(M^*) = 0$. Take any $B \subset M^*$. We have to show that $B \in \overline{\mathscr{A}}$ and $\bar\mu(B) = 0$. The latter is clear from the monotonicity of $\bar\mu$ once we have shown that $B \in \overline{\mathscr{A}}$ which means, once we know that we may plug B into $\bar\mu$.

Now, $B \subset M^*$ and $M^* = M \cup N$ for some $M \in \mathscr{A}$ and $N \in \mathfrak{N}$. As $\bar\mu(M^*) = 0$ we also know that $\mu(M) = 0$. Moreover, we know from the definition of $\mathfrak{N}$ that $N \subset N'$ for some $N' \in \mathscr{A}$ with $\mu(N') = 0$. This entails

$$\begin{aligned} B \subset M^* = M \cup N \subset M \cup N' \in \mathscr{A}\\ \text{and}\ \ \mu(M \cup N') \leqslant \mu(M) + \mu(N') = 0.\end{aligned}$$

Hence $B \in \mathfrak{N}$ as well as $B = \emptyset \cup B \in \overline{\mathscr{A}}$. In particular, $\bar\mu(B) = \mu(\emptyset) = 0$.

(v) Set $\mathscr{C} = \{A^* \subset X \,:\, \exists A, B \in \mathscr{A}, \quad A \subset A^* A \subset B, \quad \mu(B \setminus A) = 0\}$. We have to show that $\overline{\mathscr{A}} = \mathscr{C}$.

Take $A^* \in \overline{\mathscr{A}}$. Then $A^* = A \cup N$ with $A \in \mathscr{A}$, $N \in \mathfrak{N}$ and choose $N' \in \mathscr{A}$, $N \subset N'$ and $\mu(N') = 0$. This shows that

$$A \subset A^* = A \cup N \subset A \cup N' =:\, B \in \mathscr{A}$$

and that $\mu(B\setminus A) = \mu((A\cup N')\setminus A) \leqslant \mu(N') = 0$. (Note that $(A\cup N')\setminus A = (A\cup N')\cap A^c = N' \cap A^c \subset N'$ and that equality need not hold!).

Conversely, take $A^* \in \mathscr{C}$. Then, by definition, $A \subset A^* \subset B$ with $A, B \in \mathscr{A}$ and $\mu(B\setminus A) = 0$. Therefore, $N := B \setminus A$ is a null set and we see that $A^* \setminus A \subset B \setminus A$, i.e. $A^* \setminus A \in \mathfrak{N}$. So, $A^* = A \cup (A^* \setminus A)$ where $A \in \mathscr{A}$ and $A^* \setminus A \in \mathfrak{N}$ showing that $A^* \in \overline{\mathscr{A}}$.

■■

Problem 4.16 Solution: Set

$$\Sigma := \big\{F \mathbin{\triangle} N \,:\, F \in \mathscr{F},\ N \in \mathscr{N}\big\}.$$

and denote, without further mentioning, by F, F_j resp. N, N_j sets from $\mathscr{F}$ resp. $\mathscr{N}$. Since $F \mathbin{\triangle} \emptyset = F$, $\emptyset \mathbin{\triangle} N = N$ and $F \mathbin{\triangle} N \in \sigma(\mathscr{F}, \mathscr{N})$ we get

$$\mathscr{F},\ \mathscr{N} \subset \Sigma \subset \sigma(\mathscr{F}, \mathscr{N}) \tag{*}$$

and the first assertion follows if we can show that Σ is a σ-algebra. In this case, we can apply the σ-operation to the inclusions (*) and get

$$\sigma(\mathscr{F},\mathscr{N}) \subset \sigma(\Sigma) \subset \sigma(\sigma(\mathscr{F},\mathscr{N}))$$

which is just

$$\sigma(\mathscr{F},\mathscr{N}) \subset \Sigma \subset \sigma(\mathscr{F},\mathscr{N}).$$

To see that Σ is a σ-algebra, we check conditions (Σ_1)–(Σ_3).

(Σ_1): Clearly, $X \in \mathscr{F}$ and $N \in \mathscr{N}$ so that $X = X \vartriangle \emptyset \in \Sigma$;

(Σ_2): We have, using de Morgan's identities over and over again:

$$\begin{aligned}
[F \vartriangle N]^c &= [(F \setminus N) \cup (N \setminus F)]^c \\
&= (F \cap N^c)^c \cap (N \cap F^c)^c \\
&= (F^c \cup N) \cap (N^c \cup F) \\
&= (F^c \cap N^c) \cup (G^c \cap G) \cup (N \cap N^c) \cup (N \cap F) \\
&= (F^c \cap N^c) \cup (N \cap F) \\
&= (F^c \setminus N) \cup (N \setminus F^c) \\
&= \underbrace{F^c}_{\in\mathscr{F}} \vartriangle N \\
&\in \Sigma;
\end{aligned}$$

(Σ_3): We begin by a few simple observations, namely that for all $F \in \mathscr{F}$ and $N, N' \in \mathscr{N}$

$$F \cup N = F \vartriangle \underbrace{(N \setminus F)}_{\in\mathscr{N}} \in \Sigma; \tag{a}$$

$$F \setminus N = F \vartriangle \underbrace{(N \cap F)}_{\in\mathscr{N}} \in \Sigma; \tag{b}$$

$$N \setminus F = N \vartriangle \underbrace{(F \cap N)}_{\in\mathscr{N}} \in \Sigma; \tag{c}$$

$$\begin{aligned}
(F \vartriangle N) \cup N' &= \big(F \vartriangle N\big) \vartriangle \big(N' \setminus (F \vartriangle N)\big) \\
&= F \vartriangle \underbrace{\Big(N \vartriangle (N' \setminus (F \vartriangle N))\Big)}_{\in\mathscr{N}} \in \Sigma,
\end{aligned} \tag{d}$$

where we use Problem 2.6(ii) and part (a) for (d).

Now let $(F_j)_{j\in\mathbb{N}} \subset \mathscr{F}$ and $(N_j)_{j\in\mathbb{N}} \subset \mathscr{N}$ and set $F := \bigcup_j F_j \in \mathscr{F}$ and, because of σ-subadditivity of measures $N := \bigcup_j N_j \in \mathscr{N}$. Then

$$F \setminus N = \bigcup_{j\in\mathbb{N}} (F_j \setminus N) \subset \bigcup_{j\in\mathbb{N}} (F_j \setminus N_j) \subset \bigcup_{j\in\mathbb{N}} F_j = F$$

as well as

$$\emptyset \subset \bigcup_{j\in\mathbb{N}} (N_j \setminus F_j) \subset \bigcup_{j\in\mathbb{N}} N_j = N$$

which shows that

$$F \setminus N \subset \bigcup_{j\in\mathbb{N}} (F_j \vartriangle N_j) \subset F \cup N. \tag{**}$$

Since $\mathscr{F}, \mathscr{N} \subset \mathscr{A}$, and consequently $\bigcup_{j\in\mathbb{N}} (F_j \vartriangle N_j) \in \mathscr{A}$, and since $\mathscr{A}$-measurable subsets of null sets are again in $\mathscr{N}$, the inclusions (**) show that there exists some $N' \in \mathscr{N}$ so that

$$\bigcup_{j\in\mathbb{N}} (F_j \vartriangle N_j) = \underbrace{(F \setminus N)}_{\in \Sigma, \text{ cf. (b)}} \cup N' \in \Sigma$$

where we use (d) for the last inclusion.

∎∎

Problem 4.17 Solution: By definition,

$$\overline{\mathscr{A}} = \{A \cup N \,:\, A \in \mathscr{A},\ N \in \mathscr{N}\}.$$

Since

$$A \cup N = A \vartriangle \underbrace{(N \setminus A)}_{\in \mathscr{N}}$$

and since by an application of Problem 4.16 to $(X, \overline{\mathscr{A}}, \bar\mu), \mathscr{A}, \mathscr{N}$ (instead of $(X, \mathscr{A}, \mu), \mathscr{G}, \mathscr{N}$) we get

$$\sigma(\mathscr{A}, \mathscr{N}) = \{A \vartriangle N \,:\, A \in \mathscr{A},\ N \in \mathscr{N}\}$$

and we conclude that

$$\overline{\mathscr{A}} \subset \sigma(\mathscr{A}, \mathscr{N}).$$

On the other hand,

$$\mathscr{A} \subset \overline{\mathscr{A}} \quad \text{and} \quad \mathscr{N} \subset \overline{\mathscr{A}}$$

so that, since $\overline{\mathscr{A}}$ is a σ-algebra,

$$\sigma(\mathscr{A}, \mathscr{N}) \subset \sigma(\overline{\mathscr{A}}) = \overline{\mathscr{A}} \subset \sigma(\mathscr{A}, \mathscr{N}).$$

Finally, assume that $A^* \in \overline{\mathscr{A}}$ and $A \in \mathscr{A}$. Then $A = A^* \vartriangle N$ and we get

$$A^* \vartriangle A = A \vartriangle N \vartriangle A = (A \vartriangle A) \vartriangle N = N.$$

Note that this result would also follow directly from 4.15 since we know from there that $A^* = A \cup N$ so that

$$A^* \vartriangle A = (A \cup N) \vartriangle A = A \vartriangle (N \setminus A) \vartriangle A = N \setminus A$$

∎∎

Problem 4.18 Solution: Denote the completion by $\mathscr{B}^*$ and write $\mathscr{N}_x$ for all subsets of Borel null sets of δ_x. Clearly,

$$\mathscr{N}_x = \{A \subset \mathbb{R}^n : x \notin A\}.$$

Recall from Problem 4.15(i) that $\mathscr{B}^*$ contains all sets of the form $B \cup N$ with $B \in \mathscr{B}$ and $N \in \mathscr{N}_x$. Now let $C \subset \mathbb{R}^n$ be any set. If $x \in C$, then write

$$C = \underbrace{\{x\}}_{\in \mathscr{B}} \cup \underbrace{(C \setminus \{x\})}_{\in \mathscr{N}_x} \in \mathscr{B}^*;$$

Otherwise, $x \notin C$ and

$$C = C \setminus \{x\} = \underbrace{\emptyset}_{\in \mathscr{B}} \cup \underbrace{(C \setminus \{x\})}_{\in \mathscr{N}_x} \in \mathscr{B}^*.$$

This shows that $\mathscr{B}^* = \mathscr{P}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$.

Problem 4.19 Solution:

(i) Since $\mathscr{B}$ is a σ-algebra, it is closed under countable (disjoint) unions of its elements, thus ν inherits the properties (M_1), (M_2) directly from μ.

(ii) Yes [yes], since the full space $X \in \mathscr{B}$ so that $\mu(X) = \nu(X)$ is finite [resp. $= 1$].

(iii) No, σ-finiteness is also a property of the σ-algebra. Take, for example, Lebesgue measure λ on the Borel sets (this is σ-finite) and consider the σ-algebra $\mathscr{C} := \{\emptyset, (-\infty, 0), [0, \infty), \mathbb{R}\}$. Then $\lambda\big|_{\mathscr{C}}$ is not σ-finite since there is no increasing sequence of $\mathscr{C}$-sets having finite measure.

Problem 4.20 Solution: By definition, μ is σ-finite if there is an *increasing* sequence $(B_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ such that $B_j \uparrow X$ and $\mu(B_j) < \infty$. Clearly, $E_j := B_j$ satisfies the condition in the statement of the problem.

Conversely, let $(E_j)_{j\in\mathbb{N}}$ be as stated in the problem. Then $B_n := E_1 \cup \ldots \cup E_n$ is measurable, $B_n \uparrow X$ and, by subadditivity,

$$\mu(B_n) = \mu(E_1 \cup \ldots \cup E_n) \leqslant \sum_{j=1}^{n} \mu(E_j) < \infty.$$

Remark: A small change in the above argument allows to take pairwise disjoint sets E_j.

Problem 4.21 Solution:

(i) Fix $\epsilon > 0$ and choose for $A \in \Sigma$ sets $U \in \mathscr{O}$, $F \in \mathscr{F}$ such that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. Set $U' := F^c \in \mathscr{O}$ and $F' := U^c \in \mathscr{F}$. Then we have

$$F' \subset A^c \subset U' \text{ and } U' \setminus F' = F^c \setminus U^c = F^c \cap U = U \setminus F$$

and so $\mu(U' \setminus F') = \mu(U \setminus F) < \epsilon$. This means that $A^c \in \Sigma$.

Denote by $d(x, y)$ the distance of two points $x, y \in X$ and write $B_{1/n}(0)$ for the open ball $\{y \in X : d(y, 0) < \frac{1}{n}\}$. As in the solution of Problem 3.14(ii) we see that $U_n := F + B_{1/n}(0)$ is a sequence of open sets such that $U_n \downarrow F$. Because of the continuity of measures we get $\mu(U_n \setminus F) \xrightarrow[n\to\infty]{} 0$ and since $\mathscr{F} \ni F \subset F \subset U_n \in \mathscr{O}$, this means that $\mathscr{F} \subset \Sigma$.

(ii) Fix $\epsilon > 0$ and pick for $A_j \in \Sigma, j = 1, 2$, open sets U_j and closed sets F_j such that $F_j \subset A_j \subset U_j$ and $\mu(U_j \setminus F_j) < \epsilon$. Then $F_1 \cap F_2$ and $U_1 \cap U_2$ are again closed resp. open, satisfy $F_1 \cap F_2 \subset A_1 \cap A_2 \subset U_1 \cap U_2$ as well as

$$\begin{aligned}
\mu\big((U_1 \cap U_2) \setminus (F_1 \cap F_2)\big) &= \mu\big((U_1 \cap U_2) \cap (F_1^c \cup F_2^c)\big) \\
&= \mu\big([(U_1 \cap U_2) \setminus F_1] \cup [(U_1 \cap U_2) \setminus F_2]\big) \\
&\leqslant \mu\big((U_1 \cap U_2) \setminus F_1\big) + \mu\big((U_1 \cap U_2) \setminus F_2\big) \\
&< 2\epsilon.
\end{aligned}$$

This shows that Σ is $\cap$-stable.

(iii) Fix $\epsilon > 0$ and pick for a given sequence $(A_j)_{j\in\mathbb{N}} \subset \Sigma$ open sets U_j and closed sets F_j such that

$$F_j \subset A_j \subset U_j \text{ and } \mu(U_j \setminus F_j) < \epsilon 2^{-j}.$$

Set $A := \bigcup_j A_j$. Then $U := \bigcup_j U_j \supset A$ is an open set wile $F := \bigcup_j F_j$ is contained in A but it is only an increasing limit of closed sets $\Phi_n := F_1 \cup \ldots \cup F_n$. Using Problem 4.11 we get

$$\mu(U \setminus F) \leqslant \sum_j \mu(U_j \setminus F_j) \leqslant \sum_j \epsilon 2^{-j} \leqslant \epsilon.$$

Since $\Phi_n \subset A \subset U$ and $U \setminus \Phi_n \downarrow U \setminus F$, we can use the continuity of measures to conclude that $\inf_n \mu(U \setminus \Phi_n) = \mu(U \setminus F) \leqslant \epsilon$, i.e. $\mu(U \setminus \Phi_N) \leqslant 2\epsilon$ if $N = N_\epsilon$ is sufficiently large. This shows that Σ contains all countable unions of its members. Because of part (i) it is also stable under complementation and contains the empty set. Thus, Σ is a σ-algebra.

As $\mathscr{F} \subset \Sigma$ and $\mathscr{B} = \sigma(\mathscr{F})$, we have $\mathscr{B} \subset \Sigma$.

(iv) For any Borel set $B \in \Sigma$ and any $\epsilon > 0$ we can find open and closed sets U_ϵ and F_ϵ, respectively, such that $F_\epsilon \subset B \subset U_\epsilon$ and

$$\begin{aligned}
\mu(B \setminus F_\epsilon) \leqslant \mu(U_\epsilon \setminus F_\epsilon) < \epsilon &\implies \mu(B) \leqslant \epsilon + \mu(F_\epsilon), \\
\mu(U_\epsilon \setminus B) \leqslant \mu(U_\epsilon \setminus F_\epsilon) < \epsilon &\implies \mu(B) \geqslant \mu(U_\epsilon) - \epsilon.
\end{aligned}$$

Thus,

$$\sup_{F\subset B,\,F\in\mathscr{F}} \mu(F) \leqslant \mu(B) \leqslant \epsilon + \mu(F_\epsilon) \leqslant \epsilon + \sup_{F\subset B,\,F\in\mathscr{F}} \mu(F)$$
$$\inf_{U\supset B,\,U\in\mathscr{O}} \mu(U) - \epsilon \leqslant \mu(U_\epsilon) - \epsilon \leqslant \mu(b) \leqslant \inf_{U\supset B,\,U\in\mathscr{O}} \mu(U).$$

(v) For every closed $F \in \mathscr{F}$ the intersections $K_j \cap F$, $j \in \mathbb{N}$, will be compact and $K_j \cap F \uparrow F$. By the continuity of measures we get

$$\mu(F) = \sup_j \mu(K_j \cap F) \leqslant \sup_{K\subset F, K \text{ cpt}} \mu(K) \leqslant \mu(F).$$

Thus,

$$\mu(F) = \sup_{K\subset F, K \text{ cpt}} \mu(K) \qquad \forall\, F \in \mathscr{F}. \tag{*}$$

Combining (iv) and (*) we get

$$\begin{aligned}
\mu(B) &\overset{\text{(iv)}}{=} \sup_{F\subset B,\,F\in\mathscr{F}} \mu(F) \\
&\overset{(*)}{=} \sup_{F\subset B,\,F\in\mathscr{F}} \sup_{K\subset F, K \text{ cpt}} \mu(K) \\
&\leqslant \sup_{F\subset B,\,F\in\mathscr{F}} \underbrace{\sup_{K\subset B, K \text{ cpt}} \mu(K)}_{\text{note: independent of } F\subset B} \\
&= \sup_{K\subset B, K \text{ cpt}} \mu(K)
\end{aligned}$$

and since $\mu(K) \leqslant \mu(B)$ for $K \subset B$ and $\sup_{K\subset B, K \text{ cpt}} \mu(K) \leqslant \mu(B)$ are obvious, we are finished.

(vi) Assume now that μ is σ-finite. Let $(B_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ be an exhausting sequence for X such that $\mu(B_n) < \infty$. Then the measures $\mu_n(B) := \mu(B \cap B_n)$ defined on $\mathscr{B}$ are finite and regular according to part (iv). Since we may interchange any two suprema (cf. the solution of Problem 4.7) we get

$$\begin{aligned}
\mu(B) = \sup_n \mu_n(B) = \sup_n \sup_{F\subset B,\,F\in\mathscr{F}} \mu_n(F) \\
= \sup_{F\subset B,\,F\in\mathscr{F}} \sup_n \mu_n(F) \\
= \sup_{F\subset B,\,F\in\mathscr{F}} \mu(F).
\end{aligned}$$

Problem 4.21 Solution: First of all, Problem 4.21(iv) shows that

$$\mu(B) = \sup_{F\subset B, F \text{ closed}} \mu(F). \tag{*}$$

Let $(d_k)_k$ be an enumeration of the dense set $D \subset X$ and write ρ for the metric in X and $K_r(x) := \{y \in X : \rho(x, y) \leqslant r\}$ for the closed ball with centre x and radius r.

Since, for any fixed $n \in \mathbb{N}$ the sets

$$K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_m) \uparrow X \quad \text{for} \quad m \to \infty$$

we get from (*)

$$\forall \epsilon > 0 \quad \exists\, k(n) \in \mathbb{N} : \mu(F_n) + \frac{\epsilon}{2^n} \geqslant \mu(X)$$

if $F_n := K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_{k(n)})$. Setting

$$K := K_\epsilon := \bigcap_n F_n$$

it is clear that K is closed. Moreover, since K is, for every $1/n$, covered by finitely many balls of radius $1/n$, to wit,

$$K \subset K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_{k(n)}),$$

we see that K is compact. Indeed, if $(x_j)_j \subset K$ is a sequence, there is a subsequence $(x_j^n)_j$ which is completely contained in one of the balls $K_{1/n}(d_1), \ldots, K_{1/n}(d_{k(n)})$. Passing iteratively to sub-sub-etc. sequences we find a subsequence $(y_j)_j \subset (x_j)_j$ which is contained in a sequence of closed balls $K_{1/n}(c_n)$ (c_n is a suitable element from D). Thus $(y_j)_j$ is a Cauchy sequence and converges, because of completeness, to an element x^* which is, as the F_n are closed, in every F_n, hence in K. Thus K is (sequentially) compact.

Since

$$\mu(X \setminus K) = \mu\Big(\bigcup_n X \setminus F_n\Big) \leqslant \sum_n \mu(X \setminus F_n) \leqslant \sum_n \frac{\epsilon}{2^n} = \epsilon,$$

we have found a sequence of compact sets K_n such that $\mu(K_n) \to \mu(X)$ (note that the K_n need not 'converge' X as a set!). Obviously, $K_n \cap F$ is compact for every closed F and we have $\mu(K_n \cap F) \to \mu(F)$, hence

$$\mu(F) = \sup_{K \subset F, K \text{ cpt}} \mu(K) \quad \forall\, F \in \mathscr{F}.$$

Now we can use the argument from the proof of Problem 4.22(v).

5 Uniqueness of measures. Solutions to Problems 5.1–5.13

Problem 5.1 Solution: Since $X \in \mathscr{D}$ and since complements are again in $\mathscr{D}$, we have $\emptyset = X^c \in \mathscr{D}$.

If $A, B \in \mathscr{D}$ are disjoint, we set $A_1 := A, A_2 := B, A_j := \emptyset \ \forall j \geqslant 3$. Then $(A_j)_{j\in\mathbb{N}} \subset \mathscr{D}$ is a sequence of pairwise disjoint sets, and by (D_3) we find that

$$A \cup B = \bigcupdot_{j\in\mathbb{N}} A_j \in \mathscr{D}.$$

Since $(\Sigma_1) = (D_1)$, $(\Sigma_2) = (D_2)$ and since $(\Sigma_3) \implies (D_3)$, it is clear that every σ-algebra is also a Dynkin system; that the converse is, in general, wrong is seen in Problem 5.2.

Problem 5.2 Solution: Consider (D_3) only, as the other two conditions coincide: $(\Sigma_j) = (\Delta_j)$, $j = 1, 2$. We show that (Σ_3) breaks down even for finite unions. If $A, B \in \mathscr{D}$ are disjoint, it is clear that A, B *and also* $A \cupdot B$ contain an even number of elements. But if A, B have non-void intersection, and if this intersection contains an odd number of elements, then $A \cup B$ contains an odd number of elements. Here is a trivial example:

$$A = \{1, 2\} \in \mathscr{D}, \quad B = \{2, 3, 4, 5\} \in \mathscr{D},$$

whereas

$$A \cup B = \{1, 2, 3, 4, 5\} \notin \mathscr{D}.$$

This means that (D_3) holds, but (Σ_3) fails.

Problem 5.3 Solution: We verify the hint first. Using de Morgan's laws we get

$$R \setminus Q = R \setminus (R \cap Q) = R \cap (R \cap Q)^c = (R^c \cup (R \cap Q))^c = (R^c \cupdot (R \cap Q))^c$$

where the last equality follows since $R^c \cap (R \cap Q) = \emptyset$.

Now we take $A, B \in \mathscr{D}$ such that $A \subset B$. In particular $A \cap B = A$. Taking this into account and setting $Q = A$, $R = B$ we get from the above relation

$$B \setminus A = \Big(\underbrace{\underbrace{B^c}_{\in\mathscr{D}} \cupdot A}_{\in\mathscr{D}}\Big)^c \in \mathscr{D}$$

where we repeatedly use (D_2) and (D_2).

Problem 5.4 Solution:

(i) Since the σ-algebra $\mathscr{A}$ is also a Dynkin system, it is enough to prove $\delta(\mathscr{D}) = \mathscr{D}$ for any Dynkin system $\mathscr{D}$. By definition, $\delta(\mathscr{D})$ is the smallest Dynkin system containing $\mathscr{D}$, thus $\mathscr{D} \subset \delta(\mathscr{D})$. On the other hand, $\mathscr{D}$ is itself a Dynkin system, thus, because of minimality, $\mathscr{D} \supset \delta(\mathscr{D})$.

(ii) Clearly, $\mathscr{G} \subset \mathscr{H} \subset \delta(\mathscr{H})$. Since $\delta(\mathscr{H})$ is a Dynkin system containing $\mathscr{G}$, the minimality of $\delta(\mathscr{G})$ implies that $\delta(\mathscr{G}) \subset \delta(\mathscr{H})$.

(iii) Since $\sigma(\mathscr{G})$ is a σ-algebra, it is also a Dynkin system. Since $\mathscr{G} \subset \sigma(\mathscr{G})$ we conclude (again, by minimality) that $\delta(\mathscr{G}) \subset \sigma(\mathscr{G})$.

Problem 5.5 Solution: Clearly, $\delta(\{A, B\}) \subset \sigma(\{A, B\})$ is always true.

By Theorem 5.5, $\delta(\{A, B\}) = \sigma(\{A, B\})$ if $\{A, B\}$ is $\cap$-stable, i.e. if $A = B$ or $A = B^c$ or if at least one of A, B is X or $\emptyset$.

Let us exclude these cases. If $A \cap B = \emptyset$, then

$$\delta(\{A, B\}) = \sigma(\{A, B\}) = \big\{\emptyset, A, A^c, B, B^c, A \uplus B, A^c \cap B^c, X\big\}.$$

If $A \cap B \neq \emptyset$, then

$$\delta(\{A, B\}) = \big\{\emptyset, A, A^c, B, B^c, X\}$$

while $\sigma(\{A, B\})$ is much larger containing, for example, $A \cap B$.

Problem 5.6 Solution: Some authors call families of sets satisfying (D_1), (D_2'), (D_3') monotone classes (this is not the standard definition!). We will use this convention *locally for this solution only*.

Clearly, such a monotone class $\mathscr{F}$ is a Dynkin system:

$$C, D \in \mathscr{F}, \quad C \cap D = \emptyset \underset{(D_2')}{\overset{(D_1)}{\Longrightarrow}} C \uplus D = E \setminus \Big[\ \underbrace{(E \setminus C) \setminus D}_{E\setminus C\supset D \text{ as } C\cap D=\emptyset}\ \Big] \in \mathscr{F},$$

i.e., $\mathscr{F}$ is $\uplus$-stable. This and (D_3') yield (D_3); (D_2) is a special case of (D_2').

Conversely every Dynkin system $\mathscr{D}$ is a monotone class in the sense of this problem:

$$M, N \in \mathscr{D},\ M \subset N \underset{(D_3)}{\overset{(D_2)}{\Longrightarrow}} N^c \cap M = M \setminus N = \emptyset \quad \text{and} \quad N \setminus M = (N^c \uplus M)^c \in \mathscr{D},$$

i.e. (D_2') holds. Thus, (D_3) immediately implies (D_3').

Problem 5.7 Solution: We prove the hint first. Let $(G_j)_{j\in\mathbb{N}} \subset \mathscr{G}$ as stated in the problem, i.e. satisfying (1) and (2), and define the sets $F_N := G_1 \cup \ldots \cup G_N$. As $\mathscr{G} \subset \mathscr{A}$, it is clear that $F_N \in \mathscr{A}$ (but not necessarily in $\mathscr{G}$...). Moreover, it is clear that $F_N \uparrow X$.

We begin with a more general assertion: *For any finite union of $\mathscr{G}$-sets $A_1 \cup \ldots \cup A_N$ we have $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$.*

Proof. *Induction Hypothesis:* $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$ for some $N \in \mathbb{N}$ and any choice of $A_1, \ldots, A_N \in \mathscr{G}$.

Induction Start ($N = 1$)*:* is obvious.

Induction Step $N \rightsquigarrow N+1$: By the induction assumption we know that

$$\begin{aligned}
\mu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big) &= \mu\big((A_1 \cap A_N) \cup \cdots \cup (A_N \cap A_{N+1})\big)\\
&= \nu\big((A_1 \cap A_N) \cup \cdots \cup (A_N \cap A_{N+1})\big)\\
&= \nu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big).
\end{aligned}$$

If $\mu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big) < \infty$, hence $\nu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big) < \infty$, we have by the strong additivity of measures and the $\cap$-stability of $\mathscr{G}$ that

$$\begin{aligned}
&\mu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big)\\
&= \mu\big((A_1 \cup \ldots \cup A_N) \cup A_{N+1}\big)\\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big((A_1 \cup \ldots \cup A_N) \cap A_{N+1}\big)\\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big(\underbrace{(A_1 \cap A_{N+1})}_{\in\mathscr{G}} \cup \ldots \cup \underbrace{(A_N \cap A_{N+1})}_{\in\mathscr{G}}\big)\\
&= \nu\big(A_1 \cup \ldots \cup A_N\big) + \nu(A_{N+1}) - \nu\big((A_1 \cap A_{N+1}) \cup \ldots \cup (A_N \cap A_{N+1})\big)\\
&\ \ \vdots\\
&= \nu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big)
\end{aligned}$$

where we use the induction hypothesis twice, namely for the union of the N $\mathscr{G}$-sets $A_1, \ldots, A_N$ as well as for the N $\mathscr{G}$-sets $A_1 \cap A_{N+1}, \ldots, A_N \cap A_{N+1}$. The induction is complete.

If $\mu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big) = \infty$, hence $\nu\big((A_1 \cup \cdots \cup A_N) \cap A_{N+1}\big) = \infty$, there is nothing to show since the monotinicity of measures entails

$$\begin{aligned}
&(A_1 \cup \cdots \cup A_N) \cap A_{N+1} \subset (A_1 \cup \cdots \cup A_N) \cup A_{N+1}\\
\Rightarrow\ &\mu\big((A_1 \cup \cdots \cup A_N) \cup A_{N+1}\big) = \infty = \nu\big((A_1 \cup \cdots \cup A_N) \cup A_{N+1}\big).
\end{aligned}$$

In particular we see that $\mu(F_N) = \nu(F_N)$, $\nu(F_N) \leqslant \nu(G_1) + \ldots + \nu(G_N) < \infty$ by subadditivity, and that (think!) $\mu(G \cap F_N) = \nu(G \cap F_N)$ for any $G \in \mathscr{G}$ (just work out the intersection, similar to the step in the induction....). This shows that on the $\cap$-stable system

$$\widetilde{\mathscr{G}} := \{\text{all finite unions of sets in } \mathscr{G}\}$$

μ and ν coincide. Moreover, $\mathscr{G} \subset \widetilde{\mathscr{G}} \subset \mathscr{A}$ so that, by assumption $\mathscr{A} = \sigma(\mathscr{G}) \subset \sigma(\widetilde{\mathscr{G}}) \subset \sigma(\mathscr{A}) \subset \mathscr{A}$, so that equality prevails in this chain of inclusions. This means that $\widetilde{\mathscr{G}}$ is a generator of $\mathscr{A}$ satisfying all the assumptions of Theorem 5.7, and we have reduced everything to this situation.

Remark. The last step shows that we only need the induction for sets from $\mathscr{G}$ with finite μ-, hence ν-measure. Therefore, the extended discussion on finiteness is actually not needed, if the induction is only used for the sequences $(G_i)_i$ and $(F_n)_n$.

Problem 5.8 Solution: *Intuition:* in two dimensions we have rectangles. Take $I, I' \in \mathscr{J}$. Denote the lower left corner of I by $a = (a_1, a_2)$, the upper right corner by $b = (b_1, b_2)$, and do the same for I' using a', b'. This defines a rectangle uniquely. We are done, if $I \cap I' = \emptyset$. If not (draw a picture!) then we get an overlap which can be described by taking the right-and-upper-most of the two lower left corners a, a' and the left-and-lower-most of the two upper right corners b, b'. That does the trick.

Now rigorously: since $I, I' \in \mathscr{J}$, we have for suitable a_j, b_j, a'_j, b'_j's:

$$I = \mathop{\times}_{j=1}^{n} \big[a_j, b_j\big) \quad\text{and}\quad I' = \mathop{\times}_{j=1}^{n} \big[a'_j, b'_j\big).$$

We want to find $I \cap I'$, or, equivalently the condition under which $x \in I \cap I'$. Now

$$\begin{aligned} x = (x_1, \dots, x_n) \in I &\iff x_j \in [a_j, b_j) \quad \forall j = 1, 2, \dots, n \\ &\iff a_j \leqslant x_j < b_j \quad \forall j = 1, 2, \dots, n \end{aligned}$$

and the same holds for $x \in I'$ (same x, but I'—no typo). Clearly $a_j \leqslant x_j < b_j$, and, at the same time $a'_j \leqslant x_j < b'_j$ holds exactly if

$$\begin{aligned} &\max(a_j, a'_j) \leqslant x_j < \min(b_j, b'_j) \quad \forall j = 1, 2, \dots, n \\ \iff\ & x \in \mathop{\times}_{j=1}^{n} \big[\max(a_j, a'_j), \min(b_j, b'_j)\big). \end{aligned}$$

This shows that $I \cap I'$ is indeed a 'rectangle', i.e. in $\mathscr{J}$. This could be an empty set (which happens if I and I' do not meet).

Problem 5.9 Solution: First we must make sure that $t \cdot B$ is a Borel set if $B \in \mathscr{B}$. We consider first rectangles $I = [\![a, b)\!) \in \mathscr{J}$ where $a, b \in \mathbb{R}^n$. Clearly, $t \cdot I = [\![ta, tb)\!)$ where ta, tb are just the scaled vectors. So, scaled rectangles are again rectangles, and therefore Borel sets. Now fix $t > 0$ and set

$$\mathscr{B}_t := \{B \in \mathscr{B}(\mathbb{R}^n) : t \cdot B \in \mathscr{B}(\mathbb{R}^n)\}.$$

It is not hard to see that $\mathscr{B}_t$ is itself a σ-algebra and that $\mathscr{J} \subset \mathscr{B}_t \subset \mathscr{B}(\mathbb{R}^n)$. But then we get

$$\mathscr{B}(\mathbb{R}^n) = \sigma(\mathscr{J}) \subset \sigma(\mathscr{B}_t) = \mathscr{B}_t \subset \mathscr{B}(\mathbb{R}^n),$$

showing that $\mathscr{B}_t = \mathscr{B}(\mathbb{R}^n)$, i.e. scaled Borel sets are again Borel sets.

Now define a new measure $\mu(B) := \lambda^n(t \cdot B)$ for Borel sets $B \in \mathscr{B}(\mathbb{R}^n)$ (which is, because of the above, well-defined). For rectangles $[\![a, b)\!)$ we get, in particular,

$$\begin{aligned}\mu[\![a, b)\!) = \lambda^n\big((t \cdot [\![a, b)\!)\big) &= \lambda^n[\![ta, tb)\!)\\ &= \prod_{j=1}^n \big((tb_j) - (ta_j)\big)\\ &= \prod_{j=1}^n t \cdot \big(b_j - a_j\big)\\ &= t^n \cdot \prod_{j=1}^n \big(b_j - a_j\big)\\ &= t^n \lambda^n[\![a, b)\!)\end{aligned}$$

which shows that μ and $t^n\lambda^n$ coincide on the $\cap$-stable generator $\mathscr{J}$ of $\mathscr{B}(\mathbb{R}^n)$, hence they're the same everywhere. (Mind the small gap: we should make the mental step that for any measure ν a positive multiple, say, $c \cdot \nu$, is again a measure—this ensures that $t^n\lambda^n$ is a measure, and we need this in order to apply Theorem 5.7. Mind also that we need that μ is finite on all rectangles (obvious!) and that we find rectangles increasing to $\mathbb{R}^n$, e.g. $[-k, k) \times \ldots \times [-k, k)$ as in the proof of Theorem 5.8(ii).)

■■

Problem 5.10 Solution: Define $\nu(A) := \mu \circ \theta^{-1}(A)$. Obviously, ν is again a finite measure. Moreover, since $\theta^{-1}(X) = X$, we have

$$\mu(X) = \nu(X) < \infty \ \text{ and, by assumption, } \ \mu(G) = \nu(G) \quad \forall\, G \in \mathscr{G}.$$

Thus, $\mu = \nu$ on $\mathscr{G}' := \mathscr{G} \cup \{X\}$. Since $\mathscr{G}'$ is a $\cap$-stable generator of $\mathscr{A}$ containing the (trivial) exhausting sequence $X, X, X, \ldots$, the assertion follows from the uniqueness theorem for measures, Theorem 5.7.

■■

Problem 5.11 Solution: The necessity of the condition is trivial since $\mathscr{G} \subset \sigma(\mathscr{G}) = \mathscr{B}$, resp., $\mathscr{H} \subset \sigma(\mathscr{H}) = \mathscr{C}$.

Fix $H \in \mathscr{H}$ and define

$$\mu(B) := P(B \cap H) \ \text{ and } \ \nu(B) := P(B)P(H).$$

Obviously, μ and ν are finite measures on $\mathscr{B}$ having mass $P(H)$ such that μ and ν coincide on the $\cap$-stable generator $\mathscr{G} \cup \{X\}$ of $\mathscr{B}$. Note that this generator contains the exhausting sequence $X, X, X, \ldots$. By the uniqueness theorem for measures, Theorem 5.7, we conclude

$$\mu = \nu \ \text{ on the whole of } \ \mathscr{B}.$$

Now fix $B \in \mathscr{B}$ and define

$$\rho(C) := P(B \cap C) \quad \text{and} \quad \tau(C) := P(B)P(C).$$

Then the same argument as before shows that $\rho = \tau$ on $\mathscr{C}$ and, since $B \in \mathscr{B}$ was arbitrary, the claim follows.

■■

Problem 5.12 Solution:

(i) Following the hint we check that

$$\mathscr{D} := \{A \in \mathscr{A} : \forall \epsilon > 0 \; \exists \, G \in \mathscr{G} : \mu(A \triangle G) \leqslant \epsilon\}$$

is a Dynkin system.

(D_1) By assumption, $G := X \in \mathscr{G}$ and so $\mu(X \triangle G) = \mu(\emptyset) = 0$, hence $X \in \mathscr{D}$.

(D_2) Assume that $A \in \mathscr{D}$. For every $\epsilon > 0$ there is some $G \in \mathscr{G}$ such that $\mu(A \triangle G) \leqslant \epsilon$. From

$$\begin{aligned}
A^c \triangle G^c &= (G^c \setminus A^c) \cup (A^c \setminus G^c) \\
&= (G^c \cap A) \cup (A^c \cap G) \\
&= (A \setminus G) \cup (G \setminus A) \\
&= A \triangle G
\end{aligned}$$

we conclude that $\mu(A^c \triangle G^c) \leqslant \epsilon$; consequently, $A^c \in \mathscr{D}$ (observe that $G^c \in \mathscr{G}$!).

(D_3) Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{D}$ be a sequence of mutually disjoint sets and $\epsilon > 0$. Since μ is a finite measure, we get

$$\sum_{j\in\mathbb{N}} \mu(A_j) = \mu\left(\dot{\bigcup_{j\in\mathbb{N}}} A_j\right) < \infty,$$

and, in particular, we can pick $N \in \mathbb{N}$ so large, that

$$\sum_{j=N+1}^{\infty} \mu(A_j) \leqslant \epsilon.$$

For $j \in \{1, \dots, N\}$ there is some $G_j \in \mathscr{G}$ such that $\mu(A_j \triangle G_j) \leqslant \epsilon$. Thus, $G := \bigcup_{j=1}^{N} G_j \in \mathscr{G}$ satisfies

$$\begin{aligned}
\left(\dot{\bigcup_{j\in\mathbb{N}}} A_j\right) \setminus G &= \left(\dot{\bigcup_{j\in\mathbb{N}}} A_j\right) \cap G^c \\
&= \left(\dot{\bigcup_{j\in\mathbb{N}}} A_j\right) \cap \left(\bigcap_{j=1}^{N} G_j^c\right) \\
&= \dot{\bigcup_{j\in\mathbb{N}}} \left(A_j \cap \bigcap_{k=1}^{N} G_k^c\right)
\end{aligned}$$

$$\subset \bigcupdot_{j=1}^{N}(A_j \cap G_j^c) \cup \bigcupdot_{j=N+1}^{\infty} A_j.$$

In the same way we get

$$\begin{aligned}
G \setminus \left(\bigcup_{j\in\mathbb{N}} A_j\right) &\subset G \setminus \left(\bigcupdot_{j=1}^{N} A_j\right) \\
&= G \cap \bigcap_{j=1}^{N} A_j^c \\
&\subset \bigcup_{j=1}^{N}(G_j \cap A_j^c).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu\left(\left(\bigcupdot_{j\in\mathbb{N}} A_j\right) \triangle G\right) &\leqslant \mu\left(\bigcup_{j=1}^{N}(A_j \triangle G_j) \cup \bigcupdot_{j=N+1}^{\infty} A_j\right) \\
&\leqslant \sum_{j=1}^{N} \mu(A_j \triangle G_j) + \sum_{j=N+1}^{\infty} \mu\left(A_j\right) \\
&\leqslant \sum_{j=1}^{N} \epsilon 2^{-j} + \epsilon \\
&\leqslant \epsilon + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\bigcupdot_{j\in\mathbb{N}} A_j \in \mathscr{D}$.

Obviously, $\mathscr{G} \subset \mathscr{D}$ (take $G = A \in \mathscr{G}$). Since $\mathscr{G}$ is $\cap$-stable, we get

$$\mathscr{A} = \sigma(\mathscr{G}) = \delta(\mathscr{G}) \subset \mathscr{D}.$$

(ii) Using the family

$$\mathscr{D}' := \{A \in \mathscr{A} : \forall \epsilon > 0\ \exists\, G \in \mathscr{G} : \mu(A \triangle G) \leqslant \epsilon, \nu(A \triangle G) \leqslant \epsilon\},$$

we find, just as in (i), that $\mathscr{D}'$ is a Dynkin system. The rest of the proof is as before.

(iii) "$\Leftarrow$": Let $A \in \mathscr{A}$ such that $A \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\mu\left(\bigcup_{n\in\mathbb{N}} I_n\right) \leqslant \epsilon$. Because of the monotonicity of measures we get

$$\mu(A) \leqslant \mu\left(\bigcup_{n\in\mathbb{N}} I_n\right) \leqslant \epsilon,$$

and so $\mu(A) = 0$.

"$\Rightarrow$": Set $\mathscr{K} := \{A \subset \mathbb{R}^n : \exists (I_k)_{k\in\mathbb{N}} \subset \mathscr{J} : A = \bigcup_k I_k \text{ or } A^c = \bigcup_k I_k\}$ and observe that $I \in \mathscr{K} \Rightarrow I^c \in \mathscr{K}$. Define, furthermore,

$$\mathscr{D} := \{A \subset \mathbb{R}^n : \forall \epsilon \exists\, J, K \in \mathscr{K},\ J \subset A \subset K,\ \mu(K \setminus J) \leqslant \epsilon\}.$$

We claim that $\mathscr{D}$ is a Dynkin system.

(D_1) Clearly, $X = \mathbb{R}^n \in \mathscr{D}$ (take $J = K = \mathbb{R}^n$).

(D_2) Pick $A \in \mathscr{D}$ and $\epsilon > 0$. Then there are $J, K \in \mathscr{K}$ such that $J \subset A \subset K$ and $\mu(K \setminus J) \leqslant \epsilon$. From $J^c, K^c \in \mathscr{K}$, $\mu(K^c \setminus J^c) = \mu(J \setminus K) \leqslant \epsilon$ and $J^c \supset A^c \supset K^c$ we get immediately $A^c \in \mathscr{D}$.

(D_3) Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{D}$ be a sequence of mutually disjoint sets and $\epsilon > 0$. Pick $J_j \in \mathscr{K}$ and $K_j \in \mathscr{K}$ such that $J_j \subset A_j \subset K_j$, $\mu(K_j \setminus J_j) \leqslant \epsilon 2^{-j}$ and set

$$J := \bigcup_{j\in\mathbb{N}} A_j \qquad K := \bigcup_{j\in\mathbb{N}} K_j.$$

Since $\mathscr{K}$ is stable under countable unions, we get $J \in \mathscr{K}$, $K \in \mathscr{K}$. Moreover, $J \subset \dot{\bigcup}_j A_j \subset K$ and

$$\begin{aligned}
\mu(K \setminus J) &= \mu\left(\left(\bigcup_{j\in\mathbb{N}} K_j\right) \cap \left(\bigcup_{j\in\mathbb{N}} J_j\right)^c\right) \\
&= \mu\left(\left(\bigcup_{j\in\mathbb{N}} K_j\right) \cap \left(\bigcap_{j\in\mathbb{N}} J_j^c\right)\right) \\
&= \mu\left(\left[\bigcup_{j\in\mathbb{N}} \left(K_j \cap \bigcap_{k\in\mathbb{N}} J_k^c\right)\right]\right) \\
&\leqslant \mu\left(\bigcup_{j\in\mathbb{N}} (K_j \cap J_j^c)\right) \\
&\leqslant \sum_{j\in\mathbb{N}} \underbrace{\mu(K_j \cap J_j^c)}_{\mu(K_j \setminus J_j) \leqslant \epsilon 2^{-j}} \\
&\leqslant \epsilon.
\end{aligned}$$

Thus, $\dot{\bigcup}_j A_j \in \mathscr{D}$.

Finally, $\mathscr{J} \subset \mathscr{D}$ entails that $\mathscr{B}(\mathbb{R}^n) = \delta(\mathscr{J}) \subset \mathscr{D}$.

Now let A be a set satisfying $\mu(A) = 0$. Therefore, for every $\epsilon > 0$ there is a set $K_\epsilon = K \in \mathscr{K}$ such that $A \subset K$ and $\mu(K) < \epsilon$. If $K = \bigcup_i I_i$, we are done. If $K^c = \bigcup_i I_i$ we have to argue like this: Let $J := J_R := [-R, R)^d \in \mathscr{J}$. Then

$$K = \bigcap_i I_i^c \quad \text{and} \quad J \cap K = \bigcap_i I_i^c \cap J = \bigcap_i J \setminus I_i = \bigcap_k \bigcap_{i=1}^k J \setminus I_i$$

and each set $J \setminus I_i$ is a finite union of sets from $\mathscr{J}$ (since $\mathscr{J}$ is a semiring), hence $\bigcap_{i=1}^k J \setminus I_i$ is a finite union of sets from $\mathscr{J}$. Since $\mu(J \cap K) \leqslant \mu(K) \leqslant \epsilon$, a continuity-of-measure argument shows that there exists some k such that $J \cap K \subset \bigcap_{i=1}^k J \setminus I_i$ and $\mu(\bigcap_{i=1}^k J \setminus I_i) \leqslant 2\epsilon$.

If we pick $\epsilon = \epsilon/2^R$, we see that we can cover $A \cap [-R, R)^d$ by a countable union of $\mathscr{J}$-sets, call their union U_R, such that $\mu(U_R) \leqslant \epsilon/2^R$. Finally,

$$\mu(A) \leqslant \sum_{R\in\mathbb{N}} \mu(U_R) \leqslant \epsilon$$

and we can combine all covers which make up the U_R, $R \in \mathbb{N}$.

Problem 5.13 Solution:

(i) **mind the misprint: we also need stability of $\mathscr{M}$ under finite intersections.** Clearly, any σ-algebra is also a monotone class. Conversely, if $\mathscr{M}$ is a monotone class such that $M \in \mathscr{M} \implies M^c \in \mathscr{M}$, then the condition (Σ_2) holds, while (Σ_1) is satisfied by the very definition of a monotone class. If $\mathscr{M}$ is also stable under finite intersections, we get $M, N \in \mathscr{M} \implies M \cup N = (M^c \cap N^c)^c \in \mathscr{M}$, so (Σ_3) follows from the stability under finite unions and the stability of monotone classes under increasing limits of sets.

(ii) Since $\sigma(\mathscr{G})$ is a monotone class containing $\mathscr{G}$, we have – by minimality – that $\mathfrak{m}(\mathscr{G}) \subset \sigma(\mathscr{G})$. On the other hand, by the monotone class theorem, we get $\mathscr{G} \subset \mathfrak{m}(\mathscr{G}) \implies \sigma(\mathscr{G}) \subset \mathfrak{m}(\mathscr{G})$ which means that $\mathfrak{m}(\mathscr{G}) = \sigma(\mathscr{G})$.

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6 Existence of measures. Solutions to Problems 6.1–6.14

Problem 6.1 Solution:

(i) *Monotonicity:* If $x \leqslant 0 \leqslant y$, then $F_\mu(x) \leqslant 0 \leqslant F_\mu(y)$.

If $0 < x \leqslant y$, we have $[0, x) \subset [0, y)$ and so $0 \leqslant F_\mu(x) = \mu[0, x) \leqslant \mu[0, y) = F_\mu(y)$.

If $x \leqslant y < 0$, we have $[y, 0) \subset [x, 0)$ and so $0 \leqslant -F_\mu(y) = \mu[y, 0) \leqslant \mu[x, 0) = -F_\mu(x)$, i.e. $F_\mu(x) \leqslant F_\mu(y) \leqslant 0$.

Left-continuity: Let us deal with the case $x \geqslant 0$ only, the case $x < 0$ is analogous (and even easier). Assume first that $x > 0$. Take any sequence $x_k < x$ and $x_k \uparrow x$ as $k \to \infty$. Without loss of generality we can assume that $0 < x_k < x$. Then $[0, x_k) \uparrow [0, x)$ and using Proposition 4.3 (continuity of measures) implies

$$\lim_{k\to\infty} F_\mu(x_k) = \lim_{k\to\infty} \mu[0, x_k) = \mu[0, x) = F_\mu(x).$$

If $x = 0$ we must take a sequence $x_k < 0$ and we have then $[x_k, 0) \downarrow [0, 0) = \emptyset$. Again by Proposition 4.3, now (vii), we get

$$\lim_{k\to\infty} F_\mu(x_k) = -\lim_{k\to\infty} \mu[x_k, 0) = \mu(\emptyset) = 0 = F_\mu(0).$$

which shows left-continuity at this point, too.

We remark that, since for a sequence $y_k \downarrow y$, $y_k > y$ we have $[0, y_k) \downarrow [0, y]$, and not $[0, y)$, we cannot expect right-continuity in general.

(ii) Since $\mathscr{J} = \{[a, b), a \leqslant b\}$ is a semi-ring (cf. the remark preceding Proposition 6.3 or Proposition 6.5) it is enough to check that ν_F is a premeasure on $\mathscr{J}$. This again amounts to showing (M_1) and (M_2) relative to $\mathscr{J}$ (mind you: ν_F is not a *measure* as $\mathscr{J}$ is not a σ-algebra....).

(i) $\nu_F(\emptyset) = \nu_F[a, a) = F(a) - F(a) = 0$ for any a.

(ii) Let $a \leqslant b \leqslant c$ so that $[a, b), [b, c) \in \mathscr{J}$ are disjoint sets and $[a, c) = [a, b) \cup [b, c) \in \mathscr{J}$ (the latter is crucial). Then we have

$$\begin{aligned} \nu_F[a, b) + \nu_F[b, c) &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \nu_F[a, c) \\ &= \nu_F\big([a, b) \cup [b, c)\big). \end{aligned}$$

(iii) We mimick the proof of existence of Lebesgue measure. Let $I_n = [a_n, b_n) \in \mathscr{J}$ be disjoint such that $I = [a, b) = \dot\bigcup_{n=1}^\infty [a_n, b_n) \in \mathscr{J}$. Fix $\epsilon_n, \epsilon > 0$ (these values will be chosen later) and observe that

$$\bigcup_{n=1}^\infty (a_n - \epsilon_n, b_n) \supset [a, b-\epsilon]$$

is an open cover of the compact interval $[a, b-\epsilon]$. Thus, there exists a finite open subcover, hence some $N \in \mathbb{N}$ such that

$$\bigcup_{n=1}^N (a_n - \epsilon_n, b_n) \supset [a, b-\epsilon] \implies \bigcup_{n=1}^N [a_n - \epsilon_n, b_n) \supset [a, b-\epsilon).$$

We have to show that

$$\nu_F[a,b) - \sum_{n=1}^N \nu_F[a_n, b_n) \xrightarrow[N\to\infty]{} 0.$$

First note that we can de- and increase $a_n \geqslant a_n'$ and $b_n \leqslant b_n'$ such that

$$\dot\bigcup_{n=1}^N [a_n, b_n) \subset \dot\bigcup_{n=1}^N [a_n', b_n') = [a, b)$$

so that by the finite additivity of ν_F we get

$$0 = \nu_F[a,b) - \sum_{n=1}^N \nu_F[a_n', b_n') \leqslant \nu_F[a,b) - \sum_{n=1}^N \nu_F[a_n, b_n).$$

Thus, using only the finite additivity and sub-additivity of ν_F

$$\begin{aligned}
0 &\leqslant \nu_F[a,b) - \sum_{n=1}^N \nu_F[a_n, b_n) \\
&= \underbrace{\nu_F[a, b-\epsilon) - \sum_{n=1}^N \nu_F[a_n - \epsilon_n, b_n)}_{\leqslant 0 \text{ finite covering \& subadditivity}} + \nu_F[b-\epsilon, b) + \sum_{n=1}^N \nu_F[a_n - \epsilon_n, a_n) \\
&\leqslant \nu_F[b-\epsilon, b) + \sum_{n=1}^N \nu_F[a_n - \epsilon_n, a_n).
\end{aligned}$$

Now we choose ϵ and ϵ_n. For any given $\eta > 0$ we can find $\epsilon > 0$ and $\epsilon_n > 0$ such that

$$\begin{aligned}
\nu_F[b-\epsilon, b) &= F(b) - F(b-\epsilon) \leqslant \frac{\eta}{2} \\
\text{and} \quad \nu_F[a_n - \epsilon_n, a_n) &= F(a_n) - F(a_n - \epsilon_n) \leqslant 2^{-n}\frac{\eta}{2}
\end{aligned}$$

here we use the left-continuity of F. Thus,

$$0 \leqslant \nu_F[a,b) - \sum_{n=1}^N \nu_F[a_n, b_n) \leqslant \frac{\eta}{2} + \sum_{n=1}^N 2^{-n}\frac{\eta}{2} \leqslant \eta.$$

Letting first $N \to \infty$ and then $\eta \to 0$ proves the claim.

Note that ν_F takes on only positive values because F increases.

This means that we find *at least one* extension. Uniqueness follows since

$$\nu_F[-k,k) = F(k) - F(-k) < \infty \quad \text{and} \quad [-k,k) \uparrow \mathbb{R}.$$

(iii) Now let μ be a measure with $\mu[-n,n) < \infty$. The latter means that the function $F_\mu(x)$, as defined in part (i), is finite for every $x \in \mathbb{R}$. Now take this F_μ and define, as in (ii) a (uniquely defined) measure ν_{F_μ}. Let us see that $\mu = \nu_{F_\mu}$. For this, it is enough to show equality on the sets of type $[a,b)$ (since such sets generate the Borel sets and the uniqueness theorem applies....)

If $0 \leqslant a \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b) - \mu[0,a)\\ &= \mu\big([0,b) \setminus [0,a)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant b \leqslant 0$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= -\mu[b,0) - (-\mu[a,0))\\ &= \mu[a,0) - \mu[b,0)\\ &= \mu\big([a,0) \setminus [b,0)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant 0 \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b)) - (-\mu[a,0))\\ &= \mu[a,0)) + \mu[0,b)\\ &= \mu\big([a,0) \uplus [0,b)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

(iv) $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x$, since $\lambda[a,b) = b - a = F(b) - F(a)$.

(v) $F : \mathbb{R} \to \mathbb{R}$, with, say, $F(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} = \mathbb{1}_{(0,\infty)}(x)$ since $\delta_0[a,b) = 0$ whenever $a,b < 0$ or $a,b > 0$. This means that F must be constant on $(-\infty,0)$ and $(0,\infty)$ If $a \leqslant 0 < b$ we have, however, $\delta_0[a,b) = 1$ which indicates that $F(x)$ must jump by 1 at the point 0. Given the fact that F must be left-continuous, it is clear that it has, in principle, the above form. The only ambiguity is, that if $F(x)$ does the job, so does $c + F(x)$ for any constant $c \in \mathbb{R}$.

(vi) Assume that F is continuous at the point x. Then

$$
\begin{aligned}
\mu(\{x\}) &= \mu\Big(\bigcap_{k\in\mathbb{N}}\Big[x, x+\tfrac{1}{k}\Big)\Big) \\
&\overset{4.3}{=} \lim_{k\to\infty} \mu\left(\Big[x, x+\tfrac{1}{k}\Big)\right) \\
&\overset{\text{def}}{=} \lim_{k\to\infty} \left(F\left(x+\tfrac{1}{k}\right) - F(x)\right) \\
&= \lim_{k\to\infty} F\left(x+\tfrac{1}{k}\right) - F(x) \\
&\overset{(*)}{=} F(x) - F(x) = 0
\end{aligned}
$$

where we use (right-)continuity of F at x in the step marked $(*)$.

Now, let conversely $\mu(\{x\}) = 0$. A similar calculation as above shows, that for *every* sequence $\epsilon_k > 0$ with $\epsilon_k \to 0$

$$
\begin{aligned}
F(x+) - F(x) &= \lim_{k\to\infty} F\left(x+\epsilon_k\right) - F(x) \\
&\overset{\text{def}}{=} \lim_{k\to\infty} \mu[x, x+\epsilon_k) \\
&\overset{4.3}{=} \mu\Big(\bigcap_{k\in\mathbb{N}}[x, x+\epsilon_k)\Big) \\
&= \mu(\{x\}) = 0
\end{aligned}
$$

which means that $F(x) = F(x+)$ ($x+$ indicates the right limit), i.e. F is right-continuous at x, hence continuous, as F is left-continuous anyway.

■■

Problem 6.2 Solution: Using the notion of measurability we get

$$
\begin{aligned}
\mu^*\left(Q \cap \bigcup_{i=1}^{\infty} A_i\right) &= \mu^*\left(\left(Q \cap \bigcup_{i=1}^{\infty} A_i\right) \cap A_1\right) + \mu^*\left(\left(Q \cap \bigcup_{i=1}^{\infty} A_i\right) \cap A_1^c\right) \\
&= \mu^*(Q \cap A_1) + \mu^*\left(Q \cap \bigcup_{i=2}^{\infty} A_i\right) \\
&= \dots \\
&= \sum_{i=1}^{n-1} \mu^*(Q \cap A_i) + \mu^*(Q \cap (\cup_{i=n}^{\infty} A_i))
\end{aligned}
\tag{6.1}
$$

for any $n \in \mathbb{N}$. Thus, $\mu^*(Q \cap \bigcup_{i=1}^{\infty} A_i) \geqslant \sum_{i=1}^{n-1} \mu^*(Q \cap A_i)$ for all $n \in \mathbb{N}$. If $n \to \infty$ we obtain

$$
\mu^*\left(Q \cap \bigcup_{i=1}^{\infty} A_i\right) \geqslant \sum_{i=1}^{\infty} \mu^*(Q \cap A_i).
$$

Case 1: $\sum_{i=1}^{\infty} \mu^*(Q \cap A_i) = \infty$. Nothing to show.

Case 2: $\sum_{i=1}^{\infty} \mu^*(Q \cap A_i) < \infty$. Using the sub-additivity of outer measures we get

$$
\mu^*\left(Q \cap \bigcup_{i=n}^{\infty} A_i\right) \leqslant \sum_{i=n}^{\infty} \mu^*(Q \cap A_i) \xrightarrow{n\to\infty} 0
$$

and the claim follows from (6.1) as $n \to \infty$.

Problem 6.3 Solution: We know already that $\mathscr{B}[0,\infty)$ is a σ-algebra (it is a trace σ-algebra) and, by definition,

$$\Sigma = \left\{ B \cup (-B) \,:\, B \in \mathscr{B}[0,\infty) \right\}$$

if we write $-B := \{-b \,:\, b \in \mathscr{B}[0,\infty)\}$.

Since the structure $B \cup (-B)$ is stable under complementation and countable unions it is clear that Σ is indeed a σ-algebra.

One possibility to extend μ defined on Σ would be to take $B \in \mathscr{B}(\mathbb{R})$ and define $B^+ := B \cap [0,\infty)$ and $B^- := B \cap (-\infty, 0)$ and to set

$$\nu(B) := \mu(B^+ \cup (-B^+)) + \mu((-B^-) \cup B^-)$$

which is obviously a measure. We cannot expect uniqueness of this extension since Σ does not generate $\mathscr{B}(\mathbb{R})$—not all Borel sets are symmetric.

Problem 6.4 Solution: By definition we have

$$\mu^*(Q) = \inf \left\{ \sum_j \mu(B_j) \,:\, (B_j)_{j\in\mathbb{N}} \subset \mathscr{A},\ \bigcup_{j\in\mathbb{N}} B_j \supset Q \right\}.$$

(i) Assume first that $\mu^*(Q) < \infty$. By the definition of the infimum we find for every $\epsilon > 0$ a sequence $(B_j^\epsilon)_{j\in\mathbb{N}} \subset \mathscr{A}$ such that $B^\epsilon := \bigcup_j B_j^\epsilon \supset Q$ and, because of σ-subadditivity,

$$\mu(B^\epsilon) - \mu^*(Q) \leqslant \sum_j \mu(B_j^\epsilon) - \mu^*(Q) \leqslant \epsilon.$$

Set $B := \bigcap_k B^{1/k} \in \mathscr{A}$. Then $B \supset Q$ and $\mu(B) = \mu^*(B) = \mu^*(Q)$.

Now let $N \in \mathscr{A}$ and $N \subset B \setminus Q$. Then

$$\begin{aligned} B \setminus N \supset B \setminus (B \setminus Q) = B \cap [(B \cap Q^c)^c] &= B \cap [B^c \cup Q] \\ &= B \cap Q \\ &= Q. \end{aligned}$$

So,

$$\mu^*(Q) - \mu(N) = \mu(B) - \mu(N) = \mu(B \setminus N) = \mu^*(B \setminus N) \geqslant \mu^*(Q)$$

which means that $\mu(N) = 0$.

If $\mu^*(Q) = \infty$, we take the exhausting sequence $(A_k)_{k\in\mathbb{N}} \subset \mathscr{A}$ with $A_k \uparrow X$ and $\mu(A_k) < \infty$ and set $Q_k := A_k \cap Q$ for every $k \in \mathbb{N}$. By the first part we can find sets $B_k \in \mathscr{A}$ with

$B_k \supset Q_k$, $\mu(B_k) = \mu^*(Q_k)$ and $\mu(N) = 0$ for all $N \in \mathscr{A}$ with $N \subset B_k \setminus Q_k$. Without loss of generality we can assume that $B_k \subset A_k$, otherwise we replace B_k by $A_k \cap B_k$. Indeed, $B_k \cap A_k \supset Q_k$, $B_k \cap A_k \in \mathscr{A}$,

$$\mu^*(Q_k) = \mu(B_k) \geqslant \mu(A_k \cap B_k) \geqslant \mu^*(Q_k)$$

and $B_k \setminus Q_k \supset (B_k \cap A_k) \setminus Q_k$, i.e. we have again that all measurable $N \subset (B_k \cap A_k) \setminus Q_k$ satisfy $\mu(N) = 0$.

Assume now that $N \subset B \setminus Q$, $B = \bigcup_k B_k$ and $N \in \mathscr{A}$. Then $N_k := N \cap B_k \in \mathscr{A}$ and we have $N = \bigcup_k N_k$ as well as

$$N_k = N \cap B_k \subset (B \setminus Q) \cap B_k = B_k \setminus Q = B_k \setminus Q_k.$$

Thus $\mu(N_k) = 0$ and, by σ-subadditivity, $\mu(N) \leqslant \sum_{k=1}^{\infty} \mu(N_k) = 0$.

(ii) Define $\bar\mu := \mu^*\big|_{\mathscr{A}^*}$. We know from Theorem 6.1 that $\bar\mu$ is a measure on $\mathscr{A}^*$ and, because of the monotonicity of μ^*, we know that for all $N^* \in \mathscr{A}^*$ with $\bar\mu(N^*)$ we have

$$\forall\, M \subset N^* \,:\, \mu^*(M) \leqslant \mu^*(N^*) = \bar\mu(N^*) = 0.$$

It remains to show that $M \in \mathscr{A}^*$. Because of (6.2) we have to show that

$$\forall\, Q \subset X \,:\, \mu^*(Q) = \mu^*(Q \cap M) + \mu^*(Q \setminus M).$$

Since μ^* is subadditive we find for all $Q \subset X$

$$\begin{aligned}
\mu^*(Q) &= \mu^*\big((Q \cap M) \cup (Q \setminus M)\big) \\
&\leqslant \mu^*(Q \cap M) + \mu^*(Q \setminus M) \\
&= \mu^*(Q \setminus M) \\
&\leqslant \mu^*(Q),
\end{aligned}$$

which means that $M \in \mathscr{A}^*$.

(iii) Obviously, $(X, \mathscr{A}^*, \bar\mu)$ extends $(X, \mathscr{A}, \mu)$ since $\mathscr{A} \subset \mathscr{A}^*$ and $\bar\mu\big|_{\mathscr{A}} = \mu$. In view of Problem 4.15 we have to show that

$$\mathscr{A}^* = \{A \cup N \,:\, A \in \mathscr{A}, \quad N \in \mathfrak{N}\} \tag{*}$$

with $\mathfrak{N} = \{N \subset X \,:\, N$ is subset of an $\mathscr{A}$-measurable null set or, alternatively,

$$\mathscr{A}^* = \{A^* \subset X \,:\, \exists\, A, B \in \mathscr{A},\ A \subset A^* \subset B,\ \mu(B \setminus A) = 0\}. \tag{**}$$

We are going to use both equalities and show '$\supset$' in $(*)$ and '$\subset$' in $(**)$ (which is enough since, cf. Problem 4.15 asserts the equality of the right-hand sides of $(*)$, $(**)$!).

$\boxed{\text{'}\supset\text{':}}$ By part (ii), subsets of $\mathscr{A}$-null sets are in $\mathscr{A}^*$ so that every set of the form $A \cup N$ with $A \in \mathscr{A}$ and N being a subset of an $\mathscr{A}$ null set is in $\mathscr{A}^*$.

$\boxed{\text{‘}\subset\text{’:}}$ By part (i) we find for every $A^* \in \mathscr{A}^*$ some $A \in \mathscr{A}$ such that $A \supset A^*$ and $A \setminus A^*$ is an $\mathscr{A}^*$ null set. By the same argument we get $B \in \mathscr{A}$, $B \supset (A^*)^c$ and $B \setminus (A^*)^c = B \cap A^* = A^* \setminus B^c$ is an $\mathscr{A}^*$ null set. Thus,

$$B^c \subset A^* \subset A$$

and

$$A \setminus B^c \subset \big(A \setminus A^*\big) \cup \big(A^* \setminus B^c\big) = \big(A \setminus A^*\big) \cup \big(B \setminus (A^*)^c\big)$$

which is the union of two $\mathscr{A}^*$ null sets, i.e. $A \setminus B^c$ is an $\mathscr{A}$ null set.

■■

Problem 6.5 Solution: Since, by assumption, m is an additive set function such that $0 \leqslant m(X) \leqslant \mu(X) < \infty$, it is enough to show (cf. Lemma 4.9) that m is continuous at $\emptyset$ and $m(\emptyset) = 0$.

- $m(\emptyset) = 0$: This follows immediately from $m(\emptyset) \leqslant \mu(\emptyset) = 0$. (Note: $\emptyset = X^c \in \mathscr{B}$.)
- m is continuous at $\emptyset$: Let $(B_k)_{k\in\mathbb{N}} \subset \mathscr{B}$, $B_k \downarrow \emptyset$. Since $\mu(B_k) \to 0$ we get

$$m(B_k) \leqslant \mu(B_k) \xrightarrow{k\to\infty} 0.$$

  This shows that m is continuous at $\emptyset$.

Remark. In order to be self-contained, let us check that any additive set function m on a Boolean algebra $\mathscr{B}$ is a pre-measure (i.e. sigma-additive) if it is continuous at $\emptyset$:

Let $(B_n)_{n\in\mathbb{N}} \subset \mathscr{B}$ be a sequence of mutually disjoint sets and $B := \bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}$. From $B_1 \uplus \ldots \uplus B_n \in \mathscr{B}$ we get

$$A_n := B \setminus (B_1 \uplus \ldots \uplus B_n) = B \cap \underbrace{(B_1 \uplus \ldots \uplus B_n)^c}_{\in\mathscr{B}} \in \mathscr{B}.$$

Since $A_n \downarrow \emptyset$, continuity at $\emptyset$ proves $m(A_n) \to 0$. Since m is additive,

$$\begin{aligned}
m(B) &= m(B \setminus (B_1 \uplus \ldots \uplus B_n)) + m(B_1 \uplus \ldots \uplus B_n) \\
&= m(A_n) + \sum_{j=1}^{n} m(B_j) \\
&\xrightarrow{n\to\infty} 0 + \sum_{j=1}^{\infty} m(B_j).
\end{aligned}$$

■■

Problem 6.6 Solution:

(i) A little geometry first: a solid, open disk of radius r, centre 0 is the set $B_r(0) := \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < r^2\}$. The n-dimensional analogue is clearly $\{x \in \mathbb{R}^n : x_1^2+x_2^2+\ldots+x_n^2 < r^2\}$ (including $n = 1$ where it reduces to an interval). We want to inscribe a box into a ball.

Claim: $Q_\epsilon(0) := \bigtimes_{j=1}^n \left[-\frac{\epsilon}{\sqrt{n}}, \frac{\epsilon}{\sqrt{n}}\right) \subset B_{2\epsilon}(0)$. Indeed,

$$\begin{aligned} x \in Q_\epsilon(0) \;&\Rightarrow\; x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant \frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} + \ldots + \frac{\epsilon^2}{n} < (2\epsilon)^2 \\ &\Rightarrow\; x \in B_{2\epsilon}(0), \end{aligned}$$

and the claim follows.

Observe that $\lambda^n(Q_\epsilon(0)) = \prod_{j=1}^n \frac{2\epsilon}{\sqrt{n}} > 0$. Now take some open set U. By translating it we can achieve that $0 \in U$ and, as we know, this movement does not affect $\lambda^n(U)$. As $0 \in U$ we find some $\epsilon > 0$ such that $B_\epsilon(0) \subset U$, hence

$$\lambda^n(U) \geqslant \lambda^n(B_\epsilon(0)) \geqslant \lambda(Q_\epsilon(0)) > 0.$$

(ii) For closed sets this is, *in general*, wrong. Trivial counterexample: the singleton $\{0\}$ is closed, it is Borel (take a countable sequence of nested rectangles, centered at 0 and going down to $\{0\}$) and the Lebesgue measure is zero.

To get strictly positive Lebesgue measure, one possibility is to have interior points, i.e. closed sets which have non-empty interior do have positive Lebesgue measure.

Problem 6.7 Solution:

(i) Without loss of generality we can assume that $a < b$. We have $[a + \frac{1}{k}, b) \uparrow (a, b)$ as $k \to \infty$. Thus, by the continuity of measures, Proposition 4.3, we find (write $\lambda = \lambda^1$, for short)

$$\lambda(a, b) = \lim_{k\to\infty} \lambda \left[a + \frac{1}{k}, b\right) = \lim_{k\to\infty} \left(b - a - \frac{1}{k}\right) = b - a.$$

Since $\lambda[a, b) = b - a$, too, this proves again that

$$\lambda(\{a\}) = \lambda([a, b) \setminus (a, b)) = \lambda[a, b) - \lambda(a, b) = 0.$$

(ii) The hint says it all: H is contained in the union $y + \bigcup_{k\in\mathbb{N}} A_k$ for some y and we have $\lambda^2(A_k) = (2\epsilon\, 2^{-k})\cdot(2k) = 4\cdot\epsilon\cdot k2^{-k}$. Using the σ-subadditivity and monotonicity of measures (the A_k's are clearly not disjoint) as well as the translational invariance of the Lebesgue measure we get

$$0 \leqslant \lambda^2(H) \leqslant \lambda^2\left(\bigcup_{k=1}^\infty A_k\right) \leqslant \sum_{k=1}^\infty \lambda(A_k) = \sum_{k=1}^\infty 4\cdot\epsilon\cdot k2^{-k} = C\epsilon$$

where C is the finite (!) constant $4\sum_{k=1}^\infty k2^{-k}$ (check convergence!). As ϵ was arbitrary, we can let it $\to 0$ and the claim follows.

(iii) *n-dimensional version of (i):* We have $I = \bigtimes_{j=1}^n (a_j, b_j)$. Set $I_k := \bigtimes_{j=1}^n [a_j + \frac{1}{k}, b_j)$. Then $I_k \uparrow I$ as $k \to \infty$ and we have (write $\lambda = \lambda^n$, for short)

$$\lambda(I) = \lim_{k\to\infty} \lambda(I_k) = \lim_{k\to\infty} \prod_{j=1}^n \left(b_j - a_j - \frac{1}{k}\right) = \prod_{j=1}^n \left(b_j - a_j\right).$$

n-dimensional version of (ii): The changes are obvious: $A_k = [-\epsilon 2^{-k}, \epsilon 2^{-k}) \times [-k, k)^{n-1}$ and $\lambda^n(A_k) = 2^n \cdot \epsilon \cdot 2^{-k} \cdot k^{n-1}$. The rest stays as before, since the sum $\sum_{k=1}^{\infty} k^{n-1} 2^{-k}$ still converges to a finite value.

Problem 6.8 Solution:

(i) All we have to show is that $\lambda^1(\{x\}) = 0$ for any $x \in \mathbb{R}$. But this has been shown already in problem 6.6(i).

(ii) Take the Dirac measure: δ_0. Then $\{0\}$ is an atom as $\delta_0(\{0\}) = 1$.

(iii) Let C be countable and let $\{c_1, c_2, c_3, \dots\}$ be an enumeration (could be finite, if C is finite). Since singletons are in $\mathscr{A}$, so is C as a countable union of the sets $\{c_j\}$. Using the σ-additivity of a measure we get

$$\mu(C) = \mu(\cup_{j\in\mathbb{N}}\{c_j\}) = \sum_{j\in\mathbb{N}} \mu(\{c_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

(iv) If $y_1, y_2, \dots, y_N$ are atoms of mass $P(\{y_j\}) \geqslant \frac{1}{k}$ we find by the additivity and monotonicity of measures

$$\begin{aligned}\frac{N}{k} &\leqslant \sum_{j=1}^{N} P(\{x_j\}) \\ &= P\left(\bigcup_{j=1}^{N} \{y_j\}\right) \\ &= P(\{y_1, \dots, y_N\}) \leqslant P(\mathbb{R}) = 1\end{aligned}$$

so $\frac{N}{k} \leqslant 1$, i.e. $N \leqslant k$, and the claim in the hint (about the maximal number of atoms of given size) is shown.

Now denote, as in the hint, the atoms with measure of size $[\frac{1}{k}, \frac{1}{k-1})$ by $y_1^{(k)}, \dots y_{N(k)}^{(k)}$ where $N(k) \leqslant k$ is their number. Since

$$\bigcup_{k\in\mathbb{N}} \left[\frac{1}{k}, \frac{1}{k-1}\right) = (0, \infty)$$

we exhaust all possible sizes for atoms.

There are at most countably many (actually: finitely many) atoms in each size range. Since the number of size ranges is countable and since countably many countable sets make up a countable set, we can relabel the atoms as $x_1, x_2, x_3, \dots$ (could be finite) and, as we have seen in exercise 4.7(ii), the set function

$$\nu := \sum_j P(\{x_j\}) \cdot \delta_{x_j}$$

(no matter whether the sum is over a finite or countably infinite set of j's) is indeed a measure on $\mathbb{R}$. But more is true: for any Borel set A

$$\nu(A) = \sum_j P(\{x_j\}) \cdot \delta_{x_j}(A)$$

$$
\begin{aligned}
&= \sum_{j:x_j\in A} P(\{x_j\}) \\
&= P(A\cap\{x_1,x_2,\dots\}) \leqslant P(A)
\end{aligned}
$$

showing that $\mu(A) := P(A) - \nu(A)$ is a positive number for each Borel set $A \in \mathscr{B}$. This means that $\mu : \mathscr{B} \to [0,\infty]$. Let us check M_1 and M_2. Using M_1, M_2 for P and ν (for them they are clear, as P, ν are measures!) we get

$$\mu(\emptyset) = P(\emptyset) - \nu(\emptyset) = 0 - 0 = 0$$

and for a disjoint sequence $(A_j)_{j\in\mathbb{N}} \subset \mathscr{B}$ we have

$$
\begin{aligned}
\mu\Big(\bigcup_j A_j\Big) &= P\Big(\bigcup_j A_j\Big) - \nu\Big(\bigcup_j A_j\Big) \\
&= \sum_j P(A_j) - \sum_j \nu(A_j) \\
&= \sum_j \big(P(A_j) - \nu(A_j)\big) \\
&= \sum_j \mu(A_j)
\end{aligned}
$$

which is M_2 for μ.

Problem 6.9 Solution:

(i) Fix a sequence of numbers $\epsilon_k > 0, k \in \mathbb{N}_0$ such that $\sum_{k\in\mathbb{N}_0} \epsilon_k < \infty$. For example we could take a geometric series with general term $\epsilon_k := 2^{-k}$. Now define *open* intervals $I_k := (k - \epsilon_k, k + \epsilon_k)$, $k \in \mathbb{N}_0$ (these are open sets!) and call their union $I := \bigcup_{k\in\mathbb{N}_0} I_k$. As countable union of open sets I is again open. Using the σ-(sub-)additivity of $\lambda = \lambda^1$ we find

$$\lambda(I) = \lambda\Big(\bigcup_{k\in\mathbb{N}_0} I_k\Big) \overset{(*)}{\leqslant} \sum_{k\in\mathbb{N}_0} \lambda(I_k) = \sum_{k\in\mathbb{N}_0} 2\epsilon_k = 2\sum_{k\in\mathbb{N}_0} \epsilon_k < \infty.$$

By 6.7(i), $\lambda(I) > 0$.

Note that in step $(*)$ equality holds (i.e. we would use σ-additivity rather than σ-subadditivity) if the I_k are pairwise disjoint. This happens, if all $\epsilon_k < \frac{1}{2}$ (think!), but to be on the safe side and in order not to have to worry about such details we use sub-additivity.

(ii) Take the open interior of the sets A_k, $k \in \mathbb{N}$, from the hint to 6.7(ii). That is, take the open rectangles $B_k := (-2^{-k}, 2^{-k}) \times (-k, k)$, $k \in \mathbb{N}$, (we choose $\epsilon = 1$ since we are after *finiteness* and not necessarily *smallness*). That these are open sets will be seen below. Now set $B = \bigcup_{k\in\mathbb{N}} B_k$ and observe that the union of open sets is always open. B is also unbounded and it is geometrically clear that B is pathwise connected as it is some kind of lozenge-shaped 'staircase' (draw a picture!) around the y-axis. Finally, by σ-subadditivity and using 6.7(ii) we get

$$\lambda^2(B) = \lambda^2\Big(\bigcup_{k\in\mathbb{N}} B_k\Big) \leqslant \sum_{k\in\mathbb{N}} \lambda^2(B_k)$$

$$= \sum_{k\in\mathbb{N}} 2\cdot 2^{-k}\cdot 2\cdot k$$
$$= 4\sum_{k\in\mathbb{N}} k\cdot 2^{-k} < \infty.$$

It remains to check that an open rectangle is an open set. For this take any open rectangle $R = (a,b)\times(c,d)$ and pick $(x,y)\in R$. Then we know that $a < x < b$ and $c < y < d$ and since we have strict inequalities, we have that the smallest distance of this point to any of the four boundaries (draw a picture!) $h := \min\{|a-x|, |b-x|, |c-y|, |d-y|\} > 0$. This means that a square around (x,y) with side-length $2h$ is inside R and what we're going to do is to inscribe into this virtual square an open disk with radius h and centre (x,y). Since the circle is again in R, we are done. The equation for this disk is

$$(x',y')\in B_h(x,y) \iff (x-x')^2+(y-y')^2 < h^2$$

Thus,

$$|x'-x| \leqslant \sqrt{|x-x'|^2+|y-y'|^2} < h$$
$$\text{and } |y'-y| \leqslant \sqrt{|x-x'|^2+|y-y'|^2} < h$$

i.e. $x-h < x' < x+h$ and $y-h < y' < y+h$ or $(x',y')\in(x-h,x+h)\times(y-h,y+h)$, which means that (x',y') is in the rectangle of sidelength $2h$ centered at (x,y). since (x',y') was an arbitrary point of $B_h(x,y)$, we are done.

(iii) No, this is impossible. Since we are in one dimension, pathwise connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means that we must have a line of the sort (a,∞) or $(-\infty,b)$ in our set and in both cases Lebesgue measure is infinite. In all dimensions $n > 1$, see part (ii) for two dimensions, we can, however, construct pathwise connected, unbounded open sets with finite Lebesgue measure.

■■

Problem 6.10 Solution: Fix $\epsilon > 0$ and let $\{q_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,1]$. Then

$$U := U_\epsilon := \bigcup_{j\in\mathbb{N}} \left(q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1}\right)\cap[0,1]$$

is a dense open set in $[0,1]$ and, because of σ-subadditivity,

$$\lambda(U) \leqslant \sum_{j\in\mathbb{N}} \lambda\left(q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1}\right) = \sum_{j\in\mathbb{N}} \frac{\epsilon}{2^j} = \epsilon.$$

■■

Problem 6.11 Solution: Assume first that for every $\epsilon > 0$ there is some open set $U_\epsilon \supset N$ such that $\lambda(U_\epsilon) \leqslant \epsilon$. Then

$$\lambda(N) \leqslant \lambda(U_\epsilon) \leqslant \epsilon \quad \forall\epsilon > 0,$$

which means that $\lambda(N) = 0$.

Conversely, let $\lambda^*(N) = \inf\left\{ \sum_j \lambda(U_j) \;:\; U_j \in \mathscr{O},\; \cup_{j\in\mathbb{N}} U_j \supset N \right\}$. Since for the Borel set N we have $\lambda^*(N) = \lambda(N) = 0$, the definition of the infimum guarantees that for every $\epsilon > 0$ there is a sequence of open sets $(U_j^\epsilon)_{j\in\mathbb{N}}$ covering N, i.e. such that $U^\epsilon := \bigcup_j U_j^\epsilon \supset N$. Since U^ϵ is again open we find because of σ-subadditivity

$$\lambda(N) \leqslant \lambda(U^\epsilon) = \lambda\Big(\bigcup_j U_j^\epsilon\Big) \leqslant \sum_j \lambda(U_j^\epsilon) \leqslant \epsilon.$$

Attention: A construction along the lines of Problem 3.15, hint to part (ii), using open sets $U^\delta := N + B_\delta(0)$ is, in general not successful:

- it is not clear that U^δ has finite Lebesgue measure (o.k. one can overcome this by considering $N \cap [-k, k]$ and then letting $k \to \infty$...)
- $U^\delta \downarrow \bar{N}$ and *not* N (unless N is closed, of course). If, say, N is a dense set of $[0, 1]$, this approach leads nowhere.

Problem 6.12 Solution: Observe that the sets $C_k := \bigcup_{j=k}^\infty A_j$, $k \in \mathbb{N}$, decrease as $k \to \infty$—we admit less and less sets in the union, i.e. the union becomes smaller. Since P is a probability measure, $P(C_k) \leqslant 1$ and therefore Lemma 4.9 applies and shows that

$$P\Big(\bigcap_{k=1}^\infty \bigcup_{j=k}^\infty A_j\Big) = P\Big(\bigcap_{k=1}^\infty C_k\Big) = \lim_{k\to\infty} P(C_k).$$

On the other hand, we can use σ-subadditivity of the measure P to get

$$P(C_k) = P\Big(\bigcup_{j=k}^\infty A_j\Big) \leqslant \textstyle\sum_{j=k}^\infty P(A_j)$$

but this is the tail of the convergent (!) sum $\sum_{j=1}^\infty P(A_j)$ and, as such, it goes to zero as $k \to \infty$. Putting these bits together, we see

$$P\Big(\bigcap_{k=1}^\infty \bigcup_{j=k}^\infty A_j\Big) = \lim_{k\to\infty} P(C_k) \leqslant \lim_{k\to\infty} \sum_{j=k}^\infty P(A_j) = 0,$$

and the claim follows.

Problem 6.13 Solution:

(i) We can work out the 'optimal' $\mathscr{A}$-cover of (a, b):

Case 1: $a, b \in [0, 1)$. Then $[0, 1)$ is the best possible cover of (a, b), thus $\mu^*(a, b) = \mu[0, 1) = \frac{1}{2}$.

Case 2: $a, b \in [1, 2)$. Then $[1, 2)$ is the best possible cover of (a, b), thus $\mu^*(a, b) = \mu[1, 2) = \frac{1}{2}$.

Case 3: $a \in [0,1), b \in [1,2)$. Then $[0,1) \cup [1,2)$ is the best possible cover of (a,b), thus $\mu^*(a,b) = \mu[0,1) + \mu[1,2) = 1$.

And in the case of a singleton $\{a\}$ the best possible cover is always either $[0,1)$ or $[1,2)$ so that $\mu^*(\{a\}) = \frac{1}{2}$ for all a.

(ii) Assume that $(0,1) \in \mathscr{A}^*$. Since $\mathscr{A} \subset \mathscr{A}^*$, we have $[0,1) \in \mathscr{A}^*$, hence $\{0\} = [0,1) \setminus (0,1) \in \mathscr{A}^*$. Since $\mu^*(0,1) = \mu^*(\{0\}) = \frac{1}{2}$, and since μ^* is a measure on $\mathscr{A}^*$ (cf. Step 4 in the proof of Theorem 6.1), we get

$$\frac{1}{2} = \mu[0,1) = \mu^*[0,1) = \mu^*(0,1) + \mu^*\{0\} = \frac{1}{2} + \frac{1}{2} = 1$$

leading to a contradiction. Thus neither $(0,1)$ nor $\{0\}$ are elements of $\mathscr{A}^*$.

Problem 6.14 Solution: Since $\mathscr{A} \subset \mathscr{A}^*$, the only interesting sets (to which one could extend μ) are those $B \subset \mathbb{R}$ where both B and B^c are uncountable. By definition,

$$\gamma^*(B) = \inf\left\{ \sum_j \gamma(A_j) \,:\, A_j \in \mathscr{A},\ \bigcup_j A_j \supset B \right\}.$$

The infimum is obviously attained for $A_j = \mathbb{R}$, so that $\gamma^*(B) = \gamma^*(B^c) = 1$. On the other hand, since γ^* is necessarily additive on $\mathscr{A}^*$, the assumption that $B \in \mathscr{A}^*$ leads to a contradiction:

$$1 = \gamma(\mathbb{R}) = \gamma^*(\mathbb{R}) = \gamma^*(B) + \gamma^*(B^c) = 2.$$

Thus, $\mathscr{A} = \mathscr{A}^*$.

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7 Measurable mappings. Solutions to Problems 7.1–7.13

Problem 7.1 Solution: We have $\tau_x^{-1}(z) = z + x$. According to Lemma 7.2 we have to check that

$$\tau_x^{-1}([a, b)) \in \mathscr{B}(\mathbb{R}^n) \qquad \forall\, [a, b) \in \mathscr{J}$$

since the rectangles $\mathscr{J}$ generate $\mathscr{B}(\mathbb{R}^n)$. Clearly,

$$\tau_x^{-1}([a, b)) = [a, b) + x = [a + x, b + x) \in \mathscr{J} \subset \mathscr{B}(\mathbb{R}^n),$$

and the claim follows.

■■

Problem 7.2 Solution: We had $\Sigma' = \{A' \subset X' : T^{-1}(A') \in \mathscr{A}\}$ where $\mathscr{A}$ was a σ-algebra of subsets of X. Let us check the properties (Σ_1)–(Σ_3).

(Σ_1) Take $\emptyset \subset X'$. Then $T^{-1}(\emptyset) = \emptyset \in \mathscr{A}$, hence $\emptyset \in \Sigma'$.

(Σ_2) Take any $B \in \Sigma'$. Then $T^{-1}(B) \in \mathscr{A}$ and therefore $T^{-1}(B^c) = \left(T^{-1}(B)\right)^c \in \mathscr{A}$ since all set operations interchange with inverse maps and since $\mathscr{A}$ is a σ-algebra. This shows that $B^c \in \Sigma'$.

(Σ_3) Take any sequence $(B_j)_{j\in\mathbb{N}} \subset \Sigma'$. Then, using again the fact that $\mathscr{A}$ is a σ-algebra, $T^{-1}(\cup_j B_j) = \bigcup_j T^{-1}(B_j) \in \mathscr{A}$ which proves that $\bigcup_j B_j \in \Sigma'$.

■■

Problem 7.3 Solution:

(i) (Σ_1) $\emptyset \in \mathscr{A}$ is clear.

(Σ_2) Let $A \in \mathscr{A}$. If $2n \in A^c$, then $2n + 1 \in A^c$ – this follows straight from the definition of $\mathscr{A}$: if $2n+1 \in A$, then $2n \in A$. In the same way we get $2n+1 \in A^c \implies 2n \in A^c$. Consequently, $A^c \in \mathscr{A}$.

(Σ_3) Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{A}$. If $2n \in \bigcup_j A_j$, then there is some index j_0 such that $2n \in A_{j_0}$. Since $A_{j_0} \in \mathscr{A}$, we get $2n + 1 \in A_{j_0} \subseteq \bigcup_j A_j$. In the same way we find that $2n + 1 \in \bigcup_j A_j \implies 2n \in \bigcup A_j$.

(ii) It is clear that the map T is bijective as $T^{-1}(n) = n - 2$. Pick any set $A \in \mathscr{A}$. In order to verify the measurability of T, we have to show that $T^{-1}(A) \in \mathscr{A}$, i.e.

$$2n \in T^{-1}(A) \Leftrightarrow 2n + 1 \in T^{-1}(A) \quad \text{for all } n > 0.$$

If $2n \in T^{-1}(A)$, $n > 0$, then we see that $2n+2 = 2(n+1) \in A$. As $A \in \mathscr{A}$ this yields $2n+3 \in A$ and so $2n+1 = T^{-1}(2n+3) \in T^{-1}(A)$. Therefore, T is measurable.

On the other hand, T^{-1} is not measurable: the set $A = \{k; k \leqslant 0\}$ is contained in $\mathscr{A}$, but $T(A) = \{k : k \leqslant 2\} \notin \mathscr{A}$ (use $2 = 2 \cdot 1 \in A$, but $2 \cdot 1 + 1 = 3 \notin A$).

Problem 7.4 Solution:

(i) First of all we remark that $T_i^{-1}(\mathscr{A}_i)$ is itself a σ-algebra, cf. Example 3.3(vii).

If $\mathscr{C}$ is a σ-algebra of subsets of X such that $T_i : (X, \mathscr{C}) \to (X_i, \mathscr{A}_i)$ becomes measurable, we know from the very definition that $T^{-1}(\mathscr{A}_i) \subset \mathscr{C}$. From this, however, it is clear that $T^{-1}(\mathscr{A}_i)$ is the minimal σ-algebra that renders T measurable.

(ii) From part (i) we know that $\sigma(T_i, i \in I)$ necessarily contains $T_i^{-1}(\mathscr{A}_i)$ for every $i \in I$. Since $\bigcup_i T_i^{-1}(\mathscr{A}_i)$ is, in general, not a σ-algebra, we have $\sigma\Big(\bigcup_i T_i^{-1}(\mathscr{A}_i)\Big) \subset \sigma(T_i, i \in I)$. On the other hand, each T_i is, because of $T_i^{-1}(\mathscr{A}_i) \subset \bigcup_i T_i^{-1}(\mathscr{A}_i) \subset \sigma(T_i, i \in I)$ measurable w.r.t. $\sigma\Big(\bigcup_i T_i^{-1}(\mathscr{A}_i)\Big)$ and this proves the claim.

Problem 7.5 Solution:

(i), (ii)

$$\begin{aligned}\mathbb{1}_{T^{-1}(A')}(x) = 1 \ &\Leftrightarrow\ x \in T^{-1}(A') \ \Leftrightarrow\ T(x) \in A' \\ &\Leftrightarrow\ \mathbb{1}_{A'}(T(x)) = 1 \ \Leftrightarrow (\mathbb{1}_{A'} \circ T)(x) = 1\end{aligned}$$

Since an indicatior function can only assume the values 0 and 1, the claimed equality follows for the value 0 by negating the previously shown equivalence.

(iii) "$\Rightarrow$": Assume that T is measurable. We have $T^{-1}(A') \in \mathscr{A}\ \forall A' \in \mathscr{A}'$ and since $\mathscr{A}$ is a σ-algebra, we conclude

$$\sigma(T) = \sigma(\{T^{-1}(A') \mid A' \in \mathscr{A}'\}) \subset \sigma(\mathscr{A}) = \mathscr{A}.$$

"$\Leftarrow$": $\sigma(T) \subset \mathscr{A}$ implies, in particular,

$$T^{-1}(A') \in \mathscr{A}\ \forall A' \in \mathscr{A}',$$

i.e., T is measurable.

(iii) Theorem 7.6 shows that image measures are measures. By the definition of T, we have $T^{-1}(E') = E$ and $\nu \circ T^{-1}(E') < \infty$, resp., $\nu \circ T^{-1}(E') = 1$ follows from the definition of image measures.

The image measure obtained from a σ-finite measure need not be σ-finite!

Counterexample: Let μ be the counting measure on $\mathbb{Z}^2$ and define $T((x, y)) = x$. While μ is σ-finite, the image measure $T(\mu)$ isn't.

Problem 7.6 Solution: We have

$$T^{-1}(\mathscr{G}) \subset \underbrace{T^{-1}(\sigma(\mathscr{G}))}_{\text{is itself a } \sigma\text{-algebra}} \Rightarrow \sigma(T^{-1}(\mathscr{G})) \subset T^{-1}(\sigma(\mathscr{G})).$$

For the converse consider $T : (X, \sigma(T^{-1}(\mathscr{G}))) \to (Y, \sigma(\mathscr{G}))$. By the very choice of the σ-algebras and since $T^{-1}(\mathscr{G}) \subset \sigma(T^{-1}(\mathscr{G}))$ we find that T is $\sigma(T^{-1}(\mathscr{G}))/\sigma(\mathscr{G})$ measurable—mind that we only have to check measurability at a generator (here: $\mathscr{G}$) in the image region. Thus,

$$T^{-1}(\sigma(\mathscr{G})) \subset \sigma(T^{-1}(\mathscr{G})).$$

Alternative: We have

$$T^{-1}(\mathscr{G}) \subset \underbrace{T^{-1}(\sigma(\mathscr{G}))}_{\text{is itself a } \sigma\text{-algebra}} \Rightarrow \sigma(T^{-1}(\mathscr{G})) \subset T^{-1}(\sigma(\mathscr{G})).$$

For the converse, set $\Sigma := \{G \in \sigma(\mathscr{G}) : T^{-1}(G) \in \sigma(T^{-1}(\mathscr{G}))\}$. It is not hard to see that Σ is itself a σ-algebra and that $\mathscr{G} \subset \Sigma \subset \sigma(\mathscr{G})$. Thus, $\sigma(\mathscr{G}) = \Sigma$ and so $T^{-1}(\sigma(\mathscr{G})) \subset \sigma(T^{-1}(\mathscr{G}))$.

Problem 7.7 Solution: We have to show that

$$\begin{aligned} & f : (F, \mathscr{F}) \to (X, \sigma(T_i,\ i \in I)) \text{ measurable} \\ \iff \quad & \forall\, i \in I : T_i \circ f : (F, \mathscr{F}) \to (X_i, \mathscr{A}_i) \text{ measurable.} \end{aligned}$$

Now

$$\begin{aligned} \forall\, i \in I : (T_i \circ f)^{-1}(\mathscr{A}_i) \subset \mathscr{F} &\iff \forall\, i \in I : f^{-1}\big(T_i^{-1}(\mathscr{A}_i)\big) \subset \mathscr{F} \\ &\iff f^{-1}\Big(\bigcup\nolimits_{i \in I} T_i^{-1}(\mathscr{A}_i)\Big) \subset \mathscr{F} \\ &\overset{(*)}{\iff} \sigma\Big[f^{-1}\Big(\bigcup\nolimits_{i \in I} T_i^{-1}(\mathscr{A}_i)\Big)\Big] \subset \mathscr{F} \\ &\overset{(**)}{\iff} f^{-1}\Big(\sigma\Big[\bigcup\nolimits_{i \in I} T_i^{-1}(\mathscr{A}_i)\Big]\Big) \subset \mathscr{F}. \end{aligned}$$

Only (*) and (**) are not immediately clear. The direction '$\Leftarrow$' in (*) is trivial, while '$\Rightarrow$' follows if we observe that the right-hand side, $\mathscr{F}$, is a σ-algebra. The equivalence (**) is another case of Problem 7.6 (see there for the solution!).

Problem 7.8 Solution: Using the notation of the foregoing Problem 7.7 we put

$$I = \{1, 2, \ldots, m\} \quad \text{and} \quad T_j := \pi_j : \mathbb{R}^m \to \mathbb{R},\ \pi_j(x_1, \ldots, x_m) := x_j$$

i.e. π_j is the coordinate projection, $\mathscr{A}_j := \mathscr{B}(\mathbb{R})$.

Since each π_j is continuous, we have $\sigma(\pi_1, \dots, \pi_m) \subset \mathscr{B}(\mathbb{R}^m)$ so that Problem 7.7 applies and proves

$$f \text{ is } \mathscr{B}(\mathbb{R}^m)\text{-measurable} \iff$$
$$f_j = \pi_j \circ f \text{ is } \mathscr{B}(\mathbb{R})\text{-measurable for all } j = 1, 2, \dots, m.$$

Remark. We will see, in fact, in Chapter 14 (in particular in Theorem 14.17) that we have the equality $\sigma(\pi_1, \dots, \pi_m) = \mathscr{B}(\mathbb{R}^m)$.

Problem 7.9 Solution: In general the direct image $T(\mathscr{A})$ of a σ-algebra is not any longer a σ-algebra. (Σ_1) and (Σ_3) hold, but (Σ_2) will, in general, fail. Here is an example: Take $X = X' = \mathbb{N}$, take any σ-algebra $\mathscr{A}$ other than $\{\emptyset, \mathbb{N}\}$ in $\mathbb{N}$, and let $T : \mathbb{N} \to \mathbb{N}$, $T(j) = 1$ be the constant map. Then $T(\emptyset) = \emptyset$ but $T(A) = \{1\}$ whenever $A \neq \emptyset$. Thus, $\{1\} = T(A^c) \neq [T(A)]^c = \mathbb{N} \setminus \{1\}$ but equality would be needed if $T(\mathscr{A})$ were a σ-algebra. This means that Σ_2 fails.

Necessary and sufficient for $T(\mathscr{A})$ to be a σ-algebra is, clearly, that T^{-1} is a measurable map $T^{-1} : X' \to X$.

Warning. Direct images of measurable sets behave badly – even if the mapping is good. For example, the continuous (direct) image of a Borel set need not be Borel! (It is, however, analytic or Souslin).

Problem 7.10 Solution: Consider for $t > 0$ the dilation $m_t : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto t \cdot x$. Since m_t is continuous, it is Borel measurable. Moreover, $m_t^{-1} = m_{1/t}$ and so

$$t \cdot B = m_{1/t}^{-1}(B)$$

which shows that $\lambda^n(t \cdot B) = \lambda^n \circ m_{1/t}^{-1}(B) = m_{1/t}(\lambda^n)(B)$ is actually an image measure of λ^n. Now show the formula first for rectangles $B = \bigtimes_{j=1}^{n} [a_j, b_j)$ (as in Problem 5.9) and deduce the statement from the uniqueness theorem for measures.

Problem 7.11 Solution:

(i) The hint is indeed already the proof. Almost, that is... Let μ be some measure as specified in the problem. From Problam 6.1(iii) we know that the Stieltjes function $F := F_\mu$ then satisfies

$$\begin{aligned}\mu[a, b) = F(b) - F(a) &= \lambda^1[F(a), F(b)) \\ &\overset{(\#)}{=} \lambda^1(F([a, b))) \\ &\overset{(\#\#)}{=} \lambda^1 \circ F([a, b)).\end{aligned}$$

The crunching points in this argument are the steps (#) and (##).

(#) This is o.k. since F was continuous, and the intermediate value theorem for continuous functions tells us that intervals are mapped to intervals. So, no problem here, just a little thinking needed.

(##) This is more subtle. We have defined image measures *only* for inverse maps, i.e. for expressions of the type $\lambda^1 \circ G^{-1}$ where G was measurable. So our job is to see that F can be obtained in the form $F = G^{-1}$ where G is measurable. In other words, we have to invert F. The problem is that we need to understand that, if $F(x)$ is flat on some interval (a, b) inversion becomes a problem (since then F^{-1} has a jump—horizontals become verticals in inversions, as inverting is somehow the mirror-image w.r.t. the 45-degree line in the coordinate system.).

So, if there are no flat bits, then this means that F is strictly increasing, and it is clear that G exists and is even continuous there.

If we have a flat bit, let's say exactly if $x \in [a, b]$ and call $F(x) = F(a) = F(b) = C$ for those x; clearly, F^{-1} jumps at C and we must see to it that we take a version of F^{-1}, say one which makes F^{-1} left-continuous at C—note that we could assign any value from $[a, b]$ to $F^{-1}(C)$—which is accomplished by setting $F^{-1}(C) = a$. (Draw a graph to illustrate this!)

There is A canonical expression for such a 'generalized' left-continuous inverse of an increasing function (which may have jumps and flat bits—jumps of F become just flat bits in the graph of F^{-1}, think!) and this is:

$$G(y) = \inf\{x : F(x) \geqslant y\}$$

Let us check measurability:

$$\begin{aligned} y_0 \in \{G \geqslant \lambda\} &\iff G(y_0) \geqslant \lambda \\ &\overset{\text{def}}{\iff} \inf\{F \geqslant y_0\} \geqslant \lambda \\ &\overset{(\ddagger)}{\Rightarrow} F(\lambda) \leqslant y_0 \\ &\iff y_0 \in [F(\lambda), \infty). \end{aligned}$$

Since F is monotonically increasing, we find also '$\Leftarrow$' in step ($\ddagger$), hence

$$\{G \geqslant \lambda\} = [F(\lambda), \infty) \in \mathscr{B}(\mathbb{R})$$

which shows that G is measurable. Even more: it shows that $G^{-1}(x) := \inf\{G \geqslant \lambda\} = F(x)$. Thus, $\lambda^1 \circ F = \lambda^1 \circ G^{-1} = \mu$ is indeed an image measure of λ^1.

(ii) We have $F(x) = F_{\delta_0}(x) = \mathbb{1}_{(0,\infty)}(x)$ and its left-continuous inverse $G(y)$ in the sense of part

(i) is given by

$$G(y) = \begin{cases} +\infty, & y > 1 \\ 0, & 0 < y \leqslant 1 \\ -\infty, & y \leqslant 0 \end{cases} \cdot$$

This function is clearly measurable (use $\bar{\mathscr{B}}$ to accommodate $\pm\infty$) and so the claim holds in this case. Observe that in this case F is not any longer continuous but only left-continuous.

Problem 7.12 Solution:

(i) See Figure 1.4 on page 4.

(ii) Each C_n is a finite union of 2^n closed and bounded intervals. As such, C_n is itself a closed and bounded set, hence compact. The intersection of closed and bounded sets is again closed and bounded, so compact. This shows that C is compact. That C is non-empty follows from the intersection principle: if one has a nested sequence of non-empty compact sets, their intersection is not empty. (This is sometimes formulated in a somewhat stronger form and called: *finite intersection property*. The general version is then: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets such that *each finite* sub-family has non-void intersection, then $\bigcap_n K_n \neq \emptyset$). This is an obvious generalization of the interval principle: nested non-void closed and bounded intervals have a non-void intersection.

(iii) At step n we remove open middle-third intervals of length 3^{-n}. To be precise, we partition C_{n-1} in pieces of length 3^{-n} and remove every other interval. The same effect is obtained if we partition $[0,\infty)$ in pieces of length 3^{-n} and remove every other piece. Call the taken out pieces F_n and set $C_n = C_{n-1} \setminus F_n$, i.e. we remove from C_{n-1} even pieces which were already removed in previous steps. It is clear that F_n exactly consists of sets of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$, $k \in \mathbb{N}_0$ which comprises exactly 'every other' set of length 3^{-n}. Since we do this for every n, the set C is disjoint to the union of these intervals over $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(iv) Since C_n consists of 2^n intervals $J_1 \cup \ldots \cup J_{2^n}$, each of which has length 3^{-n} (prove this by a trivial induction argument!), we get

$$\lambda(C_n) = \lambda(J_1) + \ldots + \lambda(J_{2^n}) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n$$

where we also use (somewhat pedantically) that

$$\lambda[a,b] = \lambda([a,b) \cup \{b\}) = \lambda[a,b) + \lambda\{b\} = b - a + 0 = b - a.$$

Now using Proposition 4.3 we conclude that $\lambda(C) = \inf_n \lambda(C_n) = 0$.

(v) Fix $\epsilon > 0$ and choose n so big that $3^{-n} < \epsilon$. Then C_n consists of 2^n *disjoint* intervals of length $3^{-n} < \epsilon$ and cannot possibly contain a ball of radius ϵ. Since $C \subset C_n$, the same applies to C. Since ϵ was arbitrary, we are done. (Remark: an open ball in $\mathbb{R}$ with centre x is obviously an open interval with midpoint x, i.e. $(x-\epsilon, x+\epsilon)$.)

(vi) Fix n and let $k = 0, 1, 2, \dots, 3^{n-1} - 1$. We saw in (c) that at step n we remove the intervals F_n, i.e. the intervals of the form

$$\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) = \Big(0.\underbrace{*** \dots *}_{n} 1\,000\dots,\ 0.\underbrace{*** \dots *}_{n} 2\,000\dots\Big)$$

where we use the ternary representation of x. These are exactly the numbers in $[0, 1]$ whose ternary expansion has a 1 at the nth digit. As $0.\ *** \dots *\ 1 = 0.\ *** \dots *\ 022222\dots$ has two representations, the left endpoint stays in. Since we do this for every step $n \in \mathbb{N}$, the claim follows.

(vii) Take $t \in C$ with ternary representation $t = 0.t_1t_2t_3 \dots t_j \dots$, $t_j \in \{0, 2\}$ and map it to the binary number $b = 0.\frac{t_1}{2}\frac{t_2}{2}\frac{t_3}{2} \dots \frac{t_j}{2}$ with digits $b_j = \frac{t_j}{2} \in \{0, 1\}$. This gives a bijection between C and $[0, 1]$, i.e. both have 'as infinitely many' points, i.e. $\#C = \#[0, 1]$. Despite of that

$$\lambda(C) = 0 \neq 1 = \lambda([0, 1])$$

which is, by the way, another proof for the fact that σ-additivity for the Lebesgue measure does not extend to general uncountable unions.

■■

Problem 7.13 Solution:

(i) Since $\emptyset \in \mathscr{E}$ and $\emptyset \in \mathscr{F}$ we get

$$\forall E \in \mathscr{E} : E \cup \emptyset \in \mathscr{E} \Cup \mathscr{F} \implies \mathscr{E} \subset \mathscr{E} \Cup \mathscr{F}$$

and

$$\forall F \in \mathscr{F} : \emptyset \cup F \in \mathscr{E} \Cup \mathscr{F} \implies \mathscr{F} \subset \mathscr{E} \Cup \mathscr{F}$$

so that $\mathscr{E} \cup \mathscr{F} \subset \mathscr{E} \Cup \mathscr{F}$. A similar argument, using that $X \in \mathscr{E}$ and $X \in \mathscr{F}$, shows $\mathscr{E} \cup \mathscr{F} \subset \mathscr{E} \Cap \mathscr{F}$.

(ii) Let $A, B \subset X$ such that $A \cap B \neq \emptyset$, $A \cup B \neq X$ and that $A \not\subset B$, $B \not\subset A$. Then we find for $\mathscr{E} := \{\emptyset, A, A^c, X\}$ and $\mathscr{F} := \{\emptyset, B, B^c, X\}$ that

$$\mathscr{E} \cup \mathscr{F} = \{\emptyset, A, B, A^c, B^c, X\}$$

while

$$\mathscr{E} \Cup \mathscr{F} = \{\emptyset, A, B, A^c, B^c, A \cup B, A^c \cup B^c, A \cup B^c, A^c \cup B, X\}.$$

A similar example works for $\mathscr{E} \Cap \mathscr{F}$.

(iii) Part (i) shows immediately that

$$\sigma(\mathscr{E} \Cup \mathscr{F}) \supset \sigma(\mathscr{E} \cup \mathscr{F}) \quad \text{and} \quad \sigma(\mathscr{E} \Cap \mathscr{F}) \supset \sigma(\mathscr{E} \cup \mathscr{F}).$$

Conversely, it is obvious that

$$\mathscr{E} \Cup \mathscr{F} \subset \sigma(\mathscr{E} \cup \mathscr{F}) \quad \text{and} \quad \mathscr{E} \Cap \mathscr{F} \subset \sigma(\mathscr{E} \cup \mathscr{F})$$

so that

$$\sigma(\mathscr{E} \Cup \mathscr{F}) \subset \sigma(\mathscr{E} \cup \mathscr{F}) \quad \text{and} \quad \sigma(\mathscr{E} \Cap \mathscr{F}) \subset \sigma(\mathscr{E} \cup \mathscr{F})$$

which proves

$$\sigma(\mathscr{E} \Cup \mathscr{F}) = \sigma(\mathscr{E} \cup \mathscr{F}) = \sigma(\mathscr{E} \Cap \mathscr{F}).$$

8 Measurable functions. Solutions to Problems 8.1–8.26

Problem 8.1 Solution: We remark, first of all, that $\{u \geqslant \alpha\} = u^{-1}([x,\infty))$ and, similarly, for the other sets. Now assume that $\{u \geqslant \beta\} \in \mathscr{A}$ for all β. Then

$$\begin{aligned}
\{u > \alpha\} = u^{-1}((\alpha,\infty)) &= u^{-1}\left(\bigcup_{k\in\mathbb{N}} \left[\alpha + \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcup_{k\in\mathbb{N}} u^{-1}\left(\left[\alpha + \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcup_{k\in\mathbb{N}} \underbrace{\{u \geqslant \alpha + \tfrac{1}{k}\}}_{\text{by assumption } \in \mathscr{A}} \in \mathscr{A}
\end{aligned}$$

since $\mathscr{A}$ is a σ-algebra.

Conversely, assume that $\{u > \beta\} \in \mathscr{A}$ for all β. Then

$$\begin{aligned}
\{u \geqslant \alpha\} = u^{-1}([\alpha,\infty)) &= u^{-1}\left(\bigcap_{k\in\mathbb{N}} \left(\alpha - \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcap_{k\in\mathbb{N}} u^{-1}\left(\left(\alpha - \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcap_{k\in\mathbb{N}} \underbrace{\{u > \alpha - \tfrac{1}{k}\}}_{\text{by assumption } \in \mathscr{A}} \in \mathscr{A}.
\end{aligned}$$

since $\mathscr{A}$ is a σ-algebra. Finally, as

$$\{u > \alpha\}^c = \{u \leqslant \alpha\} \qquad \text{and} \qquad \{u \geqslant \alpha\}^c = \{u < \alpha\}$$

we have that $\{u > \alpha\} \in \mathscr{A}$ if, and only if, $\{u \leqslant \alpha\} \in \mathscr{A}$ and the same holds for the sets $\{u \geqslant \alpha\}, \{u < \alpha\}$.

Problem 8.2 Solution: Recall that $B^* \in \overline{\mathscr{B}}$ if, and only if $B^* = B \cup C$ where $B \in \mathscr{B}$ and C is any of the following sets: $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty,\infty\}$. Using the fact that $\mathscr{B}$ is a σ-algebra and using this notation (that is: $\overline{\mathscr{B}}$-sets carry an asterisk *) we see

(Σ_1) Take $B = \emptyset \in \mathscr{B}$, $C = \emptyset$ to see that $\emptyset^* = \emptyset \cup \emptyset \in \overline{\mathscr{B}}$;

(Σ_2) Let $B^* \in \overline{\mathscr{B}}$. Then (complements are to be taken in $\overline{\mathscr{B}}$

$$(B^*)^c = (B \cup C)^c$$

$$
\begin{aligned}
&= B^c \cap C^c \\
&= (\overline{\mathbb{R}} \setminus B) \cap (\overline{\mathbb{R}} \setminus C) \\
&= (\mathbb{R} \setminus B \cup \{-\infty, +\infty\}) \cap (\overline{\mathbb{R}} \setminus C) \\
&= ((\mathbb{R} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C)) \\
&= (\mathbb{R} \setminus B) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C))
\end{aligned}
$$

which is again of the type $\mathscr{B}$-set union a set of the list $\emptyset$, $\{-\infty\}$, $\{\infty\}$, $\{-\infty, \infty\}$, hence it is in $\overline{\mathscr{B}}$.

(Σ_3) Let $B_n^* \in \overline{\mathscr{B}}$ and $B_n^* = B_n \cup C_n$. Then

$$B^* = \bigcup_{n\in\mathbb{N}} B_n^* = \bigcup_{n\in\mathbb{N}} (B_n \cup C_n) = \bigcup_{n\in\mathbb{N}} B_n \cup \bigcup_{n\in\mathbb{N}} C_n = B \cup C$$

with $B \in \mathscr{B}$ and C from the list $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$, hence $B^* \in \overline{\mathscr{B}}$.

A problem is the notation $\overline{\mathscr{B}} = \mathscr{B}(\overline{\mathbb{R}})$. While the left-hand side can easily be defined by (8.5), $\mathscr{B}(\overline{\mathbb{R}})$ has a well-defined meaning as the (topological) Borel σ-algebra over the set $\overline{\mathbb{R}}$, i.e. the σ-algebra in $\overline{\mathbb{R}}$ which is defined via the open sets in $\overline{\mathbb{R}}$. To describe the open sets $\mathscr{O}(\overline{\mathbb{R}})$ of $\overline{\mathbb{R}}$ we use require, that each point $x \in U^* \in \mathscr{O}(\overline{\mathbb{R}})$ admits an open neighbourhood $B(x)$ inside U^*. If $x \neq \pm\infty$, we take $B(x)$ as the usual open ϵ-interval around x with $\epsilon > 0$ sufficiently small. If $x = \pm\infty$ we take half-lines $[-\infty, a)$ or $(b, +\infty]$ respectively with $|a|, |b|$ sufficiently large. Thus, $\mathscr{O}(\overline{\mathbb{R}})$ adds to $\mathscr{O}(\mathbb{R})$ a few extra sets and open sets are therefore of the form $U^* = U \cup C$ with $U \in \mathscr{O}(\mathbb{R})$ and C being of the form $[-\infty, a)$ or $(b, +\infty]$ or $\emptyset$ or $\overline{\mathbb{R}}$ or unions thereof.

Thus, $\mathscr{O}(\mathbb{R}) = \mathbb{R} \cap \mathscr{O}(\overline{\mathbb{R}})$ and therefore

$$\mathscr{B}(\mathbb{R}) = \mathbb{R} \cap \mathscr{B}(\overline{\mathbb{R}})$$

(this time in the proper topological sense).

Problem 8.3 Solution:

(i) Notice that the indicator functions $\mathbb{1}_A$ and $\mathbb{1}_{A^c}$ are measurable. By Corollary 8.11 sums and products of measurable functions are again measurable. Since $h(x)$ can be written in the form $h(x) = \mathbb{1}_A(x) f(x) + \mathbb{1}_{A^c}(x) g(x)$, the claim follows.

(ii) The condition $f_j|_{A_j \cap A_k} = f_k|_{A_j \cap A_k}$ just guarantees that $f(x)$ is well-defined if we set $f(x) = f_j(x)$ for $x \in A_j$. Using $\bigcup_j A_j = X$ we find for $B \in \mathscr{B}(\mathbb{R})$

$$f^{-1}(B) = \bigcup_{j\in\mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j\in\mathbb{N}} \underbrace{A_j \cap f_j^{-1}(B)}_{\in \mathscr{A}} \in \mathscr{A}.$$

An **alternative solution** would be to make the A_j's disjoint, e.g. by setting $C_1 := A_1$, $C_k := A_k \setminus (A_1 \cup \dots \cup A_{k-1})$. Then

$$f = \sum_j \mathbb{1}_{C_j} f = \sum_j \mathbb{1}_{C_j} f_j$$

and the claim follows from Corollaries 8.11 and 8.10.

Problem 8.4 Solution: Since $\mathbb{1}_B$ is $\mathscr{B}$-measurable if, and only if, $B \in \mathscr{B}$ the claim follows by taking $B \in \mathscr{B}$ such that $B \notin \mathscr{A}$ (this is possible as $\mathscr{B} \subsetneq \mathscr{A}$.

Problem 8.5 Solution: By definition, $f \in \mathcal{E}$ if it is a step-function of the form $f = \sum_{j=0}^N a_j \mathbb{1}_{A_j}$ with some $a_j \in \mathbb{R}$ and $A_j \in \mathscr{A}$. Since

$$f^+ = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \geqslant 0}} a_j \mathbb{1}_{A_j} \quad \text{and} \quad f^- = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \leqslant 0}} a_j \mathbb{1}_{A_j},$$

$f^\pm$ are again of this form and therefore simple functions.

The converse is also true since $f_f^+ - f^-$—see (8.8) or Problem 8.6—and since sums and differences of simple functions are again simple.

Problem 8.6 Solution: By definition

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = -\min\{u(x), 0\}.$$

Now the claim follows from the elementary identities that for any two numbers $a, b \in \mathbb{R}$

$$a + 0 = \max\{a, 0\} + \min\{a, 0\} \quad \text{and} \quad |a| = \max\{a, 0\} - \min\{a, 0\}$$

which are easily verified by considering all possible cases $a \leqslant 0$ resp. $a \geqslant 0$.

Problem 8.7 Solution: If we show that $\{u > \alpha\}$ is an open set, it is also a Borel set, hence u is measurable.

Let us first understand what openness means: $\{u > \alpha\}$ is open means that for $x \in \{u > \alpha\}$ we find some (symmetric) neighbourhood (a 'ball') of the type $(x - h, x + h) \subset \{u > \alpha\}$. What does this mean? Obviously, that $u(y) > \alpha$ for any $y \in (x - h, x + h)$ and, in other words, $u(y) > \alpha$ whenever y is such that $|x - y| < h$. And this is the hint of how to use continuity: we use it in order to find the value of h.

u being continuous at x means that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall y : |x - y| < \delta : |u(x) - u(y)| < \epsilon.$$

Since $u(x) > \alpha$ we know that for a sufficiently small ϵ we still have $u(x) \geqslant \alpha + \epsilon$. Take this ϵ and find the corresponding δ. Then

$$u(x) - u(y) \leqslant |u(x) - u(y)| < \epsilon \qquad \forall\, |x - y| < \delta$$

and since $\alpha + \epsilon \leqslant u(x)$ we get

$$\alpha + \epsilon - u(y) < \epsilon \qquad \forall\, |x-y| < \delta$$

i.e. $u(y) > \alpha$ for y such that $|x-y| < \delta$. This means, however, that $h = \delta$ does the job.

Problem 8.8 Solution: The minimum/maximum of two numbers $a, b \in \mathbb{R}$ can be written in the form

$$\begin{aligned} \min\{a,b\} &= \frac{1}{2}\big(a+b-|a-b|\big) \\ \max\{a,b\} &= \frac{1}{2}\big(a+b+|a-b|\big) \end{aligned}$$

which shows that we can write $\min\{x,0\}$ and $\max\{x,0\}$ as a combination of continuous functions. As such they are again continuous, hence measurable. Thus,

$$u^+ = \max\{u,0\}, \qquad u^- = -\min\{u,0\}$$

are compositions of measurable functions, hence measurable.

Problem 8.9 Solution:

(i) From the definition of the supremum we get

$$\begin{aligned} \sup_i f_i(x) > \lambda &\iff \exists i_0 \in I \,:\, f_{i_0}(x) > \lambda \\ &\iff \exists i_0 \in I \,:\, f_{i_0}(x) > \lambda \\ &\iff x \in \bigcup_i \{f_i > \lambda\}. \end{aligned}$$

(ii) Let $x \in \{\sup_i f_i < \lambda\}$. Then we have $f_j(x) \leqslant \sup_{i\in I} f_i(x) < \lambda$ for all $j \in I$; this means $x \in \{f_j < \lambda\}$ for all $j \in I$ and so $x \in \bigcap_{j\in I}\{f_j < \lambda\}$.

(Note: '$\supset$' is, in general, wrong. To see this, use e.g. $f_i(x) := -\frac{1}{i}$, $i \in \mathbb{N}$, and $\lambda = 0$. Then we have $\{\sup_i f_i < 0\} = \emptyset \neq \mathbb{R} = \bigcap_i \{f_i < 0\}$.)

(iii) Let $x \in \bigcup_i \{f_i \geqslant \lambda\}$. Then there is some $i_0 \in I$ such that $x \in \{f_{i_0} \geqslant \lambda\}$, hence

$$\sup_{i\in I} f(x) \geqslant f_{i_0}(x) \geqslant \lambda.$$

(iv) This follows from

$$\begin{aligned} \sup_{i\in I} f_i(x) \leqslant \lambda &\iff \forall i \in I \,:\, f_i(x) \leqslant \lambda \\ &\iff \forall i \in I \,:\, x \in \{f_i \leqslant \lambda\} \\ &\iff x \in \bigcap_{i\in I} \{f_i \leqslant \lambda\}. \end{aligned}$$

(v)–(viii) can be proved like parts (i)–(iv).

Problem 8.10 Solution: The f_j are step-functions where the bases of the steps are the sets A_k^j and A_j. Since they are of the form, e.g. $\left\{k2^{-j} \leqslant u < (k+1)2^{-j}\right\} = \left\{k2^{-j} \leqslant u\right\} \cap \left\{u < (k+1)2^{-j}\right\}$, it is clear that they are not only in $\mathscr{A}$ but in $\sigma(u)$.

Problem 8.11 Solution:

Corollary 8.12 If $u^{\pm}$ are measurable, it is clear that $u = u^+ - u^-$ is measurable since differences of measurable functions are measurable.

(For the converse we could use the previous Problem 8.10, but we give an alternative proof...) Conversely, let u be measurable. Then $s_n \uparrow u$ (this is short for: $\lim_{n\to\infty} s_n(x) = u(x)$ and this is an increasing limit) for some sequence of simple functions s_n. Now it is clear that $s_n^+ \uparrow u^+$, and s_n^+ is simple, i.e. u^+ is measurable. As $u = u^+ - u^-$ we conclude that $u^- = u^+ - u$ is again measurable as difference of two measurable functions. (Notice that in no case '$\infty - \infty$' can occur!)

Corollary 8.13 This is trivial if the difference $u - v$ is defined. In this case it is measurable as difference of measurable functions, so

$$\{u < v\} = \{0 < u - v\}$$

etc. is measurable.

Let us be a bit more careful and consider the case where we could encounter expressions of the type '$\infty - \infty$'. Since $s_n \uparrow u$ for simple functions (they are always $\mathbb{R}$-valued...) we get

$$\{u \leqslant v\} = \{\sup_n s_n \leqslant u\} \stackrel{(*)}{=} \bigcap_n \{s_n \leqslant u\} = \bigcap_n \{0 \leqslant u - s_n\}$$

and the latter is a union of measurable sets, hence measurable. Now $\{u < v\} = \{u \geqslant v\}^c$ and we get measurability after switching the roles of u and v. Finally $\{u = v\} = \{u \leqslant v\} \cap \{u \geqslant v\}$ and $\{u \neq v\} = \{u = v\}^c$.

Let me stress the importance of '$\leqslant$' in $(*)$ above: we use here

$$\begin{aligned} x \in \{\sup_n s_n \leqslant u\} &\iff \sup_n s_n(x) \leqslant u(x) \\ &\stackrel{(**)}{\iff} s_n(x) \leqslant u(x) \quad \forall\, n \\ &\iff x \in \bigcap \{s_n \leqslant u\} \end{aligned}$$

and this would be incorrect if we had had '$<$', since the argument would break down at $(**)$ (only one implication would be valid: '$\Rightarrow$').

Problem 8.12 Solution: Since X is σ-finite, there is an exhausting sequence $A_n \uparrow X$ with $\mu(A_n) < \infty$. Let $u \in \mathcal{M}(\mathscr{A})$.

- It is clearly enough to consider $u \geqslant 0$, otherwise we consider positive and negative parts separately. By the Sombrero lemma (Theorem 8.8) there is a sequence $(u_n)_n \subset \mathcal{E}(\mathscr{A})$ with $0 \leqslant u_n(x) \uparrow u(x)$ for all $x \in X$. Since $A_n \uparrow X$, we also get $u_n \mathbb{1}_{A_n} \uparrow u$, i.e. we can without loss of generality assume that the standard representation of each u_n is of the form
$$u_n = \sum_{m=1}^{M(n)} \alpha_{n,m} \mathbb{1}_{A_{n,m}}, \quad \alpha_{n,m} \geqslant 0,\ A_{n,m} \in \mathscr{A},\ \mu(A_{n,m}) < \infty.$$
- From (an obvious variant of) Problem 5.12 we know that we can approximate $A_{n,m}$ having **finite** measure by some $G_{n,m} \in \mathscr{G}$ in such a way that $\mu\{\mathbb{1}_{G_{n,m}} \neq \mathbb{1}_{A_{n,m}}\} \leqslant 2^{-n}/M(n)$ (note: $|\mathbb{1}_A - \mathbb{1}_B| = \mathbb{1}_{A \triangle B}$).

  Moreover,
$$f_n(x) := \sum_{m=1}^{M(n)} \alpha_{n,m} \mathbb{1}_{G_{n,m}}(x)$$
  and since $\{f_n \neq u_n\} \subset \bigcup_m G_{n,m} \triangle A_{n,m}$, we get $\mu\{f_n \neq u_n\} \leqslant 2^{-n}$.

  As $\lim_{n\to\infty} u_n(x) = u(x)$ for all x, we find from the continuity of the measure (from above)
$$\begin{aligned} \mu(\lim_{n\to\infty} f_n \neq u) &\leqslant \mu\left(\bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant k} \{f_n \neq u_n\}\right) \\ &\leqslant \lim_{k\to\infty} \sum_{n=k}^{\infty} \mu\{f_n \neq u_n\} \\ &\leqslant \lim_{k\to\infty} \sum_{n=k}^{\infty} 2^{-n} = 0. \end{aligned}$$
  This shows that $\mathcal{E}(\mathscr{G}) \ni f_n(x) \to u(x)$ for all $x \notin N$ with $\mu(N) = 0$.

An alternative proof can be based on the monotone class theorem. We sketch the steps below (notation as above and in Theorem 8.15):

- Set $\mathcal{V}_n := \big\{u \in \mathcal{M}(A_n \cap \mathscr{A}) \,:\, \exists (f_i)_i \subset \mathcal{E}(A_n \cap \mathscr{G}),\ \exists N_n \in \mathscr{A}, \mu(N_n) = 0,\ \forall x \notin N_n \,:\, f_i(x) \to u(x)\big\}$.

  Obviously $\mathcal{V}_n$ is a vector space which is stable under bounded suprema (use a diagonal argument and the fact that the union of countably many null sets is again a null set).
- Observe that $\mathbb{1}_{A_n}, \mathbb{1}_{A_n \cap A} \in \mathcal{V}_n$ for all $A \in \mathscr{A}$ by the result of Problem 5.12.
- Use the monotone class theorem.
- Glue together the sets $\mathcal{V}_n$ by considering $u = \lim_n u\mathbb{1}_{A_n}$. This leads again to a countable union of null sets.

∎∎

Problem 8.13 Solution: If u is differentiable, it is continuous, hence measurable. Moreover, since u' exists, we can write it in the form
$$u'(x) = \lim_{k\to\infty} \frac{u\big(x + \frac{1}{k}\big) - u(x)}{\frac{1}{k}}$$

i.e. as limit of measurable functions. Thus, u' is also measurable.

Problem 8.14 Solution: It is sometimes necessary to distinguish between domain and range. We use the subscript x to signal the domain, the subscript y for the range.

(i) Since $f : \mathbb{R}_x \to \mathbb{R}_y$ is $f(x) = x$, the inverse function is clearly $f^{-1}(y) = y$. So if we take any Borel set $B \in \mathscr{B}(\mathbb{R}_y)$ we get $B = f^{-1}(B) \subset \mathbb{R}_x$. Since, as we have seen, $\sigma(f) = f^{-1}(\mathscr{B}(\mathbb{R}_y))$, the above argument shows that $f^{-1}(\mathscr{B}(\mathbb{R}_y)) = \mathscr{B}(\mathbb{R}_x)$, hence $\sigma(f) = \mathscr{B}(\mathbb{R}_x)$.

(ii) The inverse map of $g(x) = x^2$ is multi-valued, i.e. if $y = x^2$, then $y = \pm\sqrt{x}$. So $g^{-1} : [0, \infty) \to \mathbb{R}$, $g^{-1}(y) = \pm\sqrt{y}$. Let us take some $B \in \mathscr{B}(\mathbb{R}_y)$. Since g^{-1} is only defined for positive numbers (squares yield positive numbers only!) we have that $g^{-1}(B) = g^{-1}(B \cap [0, \infty)) = \sqrt{B \cap [0, \infty)} \cup (-\sqrt{B \cap [0, \infty)})$ (where we use the obvious notation $\sqrt{A} = \{\sqrt{a} : a \in A\}$ and $-A = \{-a : a \in A\}$ whenever A is a set). This shows that

$$\begin{aligned}\sigma(g) &= \{\sqrt{B} \cup (-\sqrt{B}) : B \in \mathscr{B}, B \subset [0, \infty)\} \\ &= \{\sqrt{B} \cup (-\sqrt{B}) : B \in [0, \infty) \cap \mathscr{B}\}\end{aligned}$$

where we use the notation of trace σ-algebras in the latter identity.

(It is an instructive exercise to check that $\sigma(g)$ is indeed a σ-algebra. This is, of course, clear from the general theory since $\sigma(g) = g^{-1}([0, \infty) \cap \mathscr{B})$, i.e. it is the pre-image of the trace σ-algebra and pre-images of σ-algebras are always σ-algebras.

(iii) A very similar calculation as in part (ii) shows that

$$\begin{aligned}\sigma(h) &= \{B \cup (-B) : B \in \mathscr{B}, B \subset [0, \infty)\} \\ &= \{B \cup (-B) : B \in [0, \infty) \cap \mathscr{B}\}.\end{aligned}$$

(iv) As warm-up we follow the hint. The set $\{(x, y) : x + y = \alpha\}$ is the line $y = \alpha - x$ in the x-y-plane, i.e. a line with slope -1 and shift α. So $\{(x, y) : x + y \geqslant \alpha\}$ would be the points above this line and $\{(x, y) : \beta \geqslant x + y \geqslant \alpha\} = \{(x, y) : x + y \in [\alpha, \beta]\}$ would be the points in the strip which has the lines $y = \alpha - x$ and $y = \beta - x$ as boundaries.

More general, take a Borel set $B \in \mathscr{B}(\mathbb{R})$ and observe that

$$F^{-1}(B) = \{(x, y) : x + y \in B\}.$$

This set is, in an abuse of notation, $y = B - x$, i.e. these are all lines with slope -1 (135 degrees) and *every possible shift from the set B*—it gives a kind of stripe-pattern. To sum up:

$$\sigma(F) = \{\text{all 135-degree diagonal stripes in } \mathbb{R}^2 \text{ with 'base' } B \in \mathscr{B}(\mathbb{R})\}.$$

(v) Again follow the hint to see that $\{(x, y) : x^2 + y^2 = r\}$ is a circle, radius r, centre $(0, 0)$. So $\{(x, y) : x^2 + y^2 \leqslant r\}$ is the solid disk, radius r, centre $(0, 0)$ and $\{(x, y) : R \geqslant x^2 + y^2 \geqslant$

$r\} = \{(x, y) : x^2 + y^2 \in [r, R]\}$ is the annulus with exterior radius R and interior radius r about $(0, 0)$.

More general, take a Borel set $B \subset [0, \infty)$, $B \in \mathscr{B}(\mathbb{R})$, i.e. $B \in [0, \infty) \cap \mathscr{B}(\mathbb{R})$ (negative radii don't make sense!) and observe that the set $\{(x, y) : x^2 + y^2 \in B\}$ gives a ring-pattern which is 'supported' by the set B (i.e. we take all circles passing through B...). To sum up:

$$\begin{aligned}\sigma(G) = &\{\text{a set consists of all circles in } \mathbb{R}^2 \text{ about } (0,0)\\ &\qquad \text{passing through } B \in \mathscr{B}[0, \infty) \cap B(\mathbb{R})\}.\end{aligned}$$

Problem 8.15 Solution: Assume first that u is injective. This means that every point in the range $u(\mathbb{R})$ comes exactly from one uniquely defined $x \in \mathbb{R}$. This can be expressed by saying that $\{x\} = u^{-1}(\{u(x)\})$ — but the singleton $\{u(x)\}$ is a Borel set in the range, so $\{x\} \in \sigma(u)$ as $\sigma(u) = u^{-1}(u(\mathbb{R}) \cap \mathscr{B})$.

Conversely, assume that for each x we have $\{x\} \in \sigma(u)$. Fix an x_0 and call $u(x_0) = \alpha$. Since u is measurable, the set $\{u = \alpha\} = \{x : u(x) = \alpha\}$ is measurable and, clearly, $\{x_0\} \subset \{u = \alpha\}$. But if we had another $x_0 \neq x_1 \in \{u = \alpha\}$ this would mean that we could never 'produce' $\{x_0\}$ on its own as a pre-image of some set, but we must be able to do so as $\{x_0\} \in \sigma(u)$, by assumption. Thus, $x_1 = x_0$. To sum up, we have shown that $\{u = \alpha\}$ consists of one point only, i.e. we have shown that $u(x_0) = u(x_1)$ implies $x_0 = x_1$ which is just injectivity.

Problem 8.16 Solution: Clearly $u : \mathbb{R} \to [0, \infty)$. So let's take $I = (a, b) \subset [0, \infty)$. Then $u^{-1}((a, b)) = (-b, -a) \cup (a, b)$. This shows that for $\mu := \lambda \circ u^{-1}$

$$\begin{aligned}\mu(a, b) &= \lambda \circ u^{-1}((a, b)) = \lambda\big((-b, -a) \cup (a, b)\big) = \lambda(-b, -a) + \lambda(a, b)\\ &= (-a - (-b)) + (b - a) = 2(b - a) = 2\lambda((a, b)).\end{aligned}$$

This shows that $\mu = 2\lambda$ if we allow only intervals from $[0, \infty)$, i.e.

$$\mu(I) = 2\lambda\big(I \cap [0, \infty)\big) \quad \text{for any interval } I \subset \mathbb{R}.$$

Since a measure on the Borel sets is completely described by (either: open or closed or half-open or half-closed) intervals (the intervals generate the Borel sets!), we can invoke the uniqueness theorem to guarantee that the above equality holds for all Borel sets.

Problem 8.17 Solution:

(i) Because of Lemma 7.2 it is enough to check measurability for some generator. Let $B = [a,b) \in \mathscr{J}$, $a < b$. We have

$$Q^{-1}(B) = E \cap \begin{cases} \emptyset & \text{if } a, b \leqslant 0 \\ (-\sqrt{b}, +\sqrt{b}) & \text{if } a \leqslant 0,\ b > 0 \\ \left(-\sqrt{b}, -\sqrt{a}\right] \cup \left[\sqrt{a}, \sqrt{b}\right) & \text{if } a, b > 0 \end{cases}$$

These sets are in $\mathscr{B}(E)$, therefore Q is $\mathscr{B}(E)/\mathscr{B}(\mathbb{R})$-measurable.

(ii) Denote by T the embedding of E into $\mathbb{R}$, i.e. $T : x \mapsto x$. Formally, we get

$$\nu(T^2 \in B) = \nu(\pm T \in \sqrt{B}).$$

More precisely: we have already seen that $\nu \circ Q^{-1}$ is a measure (Theorem 7.6). Since $\mathscr{J}$ is $\cap$-stable and $\nu \circ Q^{-1}$ a finite measure (ν comes from a finite Lebesgue measure), we get uniqueness from Theorem 5.7, and it enough to consider sets of the form $B = [a,b) \in \mathscr{J}$, $a \leqslant b$.

- Part (i) gives

$$\begin{aligned} \nu(Q^{-1}(B)) &= \begin{cases} 0, & b \leqslant 0 \text{ or } a > 1 \\ \lambda([0, \sqrt{b})), & a < 0, b > 0 \\ \lambda([\sqrt{a}, \sqrt{b} \wedge 1)), & 0 < a < 1 \end{cases} \\ &= \begin{cases} 0, & b \leqslant 0 \text{ or } a > 1 \\ \sqrt{b \wedge 1} - \sqrt{0 \vee a \wedge 1}, & \text{otherwise.} \end{cases} \end{aligned}$$

- Again by part (i)

$$\begin{aligned} \nu(Q^{-1}(B)) &= \begin{cases} 0, & b \leqslant 0 \text{ or } a > 1 \\ \lambda([(-\sqrt{b}) \vee (-1), \sqrt{b} \wedge 1)), & a < 0, b > 0 \\ \frac{1}{2}\lambda([(-\sqrt{b}) \vee (-1), \sqrt{a}) \cup [\sqrt{a}, \sqrt{b} \wedge 1)), & 0 < a < 1 \end{cases} \\ &= \begin{cases} 0, & b \leqslant 0 \text{ or } a > 1 \\ 2\frac{1}{2}\lambda([0 \vee \sqrt{a} \wedge 1, \sqrt{b} \wedge 1)), & \text{otherwise} \end{cases} \\ &= \begin{cases} 0, & b \leqslant 0 \text{ or } a > 1 \\ (\sqrt{b} \wedge 1)) - (0 \wedge \sqrt{a} \wedge 1), & \text{otherwise} \end{cases} \end{aligned}$$

Problem 8.18 Solution:

- clear, since $u(x-2)$ is a combination of the measurable shift τ_2 and the measurable function u.
- this is trivial since $u \mapsto e^u$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- this is trivial since $u \mapsto \sin(u+8)$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- iterate Problem 8.13
- obviously, $\operatorname{sgn} x = (-1) \cdot \mathbb{1}_{(-\infty,0)}(x) + 0 \cdot \mathbb{1}_{\{0\}}(x) + 1 \cdot \mathbb{1}_{(0,\infty)}(x)$, i.e. a measurable function. Using the first example, we see now that $\operatorname{sgn} u(x-7)$ is a combination of three measurable functions.

∎∎

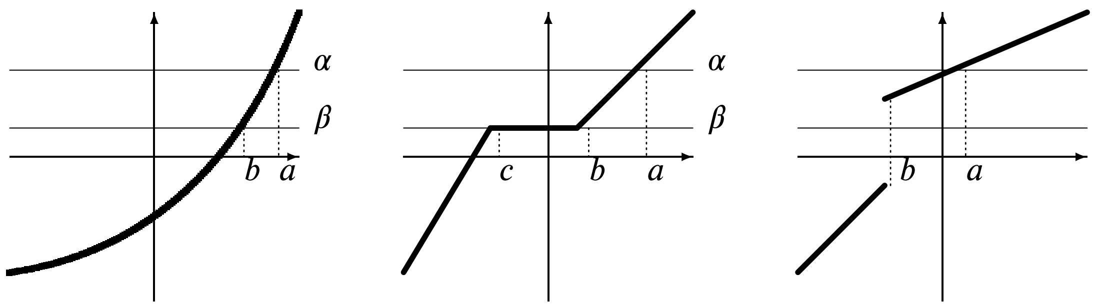
Problem 8.19 Solution: Consider, for instance, $T : [0,1) \to [0,1)$ where $T(x) = \frac{x}{2}$ and $w_n : [0,1) \to \overline{\mathbb{R}}$ with $w_n(x) = (-1)^n \mathbb{1}_{[1/2,1)}(x)$.

∎∎

Problem 8.20 Solution: Let $A \subset \mathbb{R}$ be such that $A \notin \mathscr{B}$. Then it is clear that $u(x) = \mathbb{1}_A(x) - \mathbb{1}_{A^c}(x)$ is NOT measurable (take, e.g. $A = \{f = 1\}$ which should be measurable for measurable functions), but clearly, $|f(x)| = 1$ and as constant function this IS measurable.

∎∎

Problem 8.21 Solution: We want to show that the sets $\{u \leqslant \alpha\}$ are Borel sets. We will even show that they are intervals, hence Borel sets. Imagine the graph of an increasing function and the line $y = \alpha$ cutting through. Essentially we have three scenarios: the cut happens at a point where (a) u is continuous and strictly increasing or (b) u is flat or (c) u jumps—i.e. has a gap; these three cases are shown in the following pictures: From the three pictures it is clear that we get in any case an



interval for the sub-level sets $\{u \leqslant \gamma\}$ where γ is some level (in the pic's $\gamma = \alpha$ or $= \beta$), you can read off the intervals on the abscissa where the dotted lines cross the abscissa.

Now let's look at the additional conditions: First the intuition: From the first picture, the continuous and strictly increasing case, it is clear that we can produce any interval $(-\infty, b]$ to $(-\infty, a]$ by looking at $\{u \leqslant \beta\}$ to $\{u \leqslant \alpha\}$ my moving up the β-line to level α. The point is here that we get all intervals, so we get a generator of the Borel sets, so we should get all Borel sets.

The second picture is bad: the level set $\{u \leqslant \beta\}$ is $(-\infty, b]$ and all level sets below will only come up to the point $(-\infty, c]$, so there is no chance to get any set contained in (c, b), i.e. we cannot get all Borel sets.

The third picture is good again, because the vertical jump does not hurt. The only 'problem' is whether $\{u \leqslant \beta\}$ is $(-\infty, b]$ or $(-\infty, b)$ which essentially depends on the property of the graph whether $u(b) = \beta$ or not, but this is not so relevant here, we just must make sure that we can get more or less all intervals. The reason, really, is that jumps as we described them here can only happen countably often, so this problem occurs only countably often, and we can overcome it therefore.

So the point is: we must disallow flat bits, i.e. $\sigma(u)$ is the Borel σ-algebra if, and only, if u is strictly increasing, i.e. if, and only if, u is injective. (Note that this would have been clear already from Problem 8.15, but our approach here is much more intuitive.)

Problem 8.22 Solution: For every $n \in \mathbb{N}$ the function

$$g_n(x) := \sum_{i=1}^{n} 2^{-i} \mathbb{1}_{G_i}(x), \qquad x \in X,$$

is $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable. Therefore, $g = \lim_{n\to\infty} g_n$ is $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable (pointwise limit of measurable functions), and so $\sigma(g) \subset \mathscr{A}$. For the inclusion $\mathscr{A} \subset \sigma(g)$ we define

$$\Sigma := \{A \in \mathscr{A} : A \in \sigma(g)\}.$$

Σ ist a σ-Algebra:

(Σ_1) $X \in \Sigma$ since $X \in \mathscr{A}$ and $X \in \sigma(g)$.

(Σ_2) For $A \in \Sigma$ we have $A \in \sigma(g)$; since $\sigma(g)$ is a σ-algebra, we see that $A^c \in \sigma(g)$; hence, $A^c \in \Sigma$.

(Σ_3) For $(A_n)_{n\in\mathbb{N}} \subset \Sigma$ we see $\bigcup_{n\in\mathbb{N}} A_n \in \sigma(g)$, thus $\bigcup_n A_n \in \Sigma$.

Since $G_i = \{g = 2^{-i}\} \in \sigma(g)$ we see that $\mathscr{G} \subset \Sigma$. Consequently, $\mathscr{A} = \sigma(\mathscr{G}) \subset \sigma(g)$.

Problem 8.23 Solution: Without loss of generality, assume that u is right-continuous (left-continuity works analogously). Approximate u with simple functions:

$$u_n(x) := \sum_{i=1}^{2n^2} u(x_{i+1}^n) \mathbb{1}_{[x_i^n, x_{i+1}^n)}(x)$$

where $x_i^n := -n + \frac{i}{n}$. The functions u_n are obviously Borel measurable. We claim:

$$u(x) = \lim_{n\to\infty} u_n(x).$$

Indeed: For each $x \in \mathbb{R}$ there is some $N \in \mathbb{N}$ such that $x \in [-N, N]$. By definition, we find for all $n \geqslant N$,

$$u_n(x) = u\left(\frac{\lfloor nx \rfloor + 1}{n}\right)$$

($\frac{\lfloor nx \rfloor+1}{n}$ is the smallest number of the form $\frac{k}{n}$, $k \in \mathbb{Z}$, which exceeds x.) Because of the right-continuity of u we get $u_n(x) \to u(x)$ as $n \to \infty$. Therefore, u is Borel-measurable (pointwise limit of measurable functions).

Problem 8.24 Solution: Every linear map on a finite-dimensional vector space is continuous, hence Borel measurable.

Note that $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) := (x, 0)^\top$, is continuous, hence Borel measurable. This map is, however, **not** measurable with respect to the completed Borel σ-algebras:

To see this, let $A \subset \mathbb{R}$, $A \notin \mathscr{B}(\mathbb{R})$, be a subset of a Lebesgue null set. For $A \times \{0\}$ we see that $A \times \{0\} \in \overline{\mathscr{B}(\mathbb{R}^2)}$; this follows from $A \times \{0\} \subset N := \mathbb{R} \times \{0\}$ and $\lambda^2(N) = 0$ (cf. Problem 4.15, Problem 6.7). On the other hand, $f^{-1}(A \times \{0\}) = A \notin \mathscr{B}(\mathbb{R}) \subset \overline{\mathscr{B}(\mathbb{R})}$, i.e. $f : (\mathbb{R}, \overline{\mathscr{B}(\mathbb{R})}) \to (\mathbb{R}^2, \overline{\mathscr{B}(\mathbb{R}^2)})$ is not measurable.

Problem 8.25 Solution: Without loss of generality we consider the right-continuous situation. The left-continuous counterpart is very similar.

- Fix $\omega \in \Omega$. Note that it is enough to show that $t \mapsto \xi(t, \omega)\mathbb{1}_{[a,b]}(t) =: \xi^{a,b}(t, \omega)$ is measurable for all $a < b$.

  Indeed: Because of

  $$\xi(t, \omega) = \lim_{R \to \infty} \xi^{-R,R}(t, \omega)$$

  the map $t \mapsto \xi(t, \omega)$ is measurable (pointwise limit of measurable functions, cf. Corollary 8.10.

  In order to keep notation simple, we assume that $a = 0$ and $b = 1$; the general case is similar. Define

  $$\xi_n(t, \omega) := \sum_{i=0}^{2^n - 1} \xi\left(\tfrac{i+1}{2^n}, \omega\right) \mathbb{1}_{\left[\frac{i}{2^n}, \frac{i+1}{2^n} \wedge 1\right)}(t).$$

  For any $t \in [0, 1]$ we have $\frac{\lfloor 2^n t \rfloor + 1}{2^n} \downarrow t$, and because of right-continuity,

  $$\xi_n(t, \omega) = \xi\left(\frac{\lfloor 2^n t \rfloor + 1}{2^n}, \omega\right) \xrightarrow[n \to \infty]{} \xi(t, \omega) \stackrel{t \in [0,1]}{=} \xi^{0,1}(t, \omega).$$

  For $t \notin [0, 1]$ we have $\xi_n(t, \omega) = 0 = \xi^{0,1}(t, \omega)$ and, thus,

  $$\xi^{0,1}(t, \omega) = \lim_{n \to \infty} \xi_n(t, \omega) \quad \forall t \in \mathbb{R}, \omega \in \Omega.$$

Consequently, it is enough to show (by Corollary 8.10) that each $t \mapsto \xi_n(t,\omega)$ is measurable. For $\alpha \in \mathbb{R}$ we get

$$\{t : \xi_n(t,\omega) \leqslant \alpha\} = \bigcup_{i\in I} \underbrace{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)}_{\in\mathscr{B}(\mathbb{R})} \in \mathscr{B}(\mathbb{R})$$

where

$$I := \left\{ i \in \{0,\dots,2^n-1\}; \xi\left(\frac{i+1}{2^n},\omega\right) \leqslant \alpha \right\}.$$

This proves that $t \mapsto \xi_n(t,\omega)$ is measurable.

- Since $t \mapsto \xi(t,\omega)$ is right-continuous, we have

$$\sup_{t\in\mathbb{R}} \xi(t,\omega) = \sup_{t\in\mathbb{Q}} \xi(t,\omega). \tag{$\star$}$$

Indeed: The estimate '$\geqslant$' is clear, i.e. we only have to show '$\leqslant$'. Using the definition of the supremum, there is for each $\epsilon > 0$ some $s \in \mathbb{R}$ such that

$$\xi(s,\omega) \geqslant \sup_{t\in\mathbb{R}} \xi(t,\omega) - \epsilon.$$

Because of right-continuity we find some $r \in \mathbb{Q}$, $r > s$, such that $|\xi(r,\omega) - \xi(s,\omega)| \leqslant \epsilon$. Therefore,

$$\sup_{t\in\mathbb{Q}} \xi(t,\omega) \geqslant \xi(r,\omega) \geqslant \xi(s,\omega) - \epsilon \geqslant \sup_{t\in\mathbb{R}} \xi(t,\omega) - 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the claim follows.

From ($\star$) we get that the map $\omega \mapsto \sup_{t\in\mathbb{R}} \xi(t,\omega)$ is measurable (as supremum of countably many measurable functions, cf. Corollary 8.10).

Problem 8.26 Solution: '$\Leftarrow$': Assume that there are $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable functions $f, g : X \to \mathbb{R}$ satisfying $f \leqslant \phi \leqslant g$ and $\mu\{f \neq g\} = 0$. For any $x \in \mathbb{R}$ we get

$$\begin{aligned}\{\phi \leqslant x\} &= \{\phi \leqslant x, f = g\} \cup \{\phi \leqslant x, f \neq g\} \\ &= \underbrace{\{g \leqslant x, f = g\}}_{=:A} \cup \underbrace{\{\phi \leqslant x, f \neq g\}}_{=:N}.\end{aligned}$$

Since f and g are measurable, we see that $A \in \mathscr{A}$. For N we only get $N \subset \{f \neq g\}$, i.e. N is a subset of a μ-null set. By the definition of $\overline{\mathscr{A}}$ (see Problem 4.15) we find $\{\phi \leqslant x\} \in \overline{\mathscr{A}}$.

'$\Rightarrow$': Assume, first, that ϕ is a simple function, i.e.

$$\phi(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{A_i}(x), \qquad x \in X,$$

with $c_i \in \mathbb{R}$, $A_i \in \overline{\mathscr{A}}$ $(i = 1,\dots,n)$. From the definition of $\overline{\mathscr{A}}$ we get that the A_i are of the form

$$A_i = B_i + N_i$$

with $B_i \in \mathscr{A}$ and N_i being a subset of a μ-null set $M_i \in \mathscr{A}$. Define

$$f(x) := \sum_{i=1}^{n} c_i \mathbb{1}_{B_i}(x), \qquad g(x) := \sum_{i=1}^{n} c_i \mathbb{1}_{B_i \cup M_i}(x), \qquad x \in X.$$

These are clearly $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable functions and $f \leqslant \phi \leqslant g$. Moreover,

$$\mu(f \neq g) \leqslant \mu\left(\bigcup_{i=1}^{n} M_i\right) \leqslant \sum_{i=1}^{n} \mu(M_i) = 0.$$

This proves that the claim holds for simple functions.

Let ϕ be any $\overline{\mathscr{A}}/\mathscr{B}(\mathbb{R})$-measurable function. Using Corollary 8.9, we get a sequence $(\phi_n)_{n\in\mathbb{N}}$ of $\overline{\mathscr{A}}/\mathscr{B}(\mathbb{R})$-measurable simple functions such that $\phi_n(x) \to \phi(x)$ for all $x \in X$. By the first part of this proof, there are $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable funcitons f_n, g_n, $n \in \mathbb{N}$, such that $f_n \leqslant \phi_n \leqslant g_n$ and $\mu(f_n \neq g_n) = 0$. Set

$$f(x) := \liminf_{n\to\infty} f_n(x), \qquad g(x) := \liminf_{n\to\infty} g_n(x), \qquad x \in X.$$

The functions f and g are again $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable (Corollary 8.10) and we have $f \leqslant \phi \leqslant g$. Moreover,

$$\mu(f \neq g) \leqslant \mu\left(\bigcup_{n\in\mathbb{N}} \{f_n \neq g_n\}\right) \leqslant \sum_{n\in\mathbb{N}} \mu(f_n \neq g_n) = 0.$$

■■

9 Integration of positive functions. Solutions to Problems 9.1–9.14

Problem 9.1 Solution: We know that for any two simple functions $f, g \in \mathcal{E}_+$ we have $I_\mu(f+g) = I_\mu(f) + I_\mu(g)$ (= additivity), and this is easily extended to finitely many, say, m different positive simple functions. Observe now that each $\xi_n \mathbb{1}_{A_n}$ is a positive simple function, hence

$$I_\mu\left(\sum_{n=1}^m \xi_n \mathbb{1}_{A_n}\right) = \sum_{n=1}^m I_\mu\left(\xi_n \mathbb{1}_{A_n}\right) = \sum_{n=1}^m \xi_n I_\mu\left(\mathbb{1}_{A_n}\right) = \sum_{n=1}^m \xi_n \mu\left(A_n\right).$$

Put in other words: we have used the linearity of I_μ.

Problem 9.2 Solution: We use indicator functions. Note that any fixed x can be contained in $k \in \{0, 1, \dots, N\}$ of the sets A_n. Then x is contained in $A_1 \cup \dots \cup A_N$ as well as in $\binom{k}{2}$ of the pairs $A_n \cup A_k$ where $n < k$; as usual: $\binom{m}{n} = 0$ if $m < n$. This gives

$$\begin{aligned}\sum_n \mathbb{1}_{A_n} = k \leqslant 1 + \binom{k}{2} &= \mathbb{1}_{A_1 \cup \dots \cup A_N} + \sum_{n<k} \mathbb{1}_{A_n} \mathbb{1}_{A_k} \\ &= \mathbb{1}_{A_1 \cup \dots \cup A_N} + \sum_{n<k} \mathbb{1}_{A_n \cap A_k}.\end{aligned}$$

Integrating this inequality w.r.t. μ yields the result.

Problem 9.3 Solution: We check Properties 9.8(i)–(iv).

(i) This follows from Properties 9.3 and Lemma 9.5 since $\int \mathbb{1}_A \, d\mu = I_\mu(\mathbb{1}_A) = \mu(A)$.

(ii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n \in \mathcal{E}_+$ with $u = \sup_n u_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}\int \alpha u \, d\mu = \int \alpha \sup_n u_n \, d\mu &= \sup_n I_\mu(\alpha u_n) \\ &= \sup_n \alpha I_\mu(u_n) \\ &= \alpha \sup_n I_\mu(u_n) \\ &= \alpha \int u \, d\mu.\end{aligned}$$

(iii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n, v_n \in \mathcal{E}_+$ with $u = \sup_n u_n, v = \sup_n v_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}\int (u+v)\,d\mu = \int \lim_{n\to\infty}(u_n+v_n)\,d\mu &= \lim_{n\to\infty} I_\mu(u_n+v_n)\\ &= \lim_{n\to\infty}\big(I_\mu(u_n)+I_\mu(v_n)\big)\\ &= \lim_{n\to\infty} I_\mu(u_n)+\lim_{n\to\infty} I_\mu(v_n)\\ &= \int u\,d\mu+\int v\,d\mu.\end{aligned}$$

(iv) This was shown in Step 1 of the proof of the Beppo Levi theorem 9.6

Problem 9.4 Solution: Consider on the space $([-1,0],\lambda)$, $\lambda(dx)=dx$ is Lebesgue measure on $[0,1]$, the sequence of 'tent-type' functions

$$f_k(x) = \begin{cases} 0, & -1 \leqslant x \leqslant -\frac{1}{k},\\ k^3\big(x+\frac{1}{k}\big), & -\frac{1}{k} \leqslant x \leqslant 0, \end{cases} \qquad (k\in\mathbb{N}),$$

(draw a picture!). These are clearly monotonically increasing functions but, as a sequence, we do not have $f_k(x) \leqslant f_{k+1}(x)$ for *every* x! Note also that each function is integrable (with integral $\frac{1}{2}k$) but the pointwise limit is not integrable.

Problem 9.5 Solution: The first part is trivial since it just says that the sequence becomes increasing only from index K onwards. This K does not depend on x but is uniform for the whole sequence. Since we are anyway only interested in $u = \lim_{n\to\infty} u_n = \sup_{n\geqslant K} u_n$, we can neglect the elements $u_1,\dots,u_K$ and consider only the then increasing sequence $(u_{n+K})_n$. Then we can directly apply Beppo Levi's theorem, Theorem 9.6.

The other condition says that the sequence $u_{n+K}(x)$ is increasing for some $K=K(x)$. But since K may depend on x, we will never get some overall increasing behaviour of the sequence of functions. Take, for example, on $(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda := \lambda^1)$,

$$u_n(x) = n^2(x+\tfrac{1}{n})\mathbb{1}_{(-1/n,0)}(x) - n^2(x-\tfrac{1}{n})\mathbb{1}_{(0,1/n)}(x).$$

This is a sequence of symmetric tent-like functions of tents with base $(-1/n,1/n)$ and tip at n^2 (which we take out and replace by the value 0). Clearly:

$$u_n(x)\xrightarrow[n\to\infty]{} 0 \;\text{ and }\; \int u_n(x)\,dx = 1 \quad \forall\, n.$$

Moreover, if $n \geqslant K = K(x)$ with $K(x)$ defined to be the smallest integer $> 1/|x|$, then $u_n(x)=0$ so that the second condition is clearly satisfied, but $\int u_n(x)\,dx = 1$ cannot converge to $\int 0\,dx = \int u(x)\,dx = 0$.

Problem 9.6 Solution: Following the hint we set $s_m = u_1 + u_2 + \ldots + u_m$. As a finite sum of positive measurable functions this is again positive and measurable. Moreover, s_m increases to $s = \sum_{n=1}^{\infty} u_n$ as $m \to \infty$. Using the additivity of the integral (9.8 (iii)) and the Beppo Levi theorem 9.6 we get

$$\begin{aligned}\int \sum_{n=1}^{\infty} u_n \, d\mu = \int \sup_m s_m \, d\mu = \sup_m \int s_m \, d\mu \\ &= \sup_m \int (u_1 + \ldots + u_m) \, d\mu \\ &= \sup_m \sum_{n=1}^{m} \int u_n \, d\mu \\ &= \sum_{n=1}^{\infty} \int u_n \, d\mu.\end{aligned}$$

Conversely, assume that 9.9 is true. We want to deduce from it the validity of Beppo Levi's theorem 9.6. So let $(w_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable functions with limit $w = \sup_n w$. For ease of notation we set $w_0 \equiv 0$. Then we can write each w_n as a partial sum

$$w_n = (w_n - w_{n-1}) + \cdots + (w_1 - w_0)$$

of positive measurable summands of the form $u_k := w_k - w_{k-1}$. Thus,

$$w_m = \sum_{k=1}^{m} u_k \quad \text{and} \quad w = \sum_{k=1}^{\infty} u_k$$

and, using the additivity of the integral,

$$\int w \, d\mu \overset{9.9}{=} \sum_{k=1}^{\infty} \int u_k \, d\mu = \sup_m \int \sum_{k=1}^{m} u_k \, d\mu = \sup_m \int w_m \, d\mu.$$

Problem 9.7 Solution: Set $\nu(A) := \int \mathbb{1}_A u \, d\mu$. Then ν is a $[0, \infty]$-valued set function defined for $A \in \mathscr{A}$.

(M_1) Since $\mathbb{1}_\emptyset \equiv 0$ we have clearly $\nu(\emptyset) = \int 0 \cdot u \, d\mu = 0$.

(M_1) Let $A = \bigcupdot_{n\in\mathbb{N}} A_n$ a disjoint union of sets $A_n \in \mathscr{A}$. Then

$$\sum_{n=1}^{\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

and we get from Corollary 9.9

$$\begin{aligned}\nu(A) = \int \left(\sum_{n=1}^{\infty} \mathbb{1}_{A_n} \right) \cdot u \, d\mu = \int \sum_{n=1}^{\infty} \left(\mathbb{1}_{A_n} \cdot u \right) d\mu \\ &= \sum_{n=1}^{\infty} \int \mathbb{1}_{A_n} \cdot u \, d\mu \\ &= \sum_{n=1}^{\infty} \nu(A_n).\end{aligned}$$

Problem 9.8 Solution: This is actually trivial: since our σ-algebra is $\mathscr{P}(\mathbb{N})$, all subsets of $\mathbb{N}$ are measurable. Now the sub-level sets $\{u \leqslant \alpha\} = \{k \in \mathbb{N} : u(k) \leqslant \alpha\}$ are always $\subset \mathbb{N}$ and as such they are $\in \mathscr{P}(\mathbb{N})$, hence u is always measurable.

Problem 9.9 Solution: We have seen in Problem 4.7 that μ is indeed a measure. We follow the instructions. First, for $A \in \mathscr{A}$ we get

$$\int \mathbb{1}_A \, d\mu = \mu(A) = \sum_{j\in\mathbb{N}} \mu_j(A) = \sum_{j\in\mathbb{N}} \int \mathbb{1}_A \, d\mu_j.$$

By the linearity of the integral, this easily extends to functions of the form $\alpha\mathbb{1}_A + \beta\mathbb{1}_B$ where $A, B \in \mathscr{A}$ and $\alpha, \beta \geqslant 0$:

$$\begin{aligned}
\int (\alpha\mathbb{1}_A + \beta\mathbb{1}_B)\, d\mu &= \alpha \int \mathbb{1}_A \, d\mu + \beta \int \mathbb{1}_B \, d\mu \\
&= \alpha \sum_{j\in\mathbb{N}} \int \mathbb{1}_A \, d\mu_j + \beta \sum_{j\in\mathbb{N}} \int \mathbb{1}_B \, d\mu_j \\
&= \sum_{j\in\mathbb{N}} \int (\alpha\mathbb{1}_A + \beta\mathbb{1}_B)\, d\mu_j
\end{aligned}$$

and this extends obviously to simple functions which are finite sums of the above type.

$$\int f \, d\mu = \sum_{j\in\mathbb{N}} \int f \, d\mu_j \qquad \forall f \in \mathcal{E}_+.$$

Finally, take $u \in \mathcal{M}_+$ and take an approximating sequence $u_n \in \mathcal{E}_+$ with $\sup_n u_n = u$. Then we get by Beppo Levi (indicated by an asterisk $*$)

$$\begin{aligned}
\int u \, d\mu \stackrel{*}{=} \sup_n \int u_n \, d\mu &= \sup_n \sum_{j=1}^{\infty} \int u_n \, d\mu_j \\
&= \sup_n \sup_m \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \sup_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \lim_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\
&= \sup_m \sum_{j=1}^{m} \lim_n \int u_n \, d\mu_j \\
&\stackrel{*}{=} \sup_m \sum_{j=1}^{m} \int \lim_n u_n \, d\mu_j \\
&= \sum_{j=1}^{\infty} \int u \, d\mu_j
\end{aligned}$$

where we repeatedly use that all sup's are increasing limits and that we may swap any two sup's (this was the hint to Problem 4.7.)

Problem 9.10 Solution: Set $w_n := u - u_n$. Then the w_n are a sequence of positive measurable functions. By Fatou's lemma we get

$$\begin{aligned}\int \liminf_n w_n \, d\mu &\leqslant \liminf_n \int w_n \, d\mu \\ &= \liminf_n \left(\int u \, d\mu - \int u_n \, d\mu \right) \\ &= \int u \, d\mu - \limsup_n \int u_n \, d\mu\end{aligned}$$

(see, e.g. the rules for lim inf and lim sup in Appendix A). Thus,

$$\begin{aligned}\int u \, d\mu - \limsup_n \int u_n \, d\mu &\geqslant \int \liminf_n w_n \, d\mu \\ &= \int \liminf_n (u - u_n) \, d\mu \\ &= \int \left(u - \limsup_n u_n\right) d\mu\end{aligned}$$

and the claim follows by subtracting the *finite* value $\int u \, d\mu$ on both sides.

Remark. The uniform domination of u_n by an integrable function u is really important. Have a look at the following situation: $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, $\lambda(dx) = dx$ denotes Lebesgue measure, and consider the positive measurable functions $u_n(x) = \mathbb{1}_{[n,2n]}(x)$. Then $\limsup_n u_n(x) = 0$ but $\limsup_n \int u_n \, d\lambda = \limsup_n n = \infty \neq \int 0 \, d\lambda$.

Problem 9.11 Solution:

(i) Have a look at Appendix A, Lemma A.2.

(ii) You have two possibilities: the set-theoretic version:

$$\begin{aligned}\mu\Big(\liminf_n A_n\Big) &= \mu\bigg(\bigcup_k \bigcap_{n\geqslant k} A_n\bigg) \\ &\overset{*}{=} \sup_k \quad \underbrace{\mu\bigg(\bigcap_{n\geqslant k} A_n\bigg)}_{\substack{\leqslant \mu(A_n)\ \forall\, n\geqslant k \\ \text{hence, } \leqslant \inf_{n\geqslant k} \mu(A_n)}} \\ &\leqslant \sup_k \inf_{n\geqslant k} \mu(A_n) \\ &= \liminf_n \mu(A_n)\end{aligned}$$

which uses at the point $*$ the continuity of measures, Proposition 4.3.

The *alternative* would be (i) combined with Fatou's lemma:

$$\begin{aligned}\mu\big(\liminf_n A_n\big) &= \int \mathbb{1}_{\liminf_n A_n}\,d\mu\\ &= \int \liminf_n \mathbb{1}_{A_n}\,d\mu\\ &\leqslant \liminf_n \int \mathbb{1}_{A_n}\,d\mu\end{aligned}$$

(iii) Again, you have two possibilities: the set-theoretic version:

$$\begin{aligned}\mu\big(\limsup_n A_n\big) &= \mu\Big(\bigcap_k \bigcup_{n\geqslant k} A_n\Big)\\ &\overset{\#}{=} \inf_k \underbrace{\mu\Big(\bigcup_{n\geqslant k} A_n\Big)}_{\substack{\geqslant \mu(A_n)\ \forall n\geqslant k\\ \text{hence, } \geqslant \sup_{n\geqslant k}\mu(A_n)}}\\ &\geqslant \inf_k \sup_{n\geqslant k} \mu(A_n)\\ &= \limsup_n \mu(A_n)\end{aligned}$$

which uses at the point # the continuity of measures, Proposition 4.3. This step uses the finiteness of μ.

The *alternative* would be (i) combined with the reversed Fatou lemma of Problem 9.10:

$$\begin{aligned}\mu\big(\limsup_n A_n\big) &= \int \mathbb{1}_{\limsup_n A_n}\,d\mu\\ &= \int \limsup_n \mathbb{1}_{A_n}\,d\mu\\ &\geqslant \limsup_n \int \mathbb{1}_{A_n}\,d\mu\end{aligned}$$

(iv) Take the example in the remark to the solution for Problem 9.10. We will discuss it here in its set-theoretic form: take $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$ with λ denoting Lebesgue measure $\lambda(dx) = dx$. Put $A_n = [n, 2n] \in \mathscr{B}(\mathbb{R})$. Then

$$\limsup_n A_n = \bigcap_k \bigcup_{n\geqslant k} [n, 2n] = \bigcap_k [k, \infty) = \emptyset$$

But $0 = \lambda(\emptyset) \geqslant \limsup_n \lambda(A_n) = \limsup_n n = \infty$ is a contradiction. (The problem is that $\lambda[k, \infty) = \infty$!)

■■

Problem 9.12 Solution: We use the fact that, because of disjointness,

$$1 = \mathbb{1}_X = \sum_{n=1}^{\infty} \mathbb{1}_{A_n}$$

so that, because of Corollary 9.9,

$$\int u\,d\mu = \int \Big(\sum_{n=1}^{\infty} \mathbb{1}_{A_n}\Big) \cdot u\,d\mu = \int \sum_{n=1}^{\infty} \big(\mathbb{1}_{A_n} \cdot u\big)\,d\mu$$
$$= \sum_{n=1}^{\infty} \int \mathbb{1}_{A_n} \cdot u\,d\mu.$$

Assume now that $(X, \mathscr{A}, \mu)$ is σ-finite with an exhausting sequence of sets $(B_n)_n \subset \mathscr{A}$ such that $B_n \uparrow X$ and $\mu(B_n) < \infty$. Then we make the B_n's pairwise disjoint by setting

$$A_1 := B_1, \qquad A_k := B_k \setminus (B_1 \cup \cdots \cup B_{k-1}) = B_k \setminus B_{k-1}.$$

Now take any sequence $(a_k)_k \subset (0, \infty)$ with $\sum_k a_k \mu(A_k) < \infty$—e.g. $a_k := 2^{-k}/(\mu(A_k) + 1)$—and put

$$w(x) := \sum_{n=1}^{\infty} a_k \mathbb{1}_{A_k}.$$

Then w is integrable and, obviously, $w(x) > 0$ everywhere.

Problem 9.13 Solution:

(i) We check $(M_1), (M_2)$. Using the fact that $N(x, \cdot)$ is a measure, we find

$$\mu N(\emptyset) = \int N(x, \emptyset)\,\mu(dx) = \int 0\,\mu(dx) = 0.$$

Further, let $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ be a sequence of disjoint sets and set $A = \dot{\bigcup}_n A_n$. Then

$$\mu N(A) = \int N\Big(x, \dot{\bigcup}_n A_n\Big)\,\mu(dx) = \int \sum_n N(x, A_n)\,\mu(dx)$$
$$\overset{9.9}{=} \sum_n \int N(x, A_n)\,\mu(dx)$$
$$= \sum_n \mu N(A_n).$$

(ii) We have for $A, B \in \mathscr{A}$ and $\alpha, \beta \geqslant 0$,

$$N(\alpha\mathbb{1}_A + \beta\mathbb{1}_B)(x) = \int \big(\alpha\mathbb{1}_A(y) + \beta\mathbb{1}_B(y)\big)\,N(x, dy)$$
$$= \alpha \int \mathbb{1}_A(y)\,N(x, dy) + \beta \int \mathbb{1}_B(y)\,N(x, dy)$$
$$= \alpha N\mathbb{1}_A(x) + \beta N\mathbb{1}_B(x).$$

Thus $N(f+g)(x) = Nf(x) + Ng(x)$ for positive simple $f, g \in \mathcal{E}^+(\mathscr{A})$. Moreover, since by Beppo Levi (marked by an asterisk $*$) for an increasing sequence $f_k \uparrow u$

$$\sup_k Nf_k(x) = \sup_k \int f_k(y)\,N(x, dy) \overset{*}{=} \int \sup_k f_k(y)\,N(x, dy)$$

$$= \int u(y)\, N(x, dy)$$
$$= Nu(x)$$

and since the sup is actually an increasing limit, we see for positive measurable $u, v \in \mathcal{M}^+(\mathscr{A})$ and the corresponding increasing approximations via positive simple functions f_k, g_k:

$$\begin{aligned} N(u+v)(x) &= \sup_k N(f_k + g_k)(x) \\ &= \sup_k Nf_k(x) + \sup_k Ng_k(x) \\ &= Nu(x) + Nv(x). \end{aligned}$$

Moreover, $x \mapsto N\mathbb{1}_A(x) = N(x, A)$ is a measurable function, thus $Nf(x)$ is a measurable function for all simple $f \in \mathcal{E}^+(\mathscr{A})$ and, by Beppo Levi (see above) $Nu(x)$, $u \in \mathcal{M}^+(\mathscr{A})$, is for every x an increasing limit of measurable functions $Nf_k(x)$. Therefore, $Nu \in \mathcal{M}^+(\mathscr{A})$.

(iii) If $u = \mathbb{1}_A$, $A \in \mathscr{A}$, we have

$$\begin{aligned} \int \mathbb{1}_A(y)\, \mu N(dy) = \mu N(A) &= \int N(x, A)\, \mu(dx) \\ &= \int N\mathbb{1}_A(x)\, \mu(dx). \end{aligned}$$

By linearity this carries over to $f \in \mathcal{E}^+(\mathscr{A})$ and, by a Beppo Levi-argument, to $u \in \mathcal{M}^+(\mathscr{A})$.

Problem 9.14 Solution: Put

$$\nu(A) := \int u \cdot \mathbb{1}_{A_\sigma^+}\, d\mu + \int (1-u) \cdot \mathbb{1}_{A_\sigma^-}\, d\mu.$$

If A is symmetric w.r.t. the origin, $A^+ = -A^-$ and $A_\sigma^\pm = A$. Therefore,

$$\nu(A) = \int u \cdot \mathbb{1}_A\, d\mu + \int (1-u) \cdot \mathbb{1}_A\, d\mu = \int \mathbb{1}_A\, d\mu = \mu(A).$$

This means that ν extends μ. It also shows that $\nu(\emptyset) = 0$. Since ν is defined for all sets from $\mathscr{B}(\mathbb{R})$ and since ν has values in $[0, \infty]$, it is enough to check σ-additivity.

For this, let $(A_n)_n \subset \mathscr{B}(\mathbb{R})$ be a sequence of pairwise disjoint sets. From the definitions it is clear that the sets $(A_n)_\sigma^\pm$ are again pairwise disjoint and that $\biguplus_n (A_n)_\sigma^\pm = \left(\biguplus_n A_n\right)_\sigma^\pm$. Since each of the set functions

$$B \mapsto \int u \cdot \mathbb{1}_B\, d\mu, \qquad C \mapsto \int (1-u) \cdot \mathbb{1}_C\, d\mu$$

is σ-additive, it is clear that their sum ν will be σ-additive, too.

The obvious non-uniqueness of the extension does not contradict the uniqueness theorem for extensions, since Σ does not generate $\mathscr{B}(\mathbb{R})$!

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10 Integrals of measurable functions. Solutions to Problems 10.1–10.9

Problem 10.1 Solution: Let u, v be integrable functions and $a, b \in \mathbb{R}$. Assume that either u, v are real-valued or that $au + bv$ makes sense (i.e. avoiding the case '$\infty - \infty$'). Then we have

$$|au + bv| \leqslant |au| + |bv| = |a| \cdot |u| + |b| \cdot |v| \leqslant K(|u| + |v|)$$

with $K = \max\{|a|, |b|\}$. Since the RHS is integrable (because of Theorem 10.3 and Properties 9.8) we have that $au + bv$ is integrable by Theorem 10.3. So we get from Theorem 10.4 that

$$\int (au + bv)\,d\mu = \int au\,d\mu + \int bv\,d\mu = a \int u\,d\mu + b \int v\,d\mu$$

and this is what was claimed.

Problem 10.2 Solution: Without loss of generality we consider u on $(0, 1]$ (otherwise we have to single out the point $x = 1$, and this is just awkward in the notation...) We follow the hint and show first that $u(x) := x^{-1/2}$, $0 < x \leqslant 1$, is Lebesgue integrable. The idea here is to construct a sequence of simple functions approximating u from below. Set $x_i = \left(\frac{i}{n}\right)^2$, $i = 0, 1, \dots, n$ and

$$u_n(x) := \sum_{i=0}^{n-1} u(x_{i+1}) \mathbb{1}_{(x_i, x_{i+1}]}(x) = \sum_{i=0}^{n-1} \frac{n}{i+1} \mathbb{1}_{(x_i, x_{i+1}]}(x)$$

This is clearly a simple function. Also $u_n \leqslant u$ and $\lim_{n\to\infty} u_n(x) = \sup_n u_n(x) = u(x)$ for all x. Since $P(A)$ is just $\lambda(A \cap (0, 1])$, the integral of u_n is given by

$$\begin{aligned}
\int u_n\,dP = I_P(u_n) &= \sum_{i=0}^{n-1} \frac{n}{i+1} \left[\left(\frac{i+1}{n}\right)^2 - \left(\frac{i}{n}\right)^2\right] \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{i+1} \left[(i+1)^2 - i^2\right] = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{i+1} [2i + 1] \\
&\leqslant \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{i+1} [2i + 2] = \frac{1}{n} \cdot 2n = 2
\end{aligned}$$

and is thus finite, even uniformly in n. So, Beppo Levi's theorem tells us that

$$\int u\,dP = \sup_n \int u_n\,dP \leqslant \sup_n 2 = 2 < \infty$$

showing integrability.

Now u is clearly not bounded but integrable.

Problem 10.3 Solution: Clearly, ν is defined on $\mathscr{A}$ and takes values in $[0,\infty]$. Since $\mathbb{1}_\emptyset \equiv 0$ we have

$$\nu(\emptyset) = \int \mathbb{1}_\emptyset \cdot u\,d\mu = \int 0\,d\mu = 0.$$

If $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ are mutually disjoint measurable sets, we get

$$\begin{aligned}\nu\Big(\bigcupdot_{n=1}^\infty A_n\Big) &= \int \mathbb{1}_{\bigcupdot_{n=1}^\infty A_n} \cdot u\,d\mu\\ &= \int \sum_{n=1}^\infty \mathbb{1}_{A_n} \cdot u\,d\mu\\ &= \sum_{n=1}^\infty \int \mathbb{1}_{A_n} \cdot u\,d\mu \;=\; \sum_{n=1}^\infty \nu(A_n)\end{aligned}$$

which proves σ-additivity.

Problem 10.4 Solution: '$\Rightarrow$': since the A_j are disjoint we get the identities

$$\mathbb{1}_{\bigcupdot_j A_j} = \sum_{k=1}^\infty \mathbb{1}_{A_j} \qquad \text{and so} \qquad u\cdot\mathbb{1}_{\bigcupdot_j A_j} = \sum_{k=1}^\infty u\cdot\mathbb{1}_{A_j},$$

hence $|u\mathbb{1}_{A_n}| = |u|\mathbb{1}_{A_n} \leqslant |u|\mathbb{1}_{\bigcupdot_j A_j} = |u\mathbb{1}_{\bigcupdot_j A_j}|$ showing the integrability of each $u\mathbb{1}_{A_n}$ by Theorem 10.3. By a Beppo Levi argument (Theorem 9.6) or, directly, by Corollary 9.9 we get

$$\begin{aligned}\sum_{j=1}^\infty \int_{A_j} |u|\,d\mu = \sum_{j=1}^\infty \int |u|\mathbb{1}_{A_j}\,d\mu &= \int \sum_{j=1}^\infty |u|\mathbb{1}_{A_j}\,d\mu\\ &= \int |u|\mathbb{1}_{\bigcupdot_j A_j}\,d\mu \;<\; \infty.\end{aligned}$$

The converse direction '$\Leftarrow$' follows again from Corollary 9.9, now just the other way round:

$$\begin{aligned}\int |u|\mathbb{1}_{\bigcupdot_j A_j}\,d\mu = \int \sum_{j=1}^\infty |u|\mathbb{1}_{A_j}\,d\mu &= \sum_{j=1}^\infty \int |u|\mathbb{1}_{A_j}\,d\mu\\ &= \sum_{j=1}^\infty \int_{A_j} |u|\,d\mu \;<\; \infty\end{aligned}$$

showing that $u\mathbb{1}_{\bigcupdot_j A_j}$ is integrable.

Problem 10.5 Solution: For any measurable function u we have $u \in \mathcal{L}^1(\mu) \iff |u| \in \mathcal{L}^1(\mu)$. This means that we may assume that $u \geqslant 0$. Since

$$\sum_{n=-k}^{k} \mathbb{1}_{\{2^n \leqslant u < 2^{n+1}\}} u \uparrow u\mathbb{1}_{\{u>0\}}$$

we can use Beppo Levi's theorem to conclude

$$\int u\,d\mu = \int_{\{u>0\}} u\,d\mu = \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} u\,d\mu.$$

Because of the monotonicity of the integral,

$$C := \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} 2^n\,d\mu \leqslant \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} u\,d\mu \leqslant \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} 2^{n+1}\,d\mu,$$

i.e.

$$C \leqslant \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} u\,d\mu \leqslant 2C.$$

Therefore the following assertions are equivalent:

$$\begin{aligned} u\in\mathcal{L}^1(\mu) &\iff \sum_{n\in\mathbb{Z}} \int_{\{2^n\leqslant u<2^{n+1}\}} u\,d\mu < \infty \\ &\iff C = \sum_{n\in\mathbb{Z}} 2^n \mu\{2^n \leqslant u < 2^{n+1}\} < \infty. \end{aligned}$$

Problem 10.6 Solution: Let us show the following inequalities:

$$\sum_{i=1}^\infty \mathbb{1}_{\{|u|\geqslant i\}}(x) \leqslant |u(x)| \leqslant \sum_{i=0}^\infty \mathbb{1}_{\{|u|\geqslant i\}}(x) \quad \forall x\in X.$$

First proof:

$$\sum_{i=1}^\infty \mathbb{1}_{\{|u|\geqslant i\}} = \sum_{i=1}^\infty \sum_{k=i}^\infty \mathbb{1}_{\{k+1>|u|\geqslant k\}} = \sum_{k=1}^\infty \sum_{i=1}^k \mathbb{1}_{\{k+1>|u|\geqslant k\}} = \sum_{k=1}^\infty k\mathbb{1}_{\{k+1>|u|\geqslant k\}}$$

and

$$\sum_{k=1}^\infty k\mathbb{1}_{\{k+1>|u|\geqslant k\}} \leqslant \sum_{k=1}^\infty |u|\mathbb{1}_{\{k+1>|u|\geqslant k\}} = |u|\mathbb{1}_{\{|u|\geqslant 1\}}$$

and

$$\sum_{k=1}^\infty k\mathbb{1}_{\{k+1>|u|\geqslant k\}} \geqslant \sum_{k=1}^\infty (|u|-1)\mathbb{1}_{\{k+1>|u|\geqslant k\}} = (|u|-1)\mathbb{1}_{\{|u|\geqslant 1\}} \geqslant |u|\mathbb{1}_{\{|u|\geqslant 1\}} - \mathbb{1}_{\{|u|\geqslant 0\}}.$$

So,

$$\sum_{i=1}^\infty \mathbb{1}_{\{|u|\geqslant i\}} \leqslant |u|\mathbb{1}_{\{|u|\geqslant 1\}} \leqslant |u| \leqslant 1 + \sum_{i=1}^\infty \mathbb{1}_{\{|u|\geqslant i\}} = \sum_{i=0}^\infty \mathbb{1}_{\{|u|\geqslant i\}}.$$

Second proof: For $x\in X$, there is some $k\in\mathbb{N}_0$ such that $k\leqslant |u(x)| < k+1$. Therefore,

$$x\in\{|u|\geqslant i\} \quad \forall i\in\{0,\dots,k\}$$

and

$$x\notin\{|u|\geqslant i\} \quad \forall i\geqslant k+1.$$

Thus,

$$\sum_{i\in\mathbb{N}_0} \mathbb{1}_{\{|u|\geqslant i\}}(x) = k+1.$$

Since $k \leqslant |u(x)| \leqslant k+1$ we get

$$\sum_{i\in\mathbb{N}_0} \mathbb{1}_{\{|u|\geqslant i\}}(x) = k+1 \geqslant |u(x)| \geqslant k = (k+1)-1 = \left(\sum_{i\in\mathbb{N}_0} \mathbb{1}_{\{|u|\geqslant i\}}(x)\right) - 1.$$

As $1 = \mathbb{1}_{\{|u|\geqslant 0\}}$ ($u \geqslant 0$, by assumption) we get the claimed estimates.

Integrating these inequalities we get

$$\sum_{i=1}^{\infty} \mu\{|u| \geqslant i\} \leqslant \int |u|\,d\mu \leqslant \sum_{i=0}^{\infty} \mu\{|u| \geqslant i\},$$

and (ii) follows. If $u \in \mathcal{L}^1(\mu)$, then we get $\sum_{i\geqslant 1} \mu(|u| \geqslant 1) < \infty$. On the other hand, if u is measurable, and $\sum_i \mu(|u| \geqslant i) < \infty$, then we get $\int |u|\,d\mu < \infty$, i.e. $u \in \mathcal{L}^1(\mu)$ and (i) follows.

The finiteness of the measure μ was only used for $\int 1\,d\mu < \infty$ or $\mu\{|u| \geqslant 0\} < \infty$ – which is only needed for the second estimate in (ii). Hence, the lower estimate in (ii) holds for all measures!

Problem 10.7 Solution: One possibility to solve the problem is to follow the hint. We provide an alternative (and shorter) solution.

(i) Observe that $u_j - v \geqslant 0$ is a sequence of positive and integrable functions. Applying Fatou's lemma (in the usual form) yields (observing the rules for $\liminf$, $\limsup$ from Appendix A, compare also Problem 9.10):

$$\begin{aligned}\int \liminf_j u_j\,d\mu - \int v\,d\mu &= \int \liminf_j (u_j - v)\,d\mu\\ &\leqslant \liminf_j \int (u_j - v)\,d\mu\\ &= \liminf_j \int u_j\,d\mu - \int v\,d\mu\end{aligned}$$

and the claim follows upon subtraction of the *finite (!)* number $\int v\,d\mu$.

(ii) Very similar to (i) by applying Fatou's lemma to the positive, integrable functions $w - u_j \geqslant 0$:

$$\begin{aligned}\int w\,d\mu - \int \limsup_j u_j\,d\mu &= \int \liminf_j (w - u_j)\,d\mu\\ &\leqslant \liminf_j \int (w - u_j)\,d\mu\\ &= \int w\,d\mu - \limsup_j \int u_j\,d\mu\end{aligned}$$

Now subtract the finite number $\int w\,d\mu$ on both sides.

(iii) We had the counterexample, in principle, already in Problem 9.10. Nevertheless...

Consider Lebesgue measure on $\mathbb{R}$. Put $f_j(x) = -\mathbb{1}_{[-2j,-j]}(x)$ and $g_j(x) = \mathbb{1}_{[j,2j]}(x)$.

Then $\liminf f_j(x) = 0$ and $\limsup g_j(x) = 0$ for every x and neither admits an integrable minorant resp. majorant.

Remark. Here is an even stronger version of Fatou's Lemma. For this we introduced the extended integrable functions

$$\begin{aligned}
\mathcal{L}^1(\mu) &:= \left\{ u \in \mathcal{M}(\mathscr{A}) : \int u^+ \, d\mu < \infty, \int u^- \, d\mu < \infty \right\} \\
\mathcal{L}^{1,e}(\mu) &:= \left\{ u \in \mathcal{M}(\mathscr{A}) : \int u^+ \, d\mu \in [0,\infty], \int u^- \, d\mu < \infty \right\}.
\end{aligned}$$

For $u \in \mathcal{L}^1(\mu)$ or $u \in \mathcal{L}^{1,e}(\mu)$ we may define $\int u \, d\mu = \int u^+ \, d\mu - \int u^- \, d\mu$ in $\mathbb{R}$ or $\mathbb{R} \cup \{+\infty\}$, respectively. Note that $\mathcal{L}^{1,e}(\mu)$ is not a vector space, but it is still additive and positively homogeneous. Then we have

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mathscr{A})$ such that $u_n \geqslant u$ for some $u \in \mathcal{L}^{1,e}(\mu)$.

i) $\liminf_{n\to\infty} u_n \in \mathcal{L}^{1,e}(\mu)$;

ii) $\liminf_{n\to\infty} \int u_n \, d\mu \geqslant \int \liminf_{n\to\infty} u_n \, d\mu$;

iii) *if* $\liminf_{n\to\infty} \int u_n \, d\mu < \infty$, *then* $\liminf_{n\to\infty} u_n \in \mathcal{L}^1(\mu)$.

Proof. i) We have

$$u_n \geqslant u \implies \liminf_n u_n \geqslant u \implies \begin{cases} \left(\liminf_n u_n\right)^+ \geqslant u^+ \\ \left(\liminf_n u_n\right)^- \leqslant u^- \end{cases}$$

and so $\int \left(\liminf_n u_n\right)^- \, d\mu \leqslant \int u^- \, d\mu < \infty$, i.e. $\liminf_n u_n \in \mathcal{L}^{1,e}(\mu)$.

ii) Note that $u_n - u \geqslant 0$. By (the ordinary) Fatou's lemma,

$$\liminf_n \int (u_n - u) \, d\mu \geqslant \int \liminf_n (u_n - u) \, d\mu.$$

Adding on both sides $\int u \, d\mu$ – this is possible since we do not get an expression of type "$\infty - \infty$", we get

$$\liminf_n \int u_n \, d\mu \geqslant \int \liminf_n u_n \, d\mu.$$

iii) We have

$$\begin{aligned}
\int \left(\liminf_n u_n\right)^+ \, d\mu &= \int \liminf_n u_n + \left(\liminf_n u_n\right)^- \, d\mu \\
&\leqslant \int \liminf_n u_n + u^- \, d\mu \\
&= \int \liminf_n u_n \, d\mu + \int u^- \, d\mu
\end{aligned}$$

$$\leqslant \liminf_n \int u_n \, d\mu + \int u^- \, d\mu < \infty.$$

This proves the claim. (Note that in the inequality-step in the last formula we could have used directly the ordinary Fatou lemma, and not step ii), as $u_n + u^- \geqslant 0$). ☐

Problem 10.8 Solution: For $u = \mathbb{1}_B$ and $v = \mathbb{1}_C$ we have, because of independence,

$$\int uv \, dP = P(A \cap B) = P(A)P(B) = \int u \, dP \int v \, dP.$$

For positive, simple functions $u = \sum_j \alpha_j \mathbb{1}_{B_j}$ and $v = \sum_k \beta_k \mathbb{1}_{C_k}$ we find

$$\begin{aligned}
\int uv \, dP &= \sum_{j,k} \alpha_j \beta_k \int \mathbb{1}_{A_j} \mathbb{1}_{B_k} \, dP \\
&= \sum_{j,k} \alpha_j \beta_k P(A_j \cap B_k) \\
&= \sum_{j,k} \alpha_j \beta_k P(A_j) P(B_k) \\
&= \Big(\sum_j \alpha_j P(A_j) \Big) \Big(\sum_k \beta_k P(B_k) \Big) \\
&= \int u \, dP \int v \, dP.
\end{aligned}$$

For measurable $u \in \mathcal{M}^+(\mathscr{B})$ and $v \in \mathcal{M}^+(\mathscr{C})$ we use approximating simple functions $u_k \in \mathcal{E}^+(\mathscr{B})$, $u_k \uparrow u$, and $v_k \in \mathcal{E}^+(\mathscr{C})$, $v_k \uparrow v$. Then, by Beppo Levi,

$$\begin{aligned}
\int uv \, dP = \lim_k \int u_k v_k \, dP &= \lim_k \int u_k \, dP \lim_j \int v_j \, dP \\
&= \int u \, dP \int v \, dP.
\end{aligned}$$

Integrable independent functions: If $u \in \mathcal{L}^1(\mathscr{B})$ and $v \in \mathcal{L}^1(\mathscr{C})$, the above calculation when applied to $|u|, |v|$ shows that $u \cdot v$ is integrable since

$$\int |uv| \, dP \leqslant \int |u| \, dP \int |v| \, dP < \infty.$$

Considering positive and negative parts finally also gives

$$\int uv \, dP = \int u \, dP \int v \, dP.$$

Counterexample: Just take $u = v$ which are integrable but not square integrable, e.g. $u(x) = v(x) = x^{-1/2}$. Then $\int_{(0,1)} x^{-1/2} \, dx < \infty$ but $\int_{(0,1)} x^{-1} \, dx = \infty$, compare also Problem 10.2.

Problem 10.8 Solution:

(i) Since the map $g : \mathbb{C} \to \mathbb{R}^2$ is continuous, we have $g^{-1}(\mathscr{B}(\mathbb{R}^2)) \subset \mathscr{B}(\mathbb{C})$.

On the other hand, for $z \in \mathbb{C}$ and $\epsilon > 0$ we have $B_\epsilon(z) = g^{-1}(B_{g(z)}(\epsilon)) \in g^{-1}(\mathscr{B}(\mathbb{R}^2))$; thus, $\sigma(\mathscr{O}_\mathbb{C}) \subset g^{-1}(\mathscr{B}(\mathbb{R}^2))$ (Note that the σ-algebra $\sigma(\mathscr{O}_\mathbb{C})$ is generated by the open balls $B_\epsilon(z)$, $z \in \mathbb{C}$, $\epsilon > 0$, cf. the proof of Problem 3.12.)

(ii) Part (i) shows that a map $h : E \to \mathbb{C}$ is $\mathscr{A}/\mathscr{C}$-measurable if, and only if, $g \circ h : E \to \mathbb{R}^2$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^2)$-measurable.

Indeed: The map $h : (E, \mathscr{A}) \to (\mathbb{C}, \mathscr{C})$ is, by definition, measureable if $h^{-1}(A) \in \mathscr{A}$ for all $A \in \mathscr{C}$. Since $\mathscr{C} = g^{-1}(\mathscr{B}(\mathbb{R}^2))$, this is the same as $h^{-1}(g^{-1}(B)) = (g \circ h)^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}(\mathbb{R}^2)$, hence it is the same as the measurability of $g \circ h$.

"$\Rightarrow$": Assume that $h : E \to \mathbb{C}$ is $\mathscr{A}/\mathscr{C}$-measurable. Then we have that

$$(g \circ h) = \begin{pmatrix} \operatorname{Re} h \\ \operatorname{Im} h \end{pmatrix}$$

is $\mathscr{A}/\mathscr{B}(\mathbb{R}^2)$-measurable. Since the projections $\pi_j : \mathbb{R}^2 \ni (x_1, x_2) \mapsto x_j \in \mathbb{R}$ are Borel measurable (due to continuity!), we get that $\operatorname{Re} h = \pi_1(g \circ h)$ and $\operatorname{Im} h = \pi_2(g \circ h)$ are measurable (composition of measurable functions).

"$\Leftarrow$": Assume that $\operatorname{Re} h$ and $\operatorname{Im} h$ are $\mathscr{A}/\mathscr{B}(\mathbb{R})$-measurable. Then the map $(g \circ h) = (\operatorname{Re} h, \operatorname{Im} h)$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^2)$-measurable. With the above arguments we conclude that $h : (E, \mathscr{A}) \to (\mathbb{C}, \mathscr{C})$ is measurable.

(iii) We show first additivity: let $g, h \in \mathcal{L}^1_\mathbb{C}(\mu)$. From

$$|\operatorname{Re}(g+h)| \leqslant |\operatorname{Re} g| + |\operatorname{Re} h| \in \mathcal{L}^1(\mu), \qquad |\operatorname{Im}(g+h)| \leqslant |\operatorname{Im}(g)| + |\operatorname{Im}(h)| \in \mathcal{L}^1(\mu)$$

we conclude that $g + h \in \mathcal{L}^1(\mu)$. Since $\operatorname{Re}(g + h) = \operatorname{Re}(g) + \operatorname{Re}(h)$ and $\operatorname{Im}(g + h) = \operatorname{Im}(g) + \operatorname{Im}(h)$, we get from the definition of the integral

$$\begin{aligned}
\int (g+h)\,d\mu &= \int \operatorname{Re}(g+h)\,d\mu + i \int \operatorname{Im}(g+h)\,d\mu \\
&= \int (\operatorname{Re}(g) + \operatorname{Re}(h))\,d\mu + i(\operatorname{Im}(g) + \operatorname{Im}(h))\,d\mu \\
&= \int \operatorname{Re}(g)\,d\mu + \int \operatorname{Re}(h)\,d\mu + i \int \operatorname{Im}(g)\,d\mu + i \int \operatorname{Im}(h)\,d\mu \\
&= \left(\int \operatorname{Re}(g)\,d\mu + i \int \operatorname{Im}(g)\,d\mu \right) + \left(\int \operatorname{Re}(h)\,d\mu + i \int \operatorname{Im}(h)\,d\mu \right) \\
&= \int g\,d\mu + \int h\,d\mu.
\end{aligned}$$

Note that we have used the $\mathbb{R}$-linearity of the integral for real-valued functions. The homogeneity of the complex integral is shown in a very similar way.

(iv) Since $\operatorname{Re} h$ and $\operatorname{Im} h$ are real, we get $\int \operatorname{Re} h\,d\mu \in \mathbb{R}$ and $\int \operatorname{Im} h\,d\mu \in \mathbb{R}$. Therefore,

$$\operatorname{Re}\left(\int h\,d\mu \right) = \operatorname{Re}\left(\int \operatorname{Re} h\,d\mu + i \int \operatorname{Im} h\,d\mu \right)$$

$$= \int \operatorname{Re} h \, d\mu.$$

Similarly, we see

$$\begin{aligned} \operatorname{Im}\left(\int h \, d\mu\right) &= \operatorname{Im}\left(\int \operatorname{Re} h \, d\mu + i \int \operatorname{Im} h \, d\mu\right) \\ &= \int \operatorname{Im} h \, d\mu. \end{aligned}$$

(v) We follow the hint: as $\int h \, d\mu \in \mathbb{C}$ we can pick some $\theta \in (-\pi, \pi]$ such that $e^{i\theta} \int h \, d\mu \geqslant 0$. Thus, (iii) and (iv) entail

$$\begin{aligned} \left|\int h \, d\mu\right| &= e^{i\theta} \int h \, d\mu \\ &= \operatorname{Re}\left(e^{i\theta} \int h \, d\mu\right) \\ &= \int \operatorname{Re}(e^{i\theta} h) \, d\mu \\ &\leqslant \int |e^{i\theta} h| \, d\mu \\ &= \int |h| \, d\mu. \end{aligned}$$

(vi) We know from (ii) that $h : (E, \mathscr{A}) \to (\mathbb{C}, \mathscr{C})$ is measurable if, and only if, $\operatorname{Re} h$ and $\operatorname{Im} h$ are $\mathscr{A}/\mathscr{B}(\mathbb{R}^2)$-measurable. If $\operatorname{Re} h$ and $\operatorname{Im} h$ are μ-integrable, then so is

$$|h| = \sqrt{(\operatorname{Re} h)^2 + (\operatorname{Im} h)^2} \leqslant |\operatorname{Re} h| + |\operatorname{Im} h|.$$

If $|h| \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, then we conclude from $|\operatorname{Re} h| \leqslant |h|$ and $|\operatorname{Im} h| \leqslant |h|$, that $\operatorname{Re} h$ and $\operatorname{Im} h$ are μ-integrable.

■■

11 Null sets and the 'almost everywhere'. Solutions to Problems 11.1–11.12

Problem 11.1 Solution: True, we can change an integrable function on a null set, even by setting it to the value $+\infty$ or $-\infty$ *on the null set*. This is just the assertion of Theorem 11.2 and its Corollaries 11.3, 11.4.

■■

Problem 11.2 Solution: We have seen that a single point is a Lebesgue null set: $\{x\} \in \mathscr{B}(\mathbb{R})$ for all $x \in \mathbb{R}$ and $\lambda(\{x\}) = 0$, see e.g. Problems 4.13 and 6.7. If N is countable, we know that $N = \{x_j : j \in \mathbb{N}\} = \dot{\bigcup}_{j\in\mathbb{N}}\{x_j\}$ and by the σ-additivity of measures

$$\lambda(N) = \lambda\left(\dot{\bigcup_{j\in\mathbb{N}}}\{x_j\}\right) = \sum_{j\in\mathbb{N}} \lambda\left(\{x_j\}\right) = \sum_{j\in\mathbb{N}} 0 = 0.$$

The Cantor set C from Problem 7.12 is, as we have seen, uncountable but has measure $\lambda(C) = 0$. This means that there are uncountable sets with measure zero.

In $\mathbb{R}^2$ and for two-dimensional Lebesgue measure λ^2 the situation is even easier: every line L in the plane has zero Lebesgue measure and L contains certainly uncountably many points. That $\lambda^2(L) = 0$ is seen from the fact that L differs from the ordinate $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ only by a rigid motion T which leaves Lebesgue measure invariant (see Chapter 4, Theorem 4.7) and $\lambda^2(\{x = 0\}) = 0$ as seen in Problem 6.7.

■■

Problem 11.3 Solution:

(i) Since $\{|u| > c\} \subset \{|u| \geqslant c\}$ and, therefore, $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$, this follows immediately from Proposition 11.5. Alternatively, one could also mimic the proof of this Proposition or use part (iii) of the present problem with $\phi(t) = t$, $t \geqslant 0$.

(ii) This will follow from (iii) with $\phi(t) = t^p$, $t \geqslant 0$, since $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$ as $\{|u| > c\} \subset \{|u| \geqslant c\}$.

(iii) We have, since ϕ is increasing,

$$\begin{aligned}\mu(\{|u| \geqslant c\}) &= \mu(\{\phi(|u|) \geqslant \phi(c)\}) \\ &= \int \mathbb{1}_{\{x : \phi(|u(x)|) \geqslant \phi(c)\}}(x)\, \mu(dx)\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{\phi(|u(x)|)}{\phi(|u(x)|)} \mathbb{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)} \mathbb{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)}\,\mu(dx)\\
&= \frac{1}{\phi(c)} \int \phi(|u(x)|)\,\mu(dx)
\end{aligned}
$$

(iv) Let us set $b = \alpha \int u\,d\mu$. Then we follow the argument of (iii), where we use that u and b are strictly positive.

$$
\begin{aligned}
\mu(\{u \geqslant b\}) &= \int \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&= \int \frac{u(x)}{u(x)} \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{u(x)}{b} \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{u}{b}\,d\mu\\
&= \frac{1}{b} \int u\,d\mu
\end{aligned}
$$

and substituting $\alpha \int u\,d\mu$ for b shows the inequality.

(v) Using the fact that ψ is decreasing we get $\{|u| < c\} = \{\psi(|u|) > \psi(c)\}$—mind the change of the inequality sign—and going through the proof of part (iii) again we use there that ϕ increases only in the first step in a similar role as we used the decrease of ψ here! This means that the argument of (iii) is valid after this step and we get, altogether,

$$
\begin{aligned}
\mu(\{|u| < c\}) &= \mu(\{\psi(|u|) > \psi(c)\})\\
&= \int \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx)\\
&= \int \frac{\psi(|u(x)|)}{\psi(|u(x)|)} \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\phi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)} \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx)\\
&\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)}\,\mu(dx)\\
&= \frac{1}{\psi(c)} \int \psi(|u(x)|)\,\mu(dx)
\end{aligned}
$$

(vi) This follows immediately from (ii) by taking $\mu = \mathbb{P}$, $c = \alpha\sqrt{\mathbb{V}\xi}$, $u = \xi - \mathbb{E}\xi$ and $p = 2$. Then

$$
\begin{aligned}
\mathbb{P}\big(|\xi - \mathbb{E}\xi| \geqslant \alpha\sqrt{\mathbb{V}\xi}\big) &\leqslant \frac{1}{\big(\alpha\sqrt{\mathbb{V}\xi}\big)^2} \int |\xi - \mathbb{E}\xi|^2\,d\mathbb{P}\\
&= \frac{1}{\alpha^2\mathbb{V}\xi}\,\mathbb{V}\xi \;=\; \frac{1}{\alpha^2}.
\end{aligned}
$$

Problem 11.4 Solution: We mimic the proof of Corollary 11.6. Set $N = \{|u| = \infty\} = \{|u|^p = \infty\}$. Then $N = \bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}$ and using Markov's inequality (MI) and the 'continuity' of measures, Proposition 4.3(vii), we find

$$\mu(N) = \mu\left(\bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}\right) \overset{4.3(\mathrm{vii})}{=} \lim_{k\to\infty}\mu(\{|u|^p \geqslant k\})$$
$$\overset{\mathrm{MI}}{\leqslant} \lim_{k\to\infty}\frac{1}{k}\underbrace{\int |u|^p\,d\mu}_{<\infty} = 0.$$

For arctan this is not any longer true for several reasons:

- ... arctan is odd and changes sign, so there could be cancelations under the integral.
- ... even if we had no cancelations we have the problem that the points where $u(x) = \infty$ are now transformed to points where $\arctan(u(x)) = \frac{\pi}{2}$ and we do not know how the measure μ acts under this transformation. A simple example: Take μ to be a measure of total finite mass (that is: $\mu(X) < \infty$), e.g. a probability measure, and take the function $u(x)$ which is constantly $u \equiv +\infty$. Then $\arctan(u(x)) = \frac{\pi}{2}$ throughout, and we get

$$\int \arctan u(x)\,\mu(dx) = \int \frac{\pi}{2}\,d\mu = \frac{\pi}{2}\int d\mu = \frac{\pi}{2}\,\mu(X) < \infty,$$

  but u is *nowhere* finite!

Problem 11.5 Solution:

(i) Assume that f^* is $\overline{\mathscr{A}}$-measurable. The problem at hand is to construct $\mathscr{A}$-measurable upper and lower functions g and f. For positive simple functions this is clear: if $f^*(x) = \sum_{j=0}^N \phi_j \mathbb{1}_{B_j^*}(x)$ with $\phi_j \geqslant 0$ and $B_j^* \in \overline{\mathscr{A}}$, then we can use Problem 4.15(v) to find $B_j, C_j \in \mathscr{A}$ with $\mu(C_j \setminus B_j) = 0$

$$B_j \subset B_j^* \subset C_j \implies \phi_j\mathbb{1}_{B_j} \leqslant \phi_j\mathbb{1}_{B_j^*} \leqslant \phi_j\mathbb{1}_{C_j}$$

and summing over $j = 0, 1, \dots, N$ shows that $f \leqslant f^* \leqslant g$ where f, g are the appropriate lower and upper sums which are clearly $\mathscr{A}$ measurable and satisfy

$$\begin{aligned}\mu(\{f \neq g\}) &\leqslant \mu(C_0 \setminus B_0 \cup \dots \cup C_N \setminus B_N)\\ &\leqslant \mu(C_0 \setminus B_0) + \dots + \mu(C_N \setminus B_N)\\ &= 0 + \dots + 0 = 0.\end{aligned}$$

Moreover, since by Problem 4.15 $\mu(B_j) = \mu(C_j) = \bar{\mu}(B_j^*)$, we have

$$\sum_j \phi_j\mu(B_j) = \sum_j \phi_j\bar{\mu}(B_j^*) = \sum_j \phi_j\mu(C_j)$$

which is the same as

$$\int f\,d\mu = \int f^*\,d\bar\mu = \int g\,d\mu.$$

(ii), (iii) Assume that u^* is $\mathscr{A}^*$-measurable; without loss of generality (otherwise consider positive and negative parts) we can assume that $u^* \geqslant 0$. Because of Theorem 8.8 we know that $f_k^* \uparrow u^*$ for $f_k^* \in \mathcal{E}^+(\mathscr{A}^*)$. Now choose the corresponding $\mathscr{A}$-measurable lower and upper functions f_k, g_k constructed in part (i). By considering, if necessary, $\max\{f_1, \dots, f_k\}$ we can assume that the f_k are increasing.

Set $u := \sup_k f_k$ and $v := \liminf_k g_k$. Then $u, v \in \mathcal{M}(\mathscr{A})$, $u \leqslant u^* \leqslant v$, and by Fatou's lemma

$$\begin{aligned}
\int v\,d\mu = \int \liminf_k g_k\,d\mu &\leqslant \liminf_k \int g_k\,d\mu \\
&= \liminf_k \int f_k^*\,d\bar\mu \\
&= \int u^*\,d\bar\mu \\
&\leqslant \int v\,d\mu.
\end{aligned}$$

Since $f_k \uparrow u$ we get by Beppo Levi and Fatou

$$\begin{aligned}
\int u\,d\mu = \sup_k \int f_k\,d\mu = \liminf_k \int f_k\,d\mu &\\
= \liminf_k \int g_k\,d\mu &\\
\geqslant \int \liminf_k g_k\,d\mu &\\
= \int v\,d\mu &\\
\geqslant \int u\,d\mu &
\end{aligned}$$

This proves that $\int u\,d\mu = \int v\,d\mu = \int u^*\,d\mu$. This answers part (iii) by considering positive and negative parts.

It remains to show that $\{u \neq v\}$ is a μ-null set. (This does not follow from the above integral equality, cf. Problem 11.10!) Clearly, $\{u \neq v\} = \{u < v\}$, i.e. if $x \in \{u < v\}$ is fixed, we deduce that, for sufficiently large values of k,

$$f_k(x) < g_k(x), \quad k \text{ large}$$

since $u = \sup f_k$ and $v = \liminf_k g_k$. Thus,

$$\{u \neq v\} \subset \bigcup_k \{f_k \neq g_k\}$$

but the RHS is a countable union of μ-null sets, hence a null set itself.

Conversely, assume first that $u \leqslant u^* \leqslant v$ for two $\mathscr{A}$-measurable functions u, v with $u = v$ a.e. We have to show that $\{u^* > \alpha\} \in \mathscr{A}^*$. Using that $u \leqslant u^* \leqslant v$ we find that

$$\{u > \alpha\} \subset \{u^* > \alpha\} \subset \{v > \alpha\}$$

but $\{v > \alpha\}, \{u > \alpha\} \in \mathscr{A}$ and $\{u > \alpha\} \setminus \{v > \alpha\} \subset \{u \neq v\}$ is a μ-null set. Because of Problem 4.15 we conclude that $\{u^* > \alpha\} \in \mathscr{A}^*$.

Problem 11.6 Solution: Throughout the solution the letters A, B are reserved for sets from $\mathscr{A}$.

(i) a) Let $A \subset E \subset B$. Then $\mu(A) \leqslant \mu(B)$ and going to the $\sup_{A \subset E}$ and $\inf_{E \subset B}$ proves $\mu_*(E) \leqslant \mu^*(E)$.

b) By the definition of μ_* and μ^* we find some $A \subset E$ such that

$$|\mu_*(E) - \mu(A)| \leqslant \epsilon.$$

Since $A^c \supset E^c$ we can enlarge A, if needed, and achieve

$$|\mu^*(E^c) - \mu(A^c)| \leqslant \epsilon.$$

Thus,

$$\begin{aligned} &|\mu(X) - \mu_*(E) - \mu^*(E^c)| \\ &\qquad \leqslant |\mu_*(E) - \mu(A)| + |\mu^*(E^c) - \mu(A^c)| \\ &\qquad \leqslant 2\epsilon, \end{aligned}$$

and the claim follows as $\epsilon \to 0$.

c) Let $A \supset E$ and $B \supset F$ be arbitrary majorizing $\mathscr{A}$-sets. Then $A \cup B \supset E \cup F$ and

$$\mu^*(E \cup F) \leqslant \mu(A \cup B) \leqslant \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\inf_{A \supset E}$ and $\inf_{B \supset F}$, and obtain

$$\mu^*(E \cup F) \leqslant \mu^*(E) + \mu^*(F).$$

d) Let $A \subset E$ and $B \subset F$ be arbitrary minorizing $\mathscr{A}$-sets. Then $A \uplus B \subset E \uplus F$ and

$$\mu_*(E \uplus F) \geqslant \mu(A \uplus B) = \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\sup_{A \subset E}$ and $\sup_{B \subset F}$, where we stipulate that $A \cap B = \emptyset$, and obtain

$$\mu_*(E \uplus F) \geqslant \mu_*(E) + \mu_*(F).$$

(ii) By the definition of the infimum/supremum we find sets $A_n \subset E \subset A^n$ such that

$$|\mu_*(A) - \mu(A_n)| + |\mu^*(A) - \mu(A^n)| \leqslant \frac{1}{n}.$$

Without loss of generality we can assume that the A_n increase and that the A^n decrease. Now $A_* := \bigcup_n A_n$, $A^* := \bigcap_n A^n$ are $\mathscr{A}$-sets with $A_* \subset A \subset A^*$. Now, $\mu(A^n) \downarrow \mu(A^*)$ as well as $\mu(A^n) \to \mu^*(E)$ which proves $\mu(A^*) = \mu^*(E)$. Analogously, $\mu(A_n) \uparrow \mu(A_*)$ as well as $\mu(A_n) \to \mu_*(E)$ which proves $\mu(A_*) = \mu_*(E)$.

(iii) In view of Problem 4.15 and (i), (ii), it is clear that

$$\big\{E \subset X \,:\, \mu_*(E) = \mu^*(E)\big\} = \\ \big\{E \subset X \,:\, \exists\, A, B \in \mathscr{A},\ A \subset E \subset B,\ \mu(B \setminus A) = 0\big\}$$

but the latter is the completed σ-algebra $\mathscr{A}^*$. That $\mu^*\Big|_{\mathscr{A}^*} = \mu_*\Big|_{\mathscr{A}^*} = \bar{\mu}$ is now trivial since μ_* and μ^* coincide on $\mathscr{A}^*$.

■■

Problem 11.7 Solution: Let $A \in \mathscr{A}$ and assume that there are non-measurable sets, i.e. $\mathscr{P}(X) \supsetneq \mathscr{A}$. Take some $N \notin \mathscr{A}$ which is a μ-null set. Assume also that $N \cap A = \emptyset$. Then $u = \mathbb{1}_A$ and $w := \mathbb{1}_A + 2 \cdot \mathbb{1}_N$ are a.e. identical, but w is not measurable.

This means that w is only measurable if, e.g. all (subsets of) null sets are measurable, that is if $(X, \mathscr{A}, \mu)$ is complete.

■■

Problem 11.8 Solution: The function $\mathbb{1}_{\mathbb{Q}}$ is nowhere continuous but $u = 0$ Lebesgue almost everywhere. That is

$$\{x \,:\, \mathbb{1}_{\mathbb{Q}}(x) \text{ is discontinuous}\} = \mathbb{R}$$

while

$$\{x \,:\, \mathbb{1}_{\mathbb{Q}} \neq 0\} = \mathbb{Q} \text{ is a Lebesgue null set,}$$

that is $\mathbb{1}_{\mathbb{Q}}$ coincides a.e. with a continuous function but is itself at no point continuous!

The same analysis for $\mathbb{1}_{[0,\infty)}$ yields that

$$\{x \,:\, \mathbb{1}_{[0,\infty)}(x) \text{ is discontinuous}\} = \{0\}$$

which is a Lebesgue null set, but $\mathbb{1}_{[0,\infty)}$ cannot coincide a.e. with a continuous function! This, namely, would be of the form $w = 0$ on $(-\infty, -\delta)$ and $w = 1$ on (ϵ, ∞) while it 'interpolates' somehow between 0 and 1 if $-\delta < x < \epsilon$. But this entails that

$$\{x \,:\, w(x) \neq \mathbb{1}_{[0,\infty)}(x)\}$$

cannot be a Lebesgue null set!

■■

Problem 11.9 Solution: Let $(A_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ be an exhausting sequence $A_j \uparrow X$ such that $\mu(A_j) < \infty$. Set

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \mathbb{1}_{A_j}(x).$$

Then f is measurable, $f(x) > 0$ everywhere, and using Beppo Levi's theorem

$$\begin{aligned}\int f\,d\mu &= \int \left(\sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \mathbb{1}_{A_j} \right) d\mu \\ &= \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \int \mathbb{1}_{A_j}\,d\mu \\ &= \sum_{j=1}^{\infty} \frac{\mu(A_j)}{2^j(\mu(A_j)+1)} \\ &\leqslant \sum_{j=1}^{\infty} 2^{-j} \;=\; 1.\end{aligned}$$

Thus, set $P(A) := \int_A f\,d\mu$. We know from Problem 9.7 that P is indeed a measure.

If $N \in \mathscr{N}_\mu$, then, by Theorem 11.2,

$$P(N) = \int_N f\,d\mu \stackrel{11.2}{=} 0$$

so that $\mathscr{N}_\mu \subset \mathscr{N}_P$.

Conversely, if $M \in \mathcal{M}_P$, we see that

$$\int_M f\,d\mu = 0$$

but since $f > 0$ everywhere, it follows from Theorem 11.2 that $\mathbb{1}_M \cdot f = 0$ μ-a.e., i.e. $\mu(M) = 0$. Thus, $\mathscr{N}_P \subset \mathscr{N}_\mu$.

Remark. We will see later (cf. Chapter 20 or Chapter 25, Radon–Nikodým theorem) that $\mathscr{N}_\mu = \mathscr{N}_P$ if and only if $P = f \cdot \mu$ (i.e., if P has a density w.r.t. μ) such that $f > 0$.

■■

Problem 11.10 Solution: Well, the hint given in the text should be good enough.

■■

Problem 11.11 Solution: Observe that

$$\int_C u\,d\mu = \int_C w\,d\mu \iff \int_C (u^+ + w^-)\,d\mu = \int_C (u^- + w^+)\,d\mu$$

holds for all $C \in \mathscr{C}$. The right-hand side can be read as the equality of two measures $A \mapsto \int_A (u^+ + w^-)\,d\mu$, $A \mapsto \int_A (u^- + w^+)\,d\mu$, $A \in \mathscr{A}$ which coincide on a generator $\mathscr{C}$ which satisfies the conditions of the uniqueness theorem of measures (Theorem 5.7). This shows that

$$\int_A u\,d\mu = \int_A w\,d\mu \quad \forall A \in \mathscr{A}.$$

Now the direction '⇒' follows from Corollary 11.7 where $\mathscr{B} = \mathscr{A}$.

The converse implication '⇐' follows directly from Corollary 11.6 applied to $u\mathbb{1}_C$ and $w\mathbb{1}_C$.

Problem 11.12 Solution:

(i) "⊂": Let $x \in C_f$, i.e. $f(x) = \lim_{n\to\infty} f_n(x)$ exists; in particular, $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy: for all $k \in \mathbb{N}$ there is some $\ell \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| \leqslant \frac{1}{k} \quad \forall\, m, n \geqslant \ell.$$

This shows that $x \in \bigcap_{k\in\mathbb{N}} \bigcup_{\ell\in\mathbb{N}} \bigcap_{n,m=\ell}^{\infty} \{|f_n(x) - f_m(x)| \leqslant \frac{1}{k}\}$.

"⊃": Assume that $\bigcap_{k\in\mathbb{N}} \bigcup_{\ell\in\mathbb{N}} \bigcap_{n,m=\ell}^{\infty} \{|f_n(x) - f_m(x)| \leqslant \frac{1}{k}\}$. This means that for every $k \in \mathbb{N}$ there is some $\ell \in \mathbb{N}$ with

$$|f_n(x) - f_m(x)| \leqslant \frac{1}{k} \quad \forall\, m, n \geqslant \ell.$$

This shows that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. The claim follows since $\mathbb{R}$ is complete.

(ii) From the definition of limits we get (as in part (i))

$$C_f = \bigcap_{k\in\mathbb{N}} \bigcup_{\ell\in\mathbb{N}} \bigcap_{m=\ell}^{\infty} \{|f_m(x) - f(x)| \leqslant \tfrac{1}{k}\};$$

Observe that

$$A_n^k \uparrow \bigcup_{\ell=1}^{\infty} \bigcap_{m=\ell}^{\infty} \{|f_m(x) - f(x)| \leqslant \tfrac{1}{k}\} \supset C_f$$

as $n \to \infty$. Using the continuity of measures, we get

$$\mu(A_n^k) \uparrow \mu\left(\bigcup_{\ell=1}^{\infty} \bigcap_{m=\ell}^{\infty} \left\{|f_m(x) - f(x)| \leqslant \tfrac{1}{k}\right\}\right) = \mu(X).$$

(Note: if $A \subset B$ is measurable and $\mu(A) = \mu(X)$, then we have $\mu(B) = \mu(X)$.) In particular we can pick $n = n(k, \epsilon)$ in such a way that $\mu(A_n^k) \geqslant \mu(X) - \epsilon 2^{-k}$. Therefore,

$$\mu(X \setminus A_{n(k,\epsilon)}^k) = \mu(X) - \mu(A_{n(k,\epsilon)}^k) \leqslant \epsilon 2^{-k}.$$

(iii) Fix $\epsilon > 0$, pick $n = n(k, \epsilon)$ as in part (ii), and define

$$A_\epsilon := \bigcap_{k\in\mathbb{N}} A_{n(k,\epsilon)}^k \in \mathscr{A}.$$

Using the sub-additivity of μ we get

$$\mu(X \setminus A_\epsilon) = \mu\left(\bigcup_{k\in\mathbb{N}} \left(X \setminus A_{n(k,\epsilon)}^k\right)\right) \leqslant \sum_{k\in\mathbb{N}} \mu(X \setminus A_{n(k,\epsilon)}^k) \leqslant \sum_{k\in\mathbb{N}} \epsilon 2^{-k} \leqslant \epsilon.$$

It remains to show that f_n converges uniformly to f on the set A_ϵ. By definition,

$$A_\epsilon = \bigcap_{k\in\mathbb{N}} \bigcup_{\ell=1}^{n(k,\epsilon)} \bigcap_{m=\ell}^{\infty} \{|f - f_m| \leqslant \tfrac{1}{k}\},$$

i.e. for all $x \in A_\epsilon$ and $k \in \mathbb{N}$ there is some $\ell(x) \leqslant n(k,\epsilon)$ such that

$$|f(x) - f_m(x)| \leqslant \frac{1}{k} \quad \forall\, m \geqslant \ell(x).$$

Since $\ell(x) \leqslant n(k,\epsilon)$ we get, in particular,

$$|f(x) - f_m(x)| \leqslant \frac{1}{k} \quad \forall x \in A_\epsilon,\ m \geqslant n(k,\epsilon).$$

Since $k \in \mathbb{N}$ is arbitrary, the uniform convergence A_ϵ follows.

(iv) Consider one-dimensional Lebesgue measure, set $f(x) := |x|$ and $f_n(x) := |x|\mathbb{1}_{[-n,n]}$. Then we have $f_n(x) \uparrow f(x)$ for every x, but the set $\{|f_n - f| > \epsilon\} = [-n,n]^c$ has infinite measure for any $\epsilon > 0$.

■■

12 Convergence theorems and their applications. Solutions to Problems 12.1–12.37

Problem 12.1 Solution: We start with the simple remark that

$$\begin{aligned}
|a-b|^p &\leqslant (|a|+|b|)^p \\
&\leqslant (\max\{|a|,|b|\}+\max\{|a|,|b|\})^p \\
&= 2^p \max\{|a|,|b|\}^p \\
&= 2^p \max\{|a|^p,|b|^p\} \\
&\leqslant 2^p(|a|^p+|b|^p).
\end{aligned}$$

Because of this we find that $|u_j - u|^p \leqslant 2^p g^p$ and the right-hand side is an integrable dominating function.

Proof alternative 1: Apply Theorem 12.2 on dominated convergence to the sequence $\phi_j := |u_j - u|^p$ of integrable functions. Note that $\phi_j(x) \to 0$ and that $0 \leqslant \phi_j \leqslant \Phi$ where $\Phi = 2^p g^p$ is integrable and independent of j. Thus,

$$\begin{aligned}
\lim_{j\to\infty} \int |u_j - u|^p \, d\mu = \lim_{j\to\infty} \int \phi_j \, d\mu &= \int \lim_{j\to\infty} \phi_j \, d\mu \\
&= \int 0 \, d\mu \; = \; 0.
\end{aligned}$$

Proof alternative 2: Mimic the proof of Theorem 12.2 on dominated convergence. To do so we remark that the sequence of functions

$$0 \leqslant \psi_j := 2^p g^p - |u_j - u|^p \xrightarrow[j\to\infty]{} 2^p g^p$$

Since the limit $\lim_j \psi_j$ exists, it coincides with $\liminf_j \psi_j$, and so we can use Fatou's Lemma to get

$$\begin{aligned}
\int 2^p g^p \, d\mu &= \int \liminf_{j\to\infty} \psi_j \, d\mu \\
&\leqslant \liminf_{j\to\infty} \int \psi_j \, d\mu \\
&= \liminf_{j\to\infty} \int \left(2^p g^p - |u_j - u|^p\right) d\mu
\end{aligned}$$

$$= \int 2^p g^p \, d\mu + \liminf_{j\to\infty} \left(- \int |u_j - u|^p \, d\mu \right)$$
$$= \int 2^p g^p \, d\mu - \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu$$

where we use that $\liminf_j(-\alpha_j) = -\limsup_j \alpha_j$. This shows that $\limsup_j \int |u_j - u|^p \, d\mu = 0$, hence

$$0 \leqslant \liminf_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant 0$$

showing that lower and upper limit coincide and equal to 0, hence $\lim_j \int |u_j - u|^p \, d\mu = 0$.

Problem 12.2 Solution: Assume that, as in the statement of Theorem 12.2, $u_j \to u$ and that $|u_j| \leqslant f \in \mathcal{L}^1(\mu)$. In particular,

$$-f \leqslant u_j \quad \text{and} \quad u_j \leqslant f$$

$(j \in \mathbb{N})$ is an integrable minorant resp. majorant. Thus, using Problem 10.7 at $*$ below,

$$\begin{aligned}\int u \, d\mu &= \int \liminf_{j\to\infty} u_j \, d\mu \\ &\overset{*}{\leqslant} \liminf_{j\to\infty} \int u_j \, d\mu \\ &\leqslant \limsup_{j\to\infty} \int u_j \, d\mu \\ &\overset{*}{\leqslant} \int \limsup_{j\to\infty} u_j \, d\mu \;=\; \int u \, d\mu.\end{aligned}$$

This proves $\int u \, d\mu = \lim_j \int u_j \, d\mu$.

Addition: since $0 \leqslant |u - u_j| \leqslant |\lim_j u_j| + |u_j| \leqslant 2f \in \mathcal{L}^1(\mu)$, the sequence $|u - u_j|$ has an integrable majorant and using Problem 10.7 we get

$$0 \leqslant \limsup_{j\to\infty} \int |u_j - u| \, d\mu \leqslant \int \limsup_{j\to\infty} |u_j - u| \, d\mu = \int 0 \, d\mu = 0$$

and also (i) of Theorem 12.2 follows...

Problem 12.3 Solution: By assumption we have

$$0 \leqslant f_k - g_k \xrightarrow[k\to\infty]{} f - g,$$
$$0 \leqslant G_k - f_k \xrightarrow[k\to\infty]{} G - f.$$

Using Fatou's Lemma we find

$$\int (f - g) \, d\mu = \int \lim_k (f_k - g_k) \, d\mu$$

$$
\begin{aligned}
&= \int \liminf_k (f_k - g_k)\, d\mu \\
&\leqslant \liminf_k \int (f_k - g_k)\, d\mu \\
&= \liminf_k \int f_k \, d\mu - \int g \, d\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
\int (G - f)\, d\mu &= \int \lim_k (G_k - f_k)\, d\mu \\
&= \int \liminf_k (G_k - f_k)\, d\mu \\
&\leqslant \liminf_k \int (G_k - f_k)\, d\mu \\
&= \int G \, d\mu - \limsup_k \int f_k \, d\mu.
\end{aligned}
$$

Adding resp. subtracting $\int g\, d\mu$ resp. $\int G\, d\mu$ therefore yields

$$
\limsup_k \int f_k \, d\mu \leqslant \int f \, d\mu \leqslant \liminf_k \int f_k \, d\mu
$$

and the claim follows.

Problem 12.4 Solution: Using Beppo Levi's theorem in the form of Corollary 9.9 we find

$$
\int \sum_{j=1}^{\infty} |u_j| \, d\mu = \sum_{j=1}^{\infty} \int |u_j| \, d\mu < \infty, \tag{*}
$$

which means that the positive function $\sum_{j=1}^{\infty} |u_j|$ is finite almost everywhere, i.e. the series $\sum_{j=1}^{\infty} u_j$ converges (absolutely) almost everywhere.

In order to show the second part, we want to apply dominated convergence. Set $v_k := \sum_{n=1}^{k} u_n$ and notte that

$$
|v_k| = \left| \sum_{n=1}^{k} u_n \right| \leqslant \sum_{n=1}^{k} |u_n| \leqslant \sum_{n=1}^{\infty} |u_n| \leqslant w \in \mathscr{L}^1(\mu).
$$

Clearly, $v_k \to u = \sum_{n=1}^{\infty} u_n$ as $k \to \infty$. Thus, we get with dominated convergence

$$
\begin{aligned}
\int \sum_{n=1}^{\infty} u_n \, d\mu = \int u \, d\mu = \int \lim_{k\to\infty} v_k \, d\mu = \lim_{k\to\infty} \int v_k \, d\mu &= \lim_{k\to\infty} \sum_{n=1}^{k} \int u_n \, d\mu \\
&= \sum_{n=1}^{\infty} \int u_n \, d\mu.
\end{aligned}
$$

Problem 12.5 Solution: Since $\mathcal{L}^1(\mu) \ni u_j \downarrow 0$ we find by monotone convergence, Theorem 12.1, that $\int u_j \, d\mu \downarrow 0$. Therefore,

$$
\sigma = \sum_{j=1}^{\infty} (-1)^j u_j \quad \text{and} \quad S = \sum_{j=1}^{\infty} (-1)^j \int u_j \, d\mu \quad \text{converge}
$$

(conditionally, in general). Moreover, for every $N \in \mathbb{N}$,

$$\int \sum_{j=1}^{N} (-1)^j u_j \, d\mu = \sum_{j=1}^{N} \int (-1)^j u_j \, d\mu \xrightarrow[N\to\infty]{} S.$$

All that remains is to show that the right-hand side converges to $\int \sigma \, d\mu$. Observe that for $S_N := \sum_{j=1}^{N} (-1)^j u_j$ we have

$$S_{2N} \leqslant S_{2N+2} \leqslant \ldots \leqslant S$$

and we find, as $S_j \in \mathcal{L}^1(\mu)$, by monotone convergence that

$$\lim_{N\to\infty} \int S_{2N} \, d\mu = \int \sigma \, d\mu.$$

Problem 12.6 Solution: Consider $u_j(x) := j \cdot \mathbb{1}_{(0,1/j)}(x)$, $j \in \mathbb{N}$. It is clear that u_j is measurable and Lebesgue integrable with integral

$$\int u_j \, d\lambda = j \, \frac{1}{j} = 1 \qquad \forall \, j \in \mathbb{N}.$$

Thus, $\lim_j \int u_j \, d\lambda = 1$. On the other hand, the pointwise limit is

$$u(x) := \lim_j u_j(x) \equiv 0$$

so that $0 = \int u \, d\lambda = \int \lim_j u_j \, d\lambda \neq 1$.

The example does not contradict dominated convergence as there is no uniform dominating integrable function.

Alternative: a similar situation can be found for $v_k(x) := \frac{1}{k} \mathbb{1}_{[0,k]}(x)$ and the pointwise limit $v \equiv 0$. Note that in this case the limit is even uniform and still $\lim_k \int v_k \, d\lambda = 1 \neq 0 = \int v \, d\lambda$. Again there is no contradiction to dominated convergence as there does not exist a uniform dominating integrable function.

Problem 12.7 Solution: Using the majorant ($e^{-rx} \leqslant 1 \in \mathcal{L}^1(\mu)$, $r, x \geqslant 0$) we find with dominated convergence

$$\lim_{r\to\infty} \int_{[0,\infty)} e^{-rx} \, \mu(dx) = \int_{[0,\infty)} \lim_{r\to\infty} e^{-rx} \, \mu(dx) = \int_{[0,\infty)} \mathbb{1}_{\{0\}} \, \mu(dx) = \mu\{0\}.$$

Problem 12.8 Solution:

(i) Let $\epsilon > 0$. As $u \in \mathcal{L}^1(\lambda)$, monotone convergence shows that

$$\lim_{R\to\infty} \int_{B_R(0)^c} |u| \, d\lambda = 0.$$

In particular, we can pick an $R > 0$ such that

$$\int_{B_R(0)^c} |u| \, d\lambda \leqslant \epsilon.$$

Since K is compact (in fact: bounded), there is some $r = r(R) > 0$, such that $x + K \subset B_R(0)^c$ for all x satisfying $|x| \geqslant r$. Thus, we have

$$\int_{x+K} |u| \, d\lambda \leqslant \int_{B_R(0)^c} |u| \, d\lambda \leqslant \epsilon \quad \forall x \in \mathbb{R}^n, \ |x| \geqslant r.$$

(ii) Fix $\epsilon > 0$. By assumption, u is uniformly continuous. Therefore, there is some $\delta > 0$ such that

$$|u(y) - u(x)| \leqslant \epsilon \quad \forall x \in \mathbb{R}^n, \ y \in x + K := x + \overline{B_\delta(0)} = \overline{B_\delta(x)}.$$

Hence,

$$\begin{aligned} |u(x)|^p &= \frac{1}{\lambda(K+x)} \int_{K+x} |u(x)|^p d\lambda(y) \\ &\leqslant \frac{1}{\lambda(K)} \int_{K+x} \big(\underbrace{|u(y) - u(x)|}_{\leqslant\epsilon} + |u(y)| \big)^p \, d\lambda(y). \end{aligned}$$

Using the elementary inequality

$$(a+b)^p \leqslant (2\max\{a,b\})^p \leqslant 2^p(a^p + b^p), \quad a, b \geqslant 0 \tag{$\star$}$$

we get for $C = 2^p$

$$\begin{aligned} |u(x)|^p &\leqslant \frac{C}{\lambda(K)} \left(\int_{K+x} \epsilon^p \, d\lambda(y) + \int_{K+x} |u(y)| \, d\lambda(y) \right) \\ &\leqslant C\epsilon^p \underbrace{\frac{\lambda(K+x)}{\lambda(K)}}_{1} + \frac{C}{\lambda(K)} \int_{K+x} |u(y)| \, d\lambda(y). \end{aligned}$$

Part (i) now implies

$$\limsup_{|x|\to\infty} |u(x)|^p \leqslant C\epsilon^p \xrightarrow{\epsilon\to 0} 0$$

and this is the same as to say $\lim_{|x|\to\infty} |u(x)| = 0$.

Problem 12.9 Solution:

(i) Fix $\epsilon > 0$, $R > 0$ and consider $B := \{|u| \leqslant R\}$. By definition, $\sup_{x\in B}|u(x)| < \infty$. On the other hand, dominated convergence and Corollary 11.6 show that

$$\lim_{R\to\infty}\int_{|u|>R}|u(x)|\,dx = \int_{|u|=\infty}|u(x)|\,dx = 0.$$

In particular, we can choose R so large, that $\int_B |u(x)|\,dx < \epsilon$. Using Markov's inequality (Proposition 11.5) yields

$$\lambda(B) = \lambda\{|u| \geqslant R\} \leqslant \frac{1}{R}\int |u(x)|\,dx < \infty.$$

(ii) Fix $\epsilon > 0$ and let $B \in \mathscr{B}(\mathbb{R}^n)$ be as in (i). Further, let $A \in \mathscr{B}(\mathbb{R}^n)$ with $\lambda(A) < \epsilon$. Then we have

$$\begin{aligned}\int_A |u|\,d\lambda &= \int_{A\cap B}|u|\,d\lambda + \int_{A\cap B^c}|u|\,d\lambda\\ &\leqslant \sup_{x\in B}|u(x)|\cdot\underbrace{\lambda(A\cap B)}_{\leqslant\lambda(A)} + \int_{B^c}|u|\,d\lambda\\ &\leqslant \sup_{x\in B}|u(x)|\cdot\epsilon + \epsilon.\end{aligned}$$

(Observe that $\sup_{x\in B}|u(x)| < \infty$.) This proves

$$\lim_{\lambda(A)\to 0}\int_A |u|\,d\lambda = 0.$$

■■

Problem 12.10 Solution:

(i) From $u_n \in \mathcal{L}^1(\mu)$ and $\|u_n - u\|_\infty \leqslant 1$ (for all sufficiently large n) we infer

$$\int |u|\,d\mu \leqslant \int |u_n - u|\,d\mu + \int |u_n|\,d\mu \leqslant \|u_n - u\|_\infty \mu(X) + \int |u_n|\,d\mu < \infty,$$

i.e. $u \in \mathcal{L}^1(\mu)$. A very similar argument gives

$$\left|\int u_n\,d\mu - \int u\,d\mu\right| = \left|\int (u_n - u)\,d\mu\right| \leqslant \int |u_n - u|\,d\mu \leqslant \|u_n - u\|_\infty\mu(X).$$

Since $\mu(X) < \infty$, uniform convergence $\|u_n - u\|_\infty \to 0$ implies that

$$\lim_{n\to\infty}\left|\int u_n\,d\mu - \int u\,d\mu\right| = 0.$$

(ii) False. Counterexample: $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda^1)$ and $u_n(x) := \frac{1}{2n}\mathbb{1}_{[-n,n]}(x)$, $x \in \mathbb{R}$. Clearly, $u_n \to 0$ uniformly, $u_n \in \mathcal{L}^1(\lambda^1)$, but

$$\lim_{n\to\infty}\int u_n\,d\mu = 1 \neq 0 = \int u\,d\mu.$$

■■

Problem 12.11 Solution: Without loss of generality we assume that u is increasing. Because of the monotonicity of u, we find for every sequence $(a_n)_{n\in\mathbb{N}} \subset (0,1)$ such that $a_n \downarrow 0$, that

$$u(a_n) \to u(0+) := \inf_{t>0} u(t).$$

If $a_n := t^n$, $t \in (0,1)$, we get $u(t^n) \downarrow 0$ and by monotone convergence

$$\lim_{n\to\infty} \int_0^1 u(t^n)\,dt = \inf_{n\in\mathbb{N}} \int_0^1 u(t^n)\,dt = \int_0^1 \inf_{n\in\mathbb{N}} u(t^n)\,dt = \int_0^1 u(0+)\,dt = u(0+).$$

Problem 12.12 Solution: Set $u_n(t) := t^n u(t)$, $t \in (0,1)$. Since $|t^n| \leqslant 1$ for $t \in (0,1)$, we have

$$|u_n(t)| = |t^n| \cdot |f(t)| \leqslant |f(t)| \in \mathcal{L}^1(0,1).$$

Since $t^n \xrightarrow[n\to\infty]{} 0$ for all $t \in (0,1)$ and $|f(t)| < \infty$ a.e. (Corollary 11.6), we have $|u_n(t)| \to 0$ a.e. An application of dominated convergence (Theorem 12.2 and Remark 12.3) yields

$$\lim_{n\to\infty} \int_0^1 t^n u(t)\,dt = \lim_{n\to\infty} \int_0^1 u_n(t)\,dt = \int_0^1 \underbrace{\lim_{n\to\infty} u_n(t)}_{0}\,dt = 0.$$

Problem 12.13 Solution: From the geometric series we know that $\frac{1}{1-x} = \sum_{n\geqslant 0} x^n$ for $x \in [0,1)$. This implies that for all $t > 0$

$$\frac{1}{e^t - 1} = \frac{1}{e^t}\frac{1}{1-e^{-t}} = e^{-t}\sum_{n\geqslant 0}(e^{-t})^n = \sum_{n\geqslant 1} e^{-nt}$$

(observe that $e^{-t} < 1$ for $t > 0$!). Set $u_k(t) := \sin(t) \cdot \sum_{n=1}^k e^{-nt}$, then we get the estimate

$$|u_k(t)| \leqslant |\sin t| \cdot \left|\sum_{n=1}^k e^{-nt}\right| = |\sin t| \sum_{n=1}^k e^{-nt} \leqslant |\sin t| \sum_{n\geqslant 1} e^{-nt} = \frac{|\sin t|}{e^t - 1} \tag{$*$}$$

for all $k \in \mathbb{N}$ und $t > 0$. Using the elementary inequalities $e^t - 1 \geqslant t$ ($t \geqslant 0$) and $e^t - 1 \geqslant e^{t/2}$ ($t \geqslant 1$) we see

$$|u_k(t)| \leqslant \mathbb{1}_{[0,1]}(t) + e^{-t/2}\mathbb{1}_{(1,\infty)}(t) =: w(t).$$

Let us now show that $w \in \mathcal{L}^1(0,\infty)$. This can be done with Beppo Levi's theorem:

$$\begin{aligned}\int_0^\infty w(t)\,dt &= \int_0^1 \underbrace{w(t)}_{1}\,dt + \int_1^\infty \underbrace{w(t)}_{e^{-t/2}}\,dt \\ &= 1 + \sup_{n\in\mathbb{N}} \int_1^n e^{-t/2}\,dt = 1 + \sup_{n\in\mathbb{N}} \left[-2e^{-t/2}\right]_{t=1}^n < \infty.\end{aligned}$$

We use here that every Riemann-integrable function $f : [a,b] \to \mathbb{C}$, $-\infty < a < b < \infty$, is Lebesgue integrable and that Riemann and Lebesgue intgrals coincide (in this case, see Theorem 12.8). By dominated convergence,

$$\int_0^\infty \frac{\sin(t)}{e^t - 1}\,dt = \lim_{k\to\infty} \int_0^\infty u_k(t)\,dt = \lim_{k\to\infty} \sum_{n=1}^k \int_0^\infty \sin(t)e^{-nt}\,dt.$$

With $\operatorname{Im} e^{i\,t} = \sin t$ we get

$$\int_0^\infty \sin(t)e^{-nt}\,dt = \operatorname{Im}\left(\int_0^\infty e^{t(i-n)}\,dt\right),$$

(cf. Problem 10.9). Again by dominated convergence,

$$\begin{aligned}\int_0^\infty \sin(t)e^{-nt}\,dt &= \operatorname{Im}\left(\lim_{R\to\infty}\int_{1/R}^R e^{t(i-n)}\,dt\right)\\ &= \operatorname{Im}\left(\lim_{R\to\infty}\left[\frac{e^{t(i-n)}}{i-n}\right]_{t=1/R}^R\right)\\ &= \operatorname{Im}\left(\frac{1}{n-i}\right) = \frac{1}{n^2+1}.\end{aligned}$$

Problem 12.14 Solution: We know that the exponential function is given by $e^{zx} = \sum_{n\geqslant 0} \frac{(zx)^n}{n!}$. Thus,

$$u_k(x) := u(x)\sum_{n=0}^k \frac{(zx)^n}{n!} \xrightarrow[k\to\infty]{} u(x)e^{zx}.$$

By the triangle inequality,

$$|u_k(x)| \leqslant |u(x)|\sum_{n=0}^k \left|\frac{(zx)^n}{n!}\right| \leqslant |u(x)|\sum_{n\geqslant 0}\frac{|zx|^n}{n!} = |u(x)|e^{|z||x|}.$$

As $x \mapsto e^{\lambda x}u(x)$ is integrable for fixed $\lambda = \pm|z|$, we get

$$|u_k(x)| \leqslant |u(x)|e^{-|z|x}\mathbb{1}_{(-\infty,0)}(x) + |u(x)|e^{|z|x}\mathbb{1}_{[0,\infty)}(x) \in \mathcal{L}^1(\mathbb{R}).$$

An application of dominated convergence and the linearity of the integral give

$$\begin{aligned}\int u(x)e^{zx}\,dx &= \int \lim_{k\to\infty} u_k(x)\,dx\\ &= \lim_{k\to\infty}\int u_k(x)\,dx\\ &= \lim_{k\to\infty}\sum_{n=0}^k \frac{1}{n!}\int (zx)^n u(x)\,dx\\ &= \sum_{n=0}^\infty \frac{z^n}{n!}\int x^n u(x)\,dx.\end{aligned}$$

Problem 12.15 Solution: We get $\left|\int_A u\,d\mu\right| \leqslant \int_A |u|\,d\mu$ straight from the triangle inequality. Therefore, it is enough to prove the second estimate. Fix $\epsilon > 0$.

Solution 1: The Sombrero lemma ensures that there is a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{E}(\mathscr{A})$ with $|u_n| \leqslant |u|$ and $\lim_{n\to\infty} u_n = u$ (Corollary 8.9). From dominated convergence we get $\int |u_n - u|\,d\mu \xrightarrow[n\to\infty]{} 0$; in particular, we can choose $n \in \mathbb{N}$ such that $\int |u_n - u|\,d\mu \leqslant \epsilon$. Since each u_n is bounded (b/o the definition of a simple function) we get

$$\int_A |u_n|\,d\mu \leqslant \|u_n\|_\infty \cdot \mu(A) < \epsilon$$

for any $A \in \mathscr{A}$ with $\mu(A) < \delta := \epsilon/\|u_n\|_\infty$. Using the triangle inequality we get

$$\int_A |u|\,d\mu \leqslant \int_A |u_n - u|\,d\mu + \int_A |u_n|\,d\mu \leqslant \int |u_n - u|\,d\mu + \int_A |u_n|\,d\mu \leqslant 2\epsilon$$

for any $A \in \mathscr{A}$ with $\mu(A) < \delta$.

Solution 2: Obviously,

$$\int_A |u|\,d\mu = \int_{A\cap\{|u|\geqslant R\}} |u|\,d\mu + \int_{A\cap\{|u|<R\}} |u|\,d\mu \tag{$\star$}$$

We estimate each term by itself. For the first expression on the RHS we use Beppo Levi:

$$\int_{A\cap\{|u|\geqslant R\}} |u|\,d\mu \xrightarrow[R\to\infty]{} \int_{A\cap\{|u|=\infty\}} |u|\,d\mu.$$

By assumption, $u \in \mathcal{L}^1(\mu)$, we get $\mu(|u| = \infty) = 0$ (see the proof of Corollaryr 11.6) and we get with Theorem 11.2,

$$\int_{A\cap\{|u|=\infty\}} |u|\,d\mu = 0.$$

Therefore, we can pick some $R > 0$ with

$$\int_{A\cap\{|u|\geqslant R\}} |u|\,d\mu \leqslant \epsilon.$$

For the second expression in ($\star$) we have

$$\int_{A\cap\{|u|<R\}} |u|\,d\mu \leqslant R\int_{A\cap\{|u|<R\}} 1\,d\mu \leqslant R\mu(A).$$

If $A \in \mathscr{A}$ satisfies $\mu(A) \leqslant \delta := \epsilon/R$, then

$$\int_A |u|\,d\mu = \int_{A\cap\{|u|\geqslant R\}} |u|\,d\mu + \int_{A\cap\{|u|<R\}} |u|\,d\mu \leqslant \epsilon + R\mu(A) \leqslant 2\epsilon.$$

■■

Problem 12.16 Solution: Let μ be an arbitrary Borel measure on the line $\mathbb{R}$ and define the integral function for some $u \in \mathcal{L}^1(\mu)$ through

$$I(x) := I^u_\mu(x) := \int_{(0,x)} u(t)\,\mu(dt) = \int \mathbb{1}_{(0,x)}(t)u(t)\,\mu(dt).$$

For any sequence $0 < l_j \to x$, $l_j < x$ from the left and $r_k \to x$, $r_k > x$ from the right we find

$$\mathbb{1}_{(0,l_j)}(t) \xrightarrow[j\to\infty]{} \mathbb{1}_{(0,x)}(t) \quad\text{and}\quad \mathbb{1}_{(0,r_k)}(t) \xrightarrow[k\to\infty]{} \mathbb{1}_{(0,x]}(t).$$

Since $|\mathbb{1}_{(0,x)}u| \leqslant |u| \in \mathcal{L}^1$ is a uniform dominating function, Lebesgue's dominated convergence theorem yields

$$\begin{aligned} I(x+) - I(x-) &= \lim_k I(r_k) - \lim_j I(l_j) \\ &= \int \mathbb{1}_{(0,x]}(t)u(t)\,\mu(dt) - \int \mathbb{1}_{(0,x)}(t)u(t)\,\mu(dt) \\ &= \int \big(\mathbb{1}_{(0,x]}(t) - \mathbb{1}_{(0,x)}(t)\big)u(t)\,\mu(dt) \\ &= \int \mathbb{1}_{\{x\}}(t)u(t)\,\mu(dt) \\ &= u(x)\,\mu(\{x\}). \end{aligned}$$

Thus $I(x)$ is continuous at x if, and only if, x is not an atom of μ.

Remark: the proof shows, by the way, that $I_\mu^u(x)$ is *always* left-continuous at every x, no matter what μ or u look like.

Problem 12.17 Solution:

(i) We have

$$\begin{aligned} &\int \frac{1}{x}\,\mathbb{1}_{[1,\infty)}(x)\,dx \\ &\quad= \lim_{n\to\infty} \int \frac{1}{x}\mathbb{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\ &\quad= \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x}\,dx && \text{usual shorthand} \\ &\quad= \lim_{n\to\infty} (R)\int_1^n \frac{1}{x}\,dx && \text{Riemann-}\int_1^n \text{ exists} \\ &\quad= \lim_{n\to\infty} \big[\log x\big]_1^n \\ &\quad= \lim_{n\to\infty} [\log(n) - \log(1)] = \infty \end{aligned}$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $[1,\infty)$.

(ii) We have

$$\begin{aligned} &\int \frac{1}{x^2}\,\mathbb{1}_{[1,\infty)}(x)\,dx \\ &\quad= \lim_{n\to\infty} \int \frac{1}{x^2}\mathbb{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\ &\quad= \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x^2}\,dx && \text{usual shorthand} \\ &\quad= \lim_{n\to\infty} (R)\int_1^n \frac{1}{x^2}\,dx && \text{Riemann-}\int_1^n \text{ exists} \end{aligned}$$

$$= \lim_{n\to\infty} \left[-\frac{1}{x}\right]_1^n$$
$$= \lim_{n\to\infty} [1 - \tfrac{1}{n}] = 1 < \infty$$

which means that $\frac{1}{x^2}$ is Lebesgue-integrable over $[1, \infty)$.

(iii) We have

$$\begin{aligned}
&\int \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1]}(x)\,dx \\
&\quad = \lim_{n\to\infty} \int \frac{1}{\sqrt{x}} \mathbb{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\quad = \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{\sqrt{x}}\,dx && \text{usual shorthand} \\
&\quad = \lim_{n\to\infty} (R)\int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx && \text{Riemann-}\int_{1/n}^1 \text{ exists} \\
&\quad = \lim_{n\to\infty} \left[2\sqrt{x}\right]_{1/n}^1 \\
&\quad = \lim_{n\to\infty} \left[2 - 2\sqrt{\tfrac{1}{n}}\right] \\
&\quad = 2 < \infty
\end{aligned}$$

which means that $\frac{1}{\sqrt{x}}$ is Lebesgue-integrable over $(0, 1]$.

(iv) We have

$$\begin{aligned}
&\int \frac{1}{x} \mathbb{1}_{(0,1]}(x)\,dx \\
&\quad = \lim_{n\to\infty} \int \frac{1}{x} \mathbb{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\quad = \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{x}\,dx && \text{usual shorthand} \\
&\quad = \lim_{n\to\infty} (R)\int_{1/n}^1 \frac{1}{x}\,dx && \text{Riemann-}\int_{1/n}^1 \text{ exists} \\
&\quad = \lim_{n\to\infty} \left[\log x\right]_{1/n}^1 \\
&\quad = \lim_{n\to\infty} \left[\log(1) - \log \tfrac{1}{n}\right] \\
&\quad = \infty
\end{aligned}$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $(0, 1]$.

Problem 12.18 Solution: We construct a dominating integrable function.

If $x \leqslant 1$, we have clearly $\exp(-x^\alpha) \leqslant 1$, and $\int_{(0,1]} \mathbb{1}\,dx = 1 < \infty$ is integrable.

If $x \geqslant 1$, we have $\exp(-x^\alpha) \leqslant Mx^{-2}$ for some suitable constant $M = M_\alpha < \infty$. This function is integrable in $[1, \infty)$, see e.g. Problem 12.17. The estimate is easily seen from the fact that $x \mapsto x^2 \exp(-x^\alpha)$ is continuous in $[1, \infty)$ with $\lim_{x\to\infty} x^2 \exp(-x^\alpha) = 0$.

This shows that $\exp(-x^\alpha) \leqslant \mathbb{1}_{(0,1)} + Mx^{-2}\mathbb{1}_{[1,\infty)}$ with the right-hand side being integrable.

Problem 12.19 Solution: Take $\alpha \in (a,b)$ where $0 < a < b < \infty$ are fixed (but arbitrary). We show that the function is continuous for these α. This shows the general case since continuity is a local property and we can 'catch' any given α_0 by some choice of a and b's.

We use the Continuity lemma (Theorem 12.4) and have to find uniform (for $\alpha \in (a,b)$) dominating bounds on the integrand function $f(\alpha,x) := \left(\frac{\sin x}{x}\right)^3 e^{-\alpha x}$. First of all, we remark that $\left|\frac{\sin x}{x}\right| \leqslant M$ which follows from the fact that $\frac{\sin x}{x}$ is a continuous function such that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and $\lim_{x\downarrow 0} \frac{\sin x}{x} = 1$. (Actually, we could choose $M = 1$...). Moreover, $\exp(-\alpha x) \leqslant 1$ for $x \in (0,1)$ and $\exp(-\alpha x) \leqslant C_{a,b}x^{-2}$ for $x \geqslant 1$—use for this the continuity of $x^2\exp(-\alpha x)$ and the fact that $\lim_{x\to\infty} x^2 \exp(-\alpha x) = 0$. This shows that

$$|f(\alpha,x)| \leqslant M\left(\mathbb{1}_{(0,1)}(x) + C_{a,b}x^{-2}\mathbb{1}_{[1,\infty)}(x)\right)$$

and the right-hand side is an integrable dominating function which does not depend on α—as long as $\alpha \in (a,b)$. But since $\alpha \mapsto f(\alpha,x)$ is obviously continuous, the Continuity lemma applies and proves that $\int_{(0,\infty)} f(\alpha,x)\,dx$ is continuous.

Problem 12.20 Solution: Fix some number $N > 0$ and take $x \in (-N,N)$. We show that $G(x)$ is continuous on this set. Since N was arbitrary, we find that G is continuous for every $x \in \mathbb{R}$.

Set $g(t,x) := \frac{\sin(tx)}{t(1+t^2)} = x\,\frac{\sin(tx)}{(tx)}\,\frac{1}{1+t^2}$. Then, using that $\left|\frac{\sin u}{u}\right| \leqslant M$, we have

$$|g(t,x)| \leqslant x \cdot M \cdot \frac{1}{1+t^2} \leqslant M \cdot N \cdot \left(\mathbb{1}_{(0,1)}(t) + \frac{1}{t^2}\mathbb{1}_{[1,\infty)}(t)\right)$$

and the right-hand side is a uniformly dominating function, i.e. $G(x)$ makes sense and we find $G(0) = \int_{t\neq 0} g(t,0)\,dt = 0$. To see differentiability, we use the Differentiability lemma (Theorem 12.5) and need to prove that $|\partial_x g(t,x)|$ exists (this is clear) and is uniformly dominated for $x \in (-N,N)$. We have

$$\begin{aligned}
|\partial_x g(t,x)| = \left|\partial_x \frac{\sin(tx)}{t(1+t^2)}\right| &= \left|\frac{\cos(tx)}{(1+t^2)}\right| \\
&\leqslant \frac{1}{1+t^2} \\
&\leqslant \left(\mathbb{1}_{(0,1)}(t) + \frac{1}{t^2}\mathbb{1}_{[1,\infty)}(t)\right)
\end{aligned}$$

and this allows us to apply the Differentiability lemma, so

$$\begin{aligned}
G'(x) = \partial_x \int_{t\neq 0} g(t,x)\,dt &= \int_{t\neq 0} \partial_x g(t,x)\,dt \\
&= \int_{t\neq 0} \frac{\cos(tx)}{1+t^2}\,dt \\
&= \int_{\mathbb{R}} \frac{\cos(tx)}{1+t^2}\,dt
\end{aligned}$$

(use in the last equality that $\{0\}$ is a Lebesgue null set). Thus, by a Beppo Levi-argument (and using that Riemann=Lebesgue whenever the Riemann integral over a compact interval exists...)

$$\begin{aligned} G'(0) = \int_{\mathbb{R}} \frac{1}{1+t^2}\,dt &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty} [\tan^{-1}(t)]_{-n}^{n} \\ &= \pi. \end{aligned}$$

Now observe that

$$\partial_x \sin(tx) = t\cos(tx) = \frac{t}{x}\,x\cos(tx) = \frac{t}{x}\,\partial_t \sin(tx).$$

Since the integral defining $G'(x)$ exists we can use a Beppo Levi-argument, Riemann=Lebesgue (whenever the Riemann integral over an interval exists) and integration by parts (for the Riemann integral) to find

$$\begin{aligned} xG'(x) &= \int_{\mathbb{R}} \frac{x\cos(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{x\partial_x \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{t\partial_t \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \frac{\partial_t \sin(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \partial_t \sin(tx)\cdot\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty} \left[\frac{\sin(tx)}{1+t^2}\right]_{t=-n}^{n} - \lim_{n\to\infty} (R)\int_{-n}^{n} \sin(tx)\cdot\partial_t\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty} (R)\int_{-n}^{n} \sin(tx)\cdot\frac{2t}{(1+t^2)^2}\,dt \\ &= \int_{\mathbb{R}} \frac{2t\sin(tx)}{(1+t^2)^2}\,dt. \end{aligned}$$

Problem 12.21 Solution:

(i) Note that for $0 \leqslant a, b \leqslant 1$

$$1-(1-a)^b = \int_{1-a}^{1} bt^{b-1}\,dt \geqslant \int_{1-a}^{1} b\,dt = ba$$

so that we get for $0 \leqslant x \leqslant k$ and $a := x/k$, $b := k/(k+1)$

$$\left(1-\frac{x}{k}\right)^{\frac{k}{k+1}} \leqslant 1-\frac{x}{k+1}, \qquad 0 \leqslant x \leqslant k$$

or,

$$\left(1-\frac{x}{k}\right)^{k} \mathbb{1}_{[0,k]}(x) \leqslant \left(1-\frac{x}{k+1}\right)^{k+1} \mathbb{1}_{[0,k+1]}(x).$$

Therefore we can appeal to Beppo Levi's theorem to get

$$\begin{aligned}
\lim_{k\to\infty} \int_{(1,k)} \left(1 - \frac{x}{k}\right)^k \ln x\, \lambda^1(dx) &= \sup_{k\in\mathbb{N}} \int \mathbb{1}_{(1,k)}(x)\Big(1 - \frac{x}{k}\Big)^k \ln x\, \lambda^1(dx) \\
&= \int \sup_{k\in\mathbb{N}} \left[\mathbb{1}_{(1,k)}(x)\Big(1 - \frac{x}{k}\Big)^k\right] \ln x\, \lambda^1(dx) \\
&= \int \mathbb{1}_{(1,\infty)}(x) e^{-x} \ln x\, \lambda^1(dx).
\end{aligned}$$

That $e^{-x}\ln x$ is integrable in $(1,\infty)$ follows easily from the estimates

$$e^{-x} \leqslant C_N x^{-N} \quad\text{and}\quad \ln x \leqslant x$$

which hold for all $x \geqslant 1$ and $N \in \mathbb{N}$.

(ii) Note that $x \mapsto \ln x$ is continuous and bounded in $[\epsilon, 1]$, thus Riemann integrable. It is easy to see that $x\ln x - x$ is a primitive for $\ln x$. The improper Riemann integral

$$\int_0^1 \ln x\, dx = \lim_{\epsilon\to 0}[x\ln x - x]_\epsilon^1 = -1$$

exists and, since $\ln x$ is negative throughout $(0,1)$, improper Riemann and Lebesgue integrals coincide. Thus, $\ln x \in L^1(dx, (0,1))$.

Therefore,

$$\left|\left(1 - \frac{x}{k}\right)^k \ln x\right| \leqslant |\ln x|, \quad \forall x \in (0,1)$$

is uniformly dominated by an integrable function and we can use dominated convergence to get

$$\begin{aligned}
\lim_k \int_{(0,1)} \left(1 - \frac{x}{k}\right)^k \ln x\, dx &= \int_{(0,1)} \lim_k \left(1 - \frac{x}{k}\right)^k \ln x\, dx \\
&= \int_{(0,1)} e^{-x} \ln x\, dx
\end{aligned}$$

Problem 12.22 Solution: Since the integrand of $F(t)$ is continuous and bounded by the integrable function e^{-x}, $x > 0$, it is clear that $F(t)$ exists. With the usual approximation argument,

$$\int_{(0,\infty)} e^{-x} \frac{t}{t^2 + x^2}\, \lambda(dx) = \lim_{n\to\infty} \int_{1/n}^n e^{-x} \frac{t}{t^2 + x^2}\, dx$$

(the right-hand side is a Riemann integral) we can use the classical (Riemann) rules to evaluate the integral. Thus, a change of variables $x = t\cdot y \implies dx = t\,dy$ yields

$$\begin{aligned}
F(t) &= \int_{(0,\infty)} e^{-x} \frac{t}{t^2 + x^2}\, \lambda(dx) \\
&= \int_{(0,\infty)} e^{-ty} \frac{t}{t^2 + (ty)^2}\, t\, \lambda(dy)
\end{aligned}$$

$$= \int_{(0,\infty)} e^{-ty} \frac{1}{1+y^2} \lambda(dy).$$

Observe that

$$\left| e^{-ty} \frac{1}{1+y^2} \right| \leqslant \frac{1}{1+y^2} \quad \text{uniformly for all } t > 0,$$

and that the right-hand side is Lebesgue integrable (the primitive is the arctan). Therfore, we can use dominated convergence to conclude

$$\begin{aligned} F(0+) &= \lim_{t \downarrow 0} \int_{(0,\infty)} e^{-ty} \frac{1}{1+y^2} \lambda(dy) \\ &= \int_{(0,\infty)} \lim_{t \downarrow 0} e^{-ty} \frac{1}{1+y^2} \lambda(dy) \\ &= \int_{(0,\infty)} \frac{1}{1+y^2} \lambda(dy) \\ &= \lim_{n\to\infty} \int_{1/n}^{n} \frac{1}{1+y^2} \, dy \\ &= \lim_{n\to\infty} \big[\arctan y \big]_{1/n}^{n} = \frac{\pi}{2}. \end{aligned}$$

■■

Problem 12.23 Solution: For the existence of the integrals we need $|e^{-i\cdot\xi}| \in \mathcal{L}^1(\mu)$ and $|e^{-i\cdot\xi}| \cdot |u(\cdot)| \in \mathcal{L}^1(dx)$. Since $|e^{-i\cdot\xi}| = 1$, it is reasonable to require that μ is a finite measure (such that the constant 1 is integrable) or $u \in \mathcal{L}^1(dx)$. Under these assumptions, the continuity of the Fourier transform follows directly from the continuity lemma: set

$$f(\xi, x) := \frac{1}{2\pi} e^{-ix\xi}, \qquad \xi \in \mathbb{R}, x \in \mathbb{R}.$$

By assumption, $|f(x,\xi)| \leqslant (2\pi)^{-1} \in \mathcal{L}^1(\mu)$ and $\xi \mapsto f(\xi, x)$ is continuous. Using Theorem 12.4, we get the continuity of the map

$$\xi \mapsto \int f(\xi, x) \mu(dx) = \widehat{\mu}(\xi).$$

The argument for $\hat{u}$ is similar.

Sufficient conditions for n-fold differentiability can be obtained from the differentiability lemma. Since

$$\frac{d}{d\xi} f(\xi, x) = \frac{(-ix)}{2\pi} e^{-ix\xi}$$

we get

$$\left| \frac{d}{d\xi} f(\xi, x) \right| \leqslant \frac{|x|}{2\pi}.$$

By the differentiabiliy lemma the derivative $\frac{d}{d\xi}\hat{\mu}(\xi)$ exists, if $\int |x| \, \mu(dx) < \infty$. Iterating this argument, we get that $\hat{\mu}$ is n times differentiable, if

$$\int |x|^n \, \mu(dx) < \infty.$$

Similarly one shows that $\hat{u}$ is n times differentiable, if $\int |x|^n \, |u(x)| \, dx < \infty$.

■■

Problem 12.24 Solution:

(i) Let $t \in (-R, R)$ for some $R > 0$. Since $|\phi(x) - t| \leqslant |\phi(x)| + |t| \leqslant |\phi(x)| + R \in \mathcal{L}^1([0,1], dx)$ and since $t \mapsto |\phi(x) - t|$ is continuous, the continuity lemma, Theorem 12.4, shows that the mapping

$$(-R, R) \ni t \mapsto f(t) = \int_{[0,1]} |\phi(x) - t| \, dx$$

is continuous. Since $R > 0$ is arbitrary, the claim follows.

Alternative solution: Using the lower triangle inequality we get that

$$|f(t) - f(s)| \leqslant \int_{[0,1]} \Big| |\phi(x) - t| - |\phi(x) - s| \Big| \, dx \leqslant \int_{[0,1]} |s - t| \, dx = |s - t|,$$

i.e. f is Lipschitz continuous.

(ii) '$\Leftarrow$': Let $t \in \mathbb{R}$ and assume that $\lambda\{\phi = t\} = 0$. For $h \in \mathbb{R}$ we define

$$\begin{aligned}
\frac{f(t+h) - f(t)}{h} &= \int_{\phi \leqslant t-h} \frac{|\phi(x) - (t+h)| - |\phi(x) - t|}{h} \, dx \\
&\quad + \int_{t-h<\phi<t+h} \frac{|\phi(x) - (t+h)| - |\phi(x) - t|}{h} \, dx \\
&\quad + \int_{\phi \geqslant t+h} \frac{|\phi(x) - (t+h)| - |\phi(x) - t|}{h} \, dx \\
&=: I_1(h) + I_2(h) + I_3(h).
\end{aligned}$$

and we consider the three integrals separately. We have

$$\begin{aligned}
I_1(h) &= \int_{\phi \leqslant t-h} \frac{-(\phi(x) - (t+h)) + (\phi(x) - t)}{h} \, dx \\
&= \int_{\phi \leqslant t-h} dx = \lambda(\phi \leqslant t - h) \xrightarrow[h \to 0]{} \lambda\{\phi < t\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_3(h) &= \int_{\phi \geqslant t-h} \frac{(\phi(x) - (t+h) - (\phi(x) - t)}{h} \, dx \\
&= \lambda(\phi \geqslant t + h) \xrightarrow[h \to 0]{} \lambda\{\phi > t\}.
\end{aligned}$$

By our assumptions, $\lambda\{t - h < \phi < t + h\} \xrightarrow[h \to 0]{} \lambda\{\phi = t\} = 0$, and using dominated convergence we arrive at

$$I_2(h) = \int_{t-h<\phi<t+h} \frac{|\phi(x) - (t+h)| - |\phi(x) - t|}{h} \, dx \xrightarrow[h \to 0]{} 0$$

(notice that $\frac{||\phi(x)-(t+h)|-|\phi(x)-t||}{h} \leqslant 2$ b/o the lower triangle inequality!). Putting together all calculations, we get

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lambda\{\phi > t\} + \lambda\{\phi < t\}.$$

'⇒': We use the notation introduced in the direction '⇐'. If f is differentiable at $t \in \mathbb{R}$, we find as in the first part of the proof that

$$\lim_{h\to 0} I_2(h) = f'(t) - \lim_{h\to 0} I_1(h) - \lim_{h\to 0} I_3(h)$$

exists. We split I_2 once again:

$$\begin{aligned} I_2(h) &= \int_{\{t-h<\phi<t+h\}\setminus\{\phi=t\}} \frac{|\phi(x)-(t+h)| - |\phi(x)-t|}{h}\,dx \\ &\quad + \int_{\{\phi=t\}} \frac{|\phi(x)-(t+h)| - |\phi(x)-t|}{h}\,dx \\ &=: I_2^1(h) + I_2^2(h) \end{aligned}$$

Obviously, we have

$$I_2^2(h) = \frac{|h|}{h}\int_{\{\phi=t\}} 1\,dx = \frac{|h|}{h}\lambda\{\phi=t\}$$

and with dominated convergence we get

$$\lim_{h\to 0} I_2^1(h) = 0.$$

Therefore, $\lim_{h\to 0} I_2(h)$ can only exist, if

$$\lim_{h\to 0} I_2^2(h) = \lambda(\phi=t)\lim_{h\to 0}\frac{|h|}{h}$$

exists, and this is the case if $\lambda(\phi=t)=0$.

Problem 12.25 Solution:

(i) The map $t \mapsto u(t,x) := x^{-2}\sin^2(x)e^{-tx}$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Because of the continuity and differentiability lemmas (Theorem 12.4 and 12.5) it is enough to find suitable majorants for the function and its derivatives. Fix $t \geqslant 0$. Using the elementary inequalities $\frac{\sin x}{x} \leqslant 1$ and $e^{-tx} \leqslant 1$ we get

$$|u(t,x)| \leqslant \mathbb{1}_{[0,1]}(x) + \frac{1}{x^2}\mathbb{1}_{(1,\infty)}(x) =: w(x).$$

Since $w \in \mathcal{L}^1([0,\infty))$ (cf. Beispiel 12.14), continuity follows from the continuity lemma. Assume now that $t \in (r,\infty)$ for some $r > 0$. Then we get

$$\begin{aligned} |\partial_t u(t,x)| &= \left|\frac{\sin^2(x)}{x^2}(-x)e^{-tx}\right| \\ &\leqslant \mathbb{1}_{[0,1]}(x) + xe^{-tx}\mathbb{1}_{[1,\infty)}(x) \in \mathcal{L}^1([0,\infty)) \\ |\partial_t^2 u(t,x)| &= \left|\frac{\sin^2(x)}{x^2}(-x)^2e^{-tx}\right| \\ &\leqslant \mathbb{1}_{[0,1]}(x) + x^2e^{-tx}\mathbb{1}_{[1,\infty)}(x) \in \mathcal{L}^1([0,\infty)). \end{aligned}$$

Now the differentiability lemma shows that f has two derivatives which are given by

$$f'(t) = -\int_0^\infty \frac{\sin^2(x)}{x} e^{-tx}\,dx,$$
$$f''(t) = \int_0^\infty \sin^2(x) e^{-tx}\,dx.$$

(ii) In order to calculate f'' we use that Riemann and Lebesgue integrals auszurechnen coincide if a function is Riemann integrable (Theorem 12.8).

Using $\sin^2(x) = \frac{1}{2}(1-\cos(2x)) = \frac{1}{2}\,\mathrm{Re}(1-e^{i\,2x})$ we get

$$f''(t) = \frac{1}{2}\,\mathrm{Re}\left(\int_0^\infty (1-e^{i\,2x})e^{-tx}\,dx\right),$$

(cf. Problem 10.9). Using dominated convergence, we see

$$\int_0^\infty (1-e^{i\,2x})e^{-tx}\,dx = \lim_{R\to\infty}\int_0^R (1-e^{i\,2x})e^{-tx}\,dx.$$

Since $x \mapsto (1-e^{i\,2x})e^{-tx}$ is Riemann integrable, we can integrate 'as usual':

$$\int_0^\infty (1-e^{i\,2x})e^{-tx}\,dx = \lim_{R\to\infty}\left[\frac{1}{-t}e^{-tx}\right]_{x=0}^R - \lim_{R\to\infty}\left[\frac{1}{2i-t}e^{x(2i-t)}\right]_{x=0}^R = \frac{1}{t} - \frac{1}{t-2i}.$$

Thus,

$$f''(t) = \frac{1}{2}\,\mathrm{Re}\left(\frac{1}{t} - \frac{1}{t-2i}\right) = \frac{1}{2}\left(\frac{1}{t} - \frac{t}{t^2+4}\right) = \frac{2}{t(t^2+4)}.$$

The limits $\lim_{t\to\infty} f(t)$ and $\lim_{t\to\infty} f'(t)$ follow again with dominated convergence (the necessary majorants are those from part (i)):

$$\lim_{t\to\infty} f(t) = \int_0^\infty \lim_{t\to\infty}\left(\frac{\sin^2(x)}{x^2}e^{-tx}\right)dx = 0,$$
$$\lim_{t\to\infty} f'(t) = -\int_0^\infty \lim_{t\to\infty}\left(\frac{\sin^2(x)}{x}e^{-tx}\right)dx = 0.$$

(iii) We begin with a closed expression for f': from the fundamental theorem of (Riemann) integration we know

$$f'(R) - f'(t) = \int_t^R f''(s)\,ds.$$

Letting $R\to\infty$ we get using (ii)

$$\begin{aligned} f'(t) &= -\lim_{R\to\infty}\int_t^R f''(s)\,ds \\ &= -\frac{1}{2}\lim_{R\to\infty}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_{s=t}^R \\ &= \frac{1}{2}\left(\log t - \frac{1}{2}\log(t^2+4)\right) \\ &= \frac{1}{2}\log\frac{t}{\sqrt{t^2+4}}. \end{aligned}$$

Finally,

$$f(t) = -\lim_{R\to\infty} \int_t^R f'(s)\,ds = -\frac{1}{2}\int_t^\infty \log\frac{s}{\sqrt{s^2+4}}\,ds.$$

(In this part we have again used the fact that the Lebesgue integral extends the Riemann integral.)

Problem 12.26 Solution: We follow the hint: since $e^{-tx} \geqslant 0$ we can use Beppo Levi to get

$$\int_0^\infty e^{-xt}\,dx = \sup_{n\in\mathbb{N}} \int_0^n e^{-xt}\,dx = \lim_{n\to\infty}\int_0^n e^{-xt}\,dx.$$

Moreover, $x \mapsto e^{-tx}$ is continuous, hence measurable and Riemann-integrable on compact intervals, and we may (Theorem 12.8) use the Riemann integral to evaluate things.

$$\int_0^n e^{-xt}\,dx = \left[\frac{e^{-tx}}{-t}\right]_{x=0}^n \xrightarrow{n\to\infty} \frac{1}{t}.$$

Thus, $e^{-tx} \in \mathcal{L}^1(0,\infty)$ and $\int_0^\infty e^{-xt}dx = \frac{1}{t}$. Now we use the differentiability lemma, Theorem 12.5. For $u(t,x) := e^{-tx}$ we have

$$|\partial_t u(t,x)| = |x|e^{-tx} \leqslant |x|e^{-ax} \in \mathcal{L}^1(0,\infty) \quad \forall\, t\in(a,\infty),\ a>0,$$

(cf. Example 12.14). Therefore (use the differentiability lemma)

$$\frac{d}{dt}\int_0^\infty e^{-tx}\,dx = \int_0^\infty (-x)e^{-tx}\,dx \qquad \forall\, t\in(a,\infty).$$

Since $a>0$ is arbitrary, we get differentiability on $(0,\infty)$. Iterating this argument, we inver that we can swap derivatives of any order with the integral. Morover,

$$\begin{aligned}\frac{d^n}{dt^n}\left(\int_0^\infty e^{-xt}\,dx\right) &= \frac{d^n}{dt^n}\left(\frac{1}{t}\right)\\ \Rightarrow \left(\int_0^\infty (-x)^n e^{-xt}\,dx\right) &= \frac{(-1)^n n!}{t^{n+1}}.\end{aligned}$$

If $t=1$, the claim follows.

Problem 12.27 Solution: Throughout we fix $(a,b)\subset(0,\infty)$ and take $t\in(a,b)$. As in Problem 12.17 we get

$$\int_{(0,1)} x^{-\delta}\,dx < \infty \quad \forall\,\delta<1 \quad \text{and} \quad \int_{(1,\infty)} x^{-\delta}\,dx < \infty \quad \forall\,\delta>1.$$

(i) Note that differentiability implies continuity, so it suffices to show that Γ is m times differentiable for every m.

Induction Hypothesis: $\Gamma^{(m)}$ exists and is of the form as claimed in the statement of the problem.

Induction Start $m = 1$: We have to show that $\Gamma(t)$ is differentiable. We want to use the differentiability lemma. For this we remark first of all, that the integrand function $t \mapsto \gamma(t,x)$ is differentiable on (a,b) and that

$$\partial_t \gamma(t,x) = \partial_t e^{-x}\, x^{t-1} = e^{-x}\, x^{t-1}\, \log x.$$

We have now to find a uniform (for $t \in (a,b)$) integrable dominating function for $|\partial_t \gamma(t,x)|$. Since $\log x \leqslant x$ for all $x > 0$ (the logarithm is a concave function!),

$$\begin{aligned} \left| e^{-x}\, x^{t-1}\, \log x \right| &= e^{-x}\, x^{t-1}\, \log x \\ &\leqslant e^{-x} x^t \;\leqslant\; e^{-x} x^b \;\leqslant\; C_b\, x^{-2} \qquad \forall\, x \geqslant 1, \quad t \in (a,b) \end{aligned}$$

(for the last step multiply with x^2 and use that $x^\rho e^{-x}$ is continuous for every $\rho > 0$ and $\lim_{x\to\infty} x^\rho e^{-x} = 0$ to find C_b). Moreover,

$$\begin{aligned} \left| e^{-x}\, x^{t-1}\, \log x \right| &\leqslant x^{a-1}\, |\log x| \\ &= x^{a-1}\, \log \frac{1}{x} \;\leqslant\; C_a\, x^{-1/2} \qquad \forall\, x \in (0,1), \quad t \in (a,b) \end{aligned}$$

where we use the fact that $\lim_{x\to 0} x^\rho \log \frac{1}{x} = 0$ which is easily seen by the substitution $x = e^{-u}$ and $u \to \infty$ and the continuity of the function $x^\rho\, \log \frac{1}{x}$.

Both estimates together furnish an integrable dominating function, so the differentiability lemma applies and shows that

$$\Gamma'(t) = \int_{(0,\infty)} \partial_t \gamma(t,x)\, dx = \int_{(0,\infty)} e^{-x}\, x^{t-1}\, \log x\, dx = \Gamma^{(1)}(x).$$

Induction Step $m \rightsquigarrow m+1$: Set $\gamma^{(m)}(t,x) = e^{-x}\, x^{t-1}\, (\log x)^m$. We want to apply the differentiability lemma to $\Gamma^{(m)}(x)$. With very much the same arguments as in the induction start we find that $\gamma^{(m+1)}(t,x) = \partial_t \gamma^{(m)}(t,x)$ exists (obvious) and satisfies the following bounds

$$\begin{aligned} \left| e^{-x}\, x^{t-1}\, (\log x)^{m+1} \right| &= e^{-x}\, x^{t-1}\, (\log x)^{m+1} \\ &\leqslant e^{-x} x^{t+m} \\ &\leqslant e^{-x} x^{b+m} \\ &\leqslant C_{b,m}\, x^{-2} \qquad \forall\, x \geqslant 1, \quad t \in (a,b) \\ \left| e^{-x}\, x^{t-1}\, (\log x)^{m+1} \right| &\leqslant x^{a-1}\, |\log x|^{m+1} \\ &= x^{a-1} \left(\log \frac{1}{x} \right)^{m+1} \\ &\leqslant C_{a,m}\, x^{-1/2} \qquad \forall\, x \in (0,1), \quad t \in (a,b) \end{aligned}$$

and the differentiability lemma applies completing the induction step.

(ii) Using a combination of Beppo Levi (indicated by 'BL'), Riemann=Lebesgue (if the Riemann integral over an interval exists) and integration by parts (for the Riemann integral, indicated by 'parts') techniques we get

$$\begin{aligned}
t\Gamma(t) &= \lim_{n\to\infty} \int_{(1/n,n)} e^{-x}\, t x^{t-1}\, dx && \text{(BL)}\\
&= \lim_{n\to\infty} (R)\int_{1/n}^{n} e^{-x}\, \partial_x x^t\, dx\\
&= \lim_{n\to\infty} \left[e^{-x}\, x^t\right]_{x=1/n}^{n} - \lim_{n\to\infty}(R)\int_{1/n}^{n} \partial_x e^{-x}\, x^t\, dx && \text{(parts)}\\
&= \lim_{n\to\infty} (R)\int_{1/n}^{n} e^{-x}\, x^{(t+1)-1}\, dx\\
&= \lim_{n\to\infty} \int_{(1/n,n)} e^{-x}\, x^{(t+1)-1}\, dx\\
&= \int_{(0,\infty)} e^{-x}\, x^{(t+1)-1}\, dx && \text{(BL)}\\
&= \Gamma(t+1).
\end{aligned}$$

(iii) We have to show that

$$\log\Gamma(\lambda t + (1-\lambda)s) \leqslant \lambda \log\Gamma(t) + (1-\lambda)\log\Gamma(s) \quad \forall s,t>0,\ \lambda\in(0,1).$$

This is clearly equivalent to

$$\Gamma(\lambda t + (1-\lambda)s) \leqslant [\Gamma(t)]^{\lambda}\, [\Gamma(s)]^{1-\lambda} \quad \forall s,t>0,\ \lambda\in(0,1).$$

Fix $s,t>0$ and write $\lambda = \frac{1}{p}$ and $1-\lambda = \frac{1}{q} = 1-\frac{1}{p}$ where $p,q\in(1,\infty)$ are conjugate exponents. We get using Hölder's inequality

$$\begin{aligned}
\Gamma(\lambda t + (1-\lambda)s) &= \int_0^\infty e^{-x} x^{\lambda t + (1-\lambda)s - 1}\, dx\\
&= \int_0^\infty e^{-\frac{1}{p}x} x^{\frac{1}{p}(t-1)} e^{-\frac{1}{q}x} x^{\frac{1}{q}(s-1)}\, dx\\
&\leqslant \left[\int_0^\infty e^{-x} x^{t-1}\, dx\right]^{\frac{1}{p}} \left[\int_0^\infty e^{-x} x^{s-)}\, dx\right]^{\frac{1}{q}}\\
&\leqslant [\Gamma(t)]^{\lambda}\, [\Gamma(s)]^{1-\lambda}.
\end{aligned}$$

(iii) **Alternative – direct calculuation** Since log and Γ are in C^2 we can apply the convexity criterion: $\log\Gamma$ is convex if, and only if, $\frac{d^2}{dt^2}\log\Gamma(t) \geqslant 0$ holds. We have

$$\begin{aligned}
\frac{d}{dt}\log\Gamma(t) &= \frac{\Gamma'(t)}{\Gamma(t)}\\
\frac{d^2}{dt^2}\log\Gamma(t) &= \frac{\Gamma(t)\Gamma''(t) - (\Gamma'(t))^2}{(\Gamma(t))^2}
\end{aligned}$$

which is non-negative iff

$$0 \overset{!}{\leqslant} \Gamma(t)\Gamma''(t) - (\Gamma'(t))^2$$

So with the notation from part (ii), along with the dominated convergence theorem (indicated by 'DC' – this is needed for Γ', since its integrand will take negative values, so Beppo Levi does not apply), we get

$$
\begin{aligned}
\Gamma(t)\Gamma''(t) - (\Gamma'(t))^2 &= \lim_{n\to\infty} \int_{(1/n,n)} \int_{(1/n,n)} e^{-x-y}\,(xy)^{t-1}(\log y)^2\,dx\,dy && \text{(BL)}\\
&\quad - \lim_{n\to\infty} \int_{(1/n,n)} \int_{(1/n,n)} e^{-x-y}\,(xy)^{t-1}\log x \log y\,dx\,dy && \text{(DC)}\\
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1} \log y\,(\log y - \log x)\,dx\,dy\\
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1} \log y \log \frac{y}{x}\,dx\,dy
\end{aligned}
$$

In the last expression we can change the roles of x and y without changing the value of the integrals (Fubini), so we get

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{2}(R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1} \log y \log \frac{y}{x}\,dx\,dy\\
&\quad + \lim_{n\to\infty} \frac{1}{2}(R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1} \log x \log \frac{x}{y}\,dx\,dy\\
&= \lim_{n\to\infty} \frac{1}{2}(R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1}(\log y \log \frac{y}{x} + \log x \log \frac{x}{y})\,dx\,dy.
\end{aligned}
$$

At last, using well-known logarithmic identities, we get

$$
\begin{aligned}
\log y \log \frac{y}{x} + \log x \log \frac{x}{y} &= \log y \log \frac{y}{x} - \log x \log \frac{y}{x}\\
&= \log \frac{y}{x}(\log y - \log x)\\
&= \left(\log \frac{y}{x}\right)^2
\end{aligned}
$$

and inserting this into the above integral gives

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{2}(R) \int_{1/n}^{n} \int_{1/n}^{n} e^{-x-y}(xy)^{t-1} \left(\log \frac{y}{x}\right)^2\,dx\,dy\\
&= \frac{1}{2} \int_{(0,\infty)} \int_{(0,\infty)} \underbrace{e^{-x-y}(xy)^{t-1}(\log \frac{y}{x})^2}_{\geqslant 0}\,dx\,dy \geqslant 0. && \text{(BL)}
\end{aligned}
$$

This finishes the proof.

Problem 12.28 Solution:

(i) The function $x \mapsto x \ln x$ is bounded and continuous in $[0,1]$, hence Riemann integrable. Since in this case Riemann and Lebesgue integrals coincide, we may use Riemann's integral and the usual rules for integration. Thus, changing variables according to $x = e^{-t}$, $dx = -e^{-t}\,dt$ and then $s = (k+1)t$, $ds = (k+1)\,ds$ we find,

$$
\int_0^1 (x \ln x)^k\,dx = \int_0^\infty \left[e^{-t}(-t)\right]^k e^{-t}\,dt
$$

$$= (-1)^k \int_0^\infty t^k e^{-t(k+1)}\,dt$$
$$= (-1)^k \int_0^\infty \Big(\frac{s}{k+1}\Big)^k e^{-s}\,\frac{ds}{k+1}$$
$$= (-1)^k \Big(\frac{1}{k+1}\Big)^{k+1} \int_0^\infty s^{(k+1)-1} e^{-s}\,ds$$
$$= (-1)^k \Big(\frac{1}{k+1}\Big)^{k+1} \Gamma(k+1).$$

(ii) Following the hint we write

$$x^{-x} = e^{-x\ln x} = \sum_{k=0}^\infty (-1)^k \frac{(x\ln x)^k}{k!}.$$

Since for $x \in (0,1)$ the terms under the sum are all positive, we can use Beppo Levi's theorem and the formula $\Gamma(k+1) = k!$ to get

$$\begin{aligned}\int_{(0,1)} x^{-x}\,dx &= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} \int_{(0,1)} (x\ln x)^k\,dx \\ &= \sum_{k=0}^\infty (-1)^k \frac{1}{k!} (-1)^k \Big(\frac{1}{k+1}\Big)^{k+1} \Gamma(k+1) \\ &= \sum_{k=0}^\infty \Big(\frac{1}{k+1}\Big)^{k+1} \\ &= \sum_{n=1}^\infty \Big(\frac{1}{n}\Big)^n.\end{aligned}$$

Problem 12.29 Solution: Fix $(a,b) \subset (0,1)$ and let always $u \in (a,b)$. We have for $x \geqslant 0$ and $L \in \mathbb{N}_0$

$$\begin{aligned}|x^L f(u,x)| &= |x|^L \left|\frac{e^{ux}}{e^x+1}\right| \\ &= x^L \frac{e^{ux}}{e^x+1} \\ &\leqslant x^L \frac{e^{ux}}{e^x} \\ &= x^L e^{(u-1)x} \\ &\leqslant \mathbb{1}_{[0,1]}(x) + M_{a,b}\mathbb{1}_{(1,\infty)}(x)\,x^{-2}\end{aligned}$$

where we use that $u - 1 < 0$, the continuity and boundedness of $x^\rho e^{-ax}$ for $x \in [1,\infty)$ and $\rho \geqslant 0$. If $x \leqslant 0$ we get

$$\begin{aligned}|x^L f(u,x)| &= |x|^L \left|\frac{e^{ux}}{e^x+1}\right| \\ &= |x|^L e^{-u|x|} \\ &\leqslant \mathbb{1}_{[-1,0]}(x) + N_{a,b}\mathbb{1}_{(-\infty,1)}(x)\,|x|^{-2}.\end{aligned}$$

Both inequalities give dominating functions which are integrable; therefore, the integral $\int_{\mathbb{R}} x^L f(u,x)\,dx$ exists.

To see m-fold differentiability, we use the Differentiability lemma (Theorem 12.5) m-times. Formally, we have to use induction. Let us only make the induction step (the start is very similar!). For this, observe that

$$\partial_u^m(x^n f(u,x)) = \partial_u^m \frac{x^n e^{ux}}{e^x+1} = \frac{x^{n+m}e^{ux}}{e^x+1}$$

but, as we have seen in the first step with $L = n+m$, this is uniformly bounded by an integrable function. Therefore, the Differentiability lemma applies and shows that

$$\partial_u^m \int_{\mathbb{R}} x^n f(u,x)\,dx = \int_{\mathbb{R}} x^n \partial_u^m f(u,x)\,dx = \int_{\mathbb{R}} x^{n+m} f(u,x)\,dx.$$

Problem 12.30 Solution: Because of the binomial formual we have $(1+x^2)^n \geqslant 1+nx^2$; this yields, in particular,

$$\left|\frac{1+nx^2}{(1+x^2)^n}\right| \leqslant 1.$$

Since

$$\lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} = 0 \quad \forall\, x \in (0,1)$$

(exponential growth is always stronger than polynomial growth!) we can use dominated convergence and find

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx = 0.$$

Problem 12.31 Solution:

(i) We begin by showing that f is well defined, i.e. the integral expression makes sense. Recall the following estimates

$$|\arctan(y)| \leqslant |y|, \qquad |\arctan(y)| \leqslant \frac{\pi}{2}, \qquad y \in \mathbb{R},$$

(the first inequality follows from the mean value theorem, the second from the definition of arctan.) Moreover,

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \geqslant \frac{1}{2}(e^x - 1) \geqslant \frac{1}{2}\frac{1}{2}x^2 \quad \forall x \geqslant 1.$$

For $u(t,x) := \arctan\left(\frac{t}{\sinh x}\right)$ we see

$$\begin{aligned}|u(t,x)| &\leqslant \frac{\pi}{2}\mathbb{1}_{(0,1)}(x) + \left|\frac{t}{\sinh x}\right| \mathbb{1}_{[1,\infty)}(x)\\ &\leqslant \frac{\pi}{2}\mathbb{1}_{[0,1]}(x) + \frac{1}{4}\frac{1}{x^2}\mathbb{1}_{[1,\infty)}(x) \in \mathcal{L}^1((0,\infty)).\end{aligned}$$

This proves that the integral $f(t) = \int_{(0,\infty)} u(t,x)\,dx$ exists. In order to check differentiability of f, we have to find (Theorem 12.5) a majorizing function for the derivative of the integrand. Fix $R > 0$ and let $t \in (R^{-1}, R)$. By the chain rule

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{1+\left(\frac{t}{\sinh x}\right)^2}\frac{1}{\sinh x}$$

$$= \frac{1}{\frac{t^2}{\sinh x} + \sinh x}.$$

Since $x \mapsto \frac{1}{R^{-2}+\sinh x}$ is continuous, there is a constant $C_1 > 0$ such that

$$\sup_{x\in[0,1]} \frac{1}{R^{-2} + \sinh x} \leqslant C_1.$$

Using $0 \leqslant \sinh x \leqslant 1$ for $x \in [0,1]$, we get

$$|\partial_t u(t,x)| \leqslant \frac{1}{\frac{R^{-2}}{\sinh x} + \sinh x} \leqslant \frac{1}{R^{-2} + \sinh x} \leqslant C_1 \qquad \forall\, x \in [0,1].$$

Similarly we get for $x > 1$

$$|\partial_t u(t,x)| \geqslant \frac{1}{\sinh x} = 2\frac{1}{e^x - e^{-x}} = \frac{2}{e^x}\underbrace{\frac{1}{1-e^{-2x}}}_{\leqslant C_2 < \infty} \in \mathcal{L}^1((1,\infty)).$$

Therefore,

$$|\partial_t u(t,x)| \leqslant C_1 \mathbb{1}_{(0,1]}(x) + 2C_2 \frac{1}{e^x}\mathbb{1}_{(1,\infty)}(x) \in \mathcal{L}^1((0,\infty)).$$

Using the differentiability lemma, Theorem 12.5, we find that f is differentiable on (R^{-1}, R) and that

$$f'(t) = \int_{(0,\infty)} \frac{1}{\frac{t^2}{\sinh x} + \sinh x}\, dx \quad \forall\, t \in (R^{-1}, R).$$

Since $R > 0$ is arbitrary, f is differentiable on $(0,\infty)$. That $\lim_{t\downarrow 0} f'(t)$ does not exist, follows directly from the closed expresson for f' in part (ii).

(ii) Note that $f(0) = 0$. In order to find an expression for f', we perform the following substitution: $u = \cosh x$ and we get, observing that $\cosh^2 x - \sinh^2 x = 1$:

$$\begin{aligned} f'(t) &= \int_{(1,\infty)} \frac{1}{\frac{t^2}{\sqrt{u^2-1}} + \sqrt{u^2-1}} \frac{1}{\sqrt{u^2-1}}\, du \\ &= \int_{(1,\infty)} \frac{1}{t^2 - 1 + u^2}\, du. \end{aligned}$$

(Observe: $x \mapsto \frac{1}{\frac{t^2}{\sinh x}+\sinh x}$ is continuous, hence Riemann-integrable. Since we have established in part (i) the existence of the Lebesgue integral, we can use Riemann integrals (b/o Theorem 12.8).) There are two cases:

- $t > 1$: We have $t^2 - 1 > 0$ and so

$$\begin{aligned} f'(t) &= \frac{1}{t^2-1}\int_{(1,\infty)} \frac{1}{1 + \left(\frac{u}{\sqrt{t^2-1}}\right)^2}\, du \\ &= \frac{1}{t^2-1}\left[\sqrt{t^2-1}\arctan\left(\frac{u}{\sqrt{t^2-1}}\right)\right]_{u=1}^{\infty} \\ &= \frac{1}{\sqrt{t^2-1}}\left(\frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt{t^2-1}}\right)\right) \\ &= \frac{1}{\sqrt{t^2-1}}\arctan\left(\sqrt{t^2-1}\right). \end{aligned}$$

- $t < 1$: Then $C := \sqrt{1-t^2}$ makes sense and we get

$$u^2 + t^2 - 1 = u^2 - C^2 = (u+C)(u-c).$$

Moreover, by partial fractions,

$$\frac{1}{u^2 - C^2} = \frac{1}{2C}\frac{1}{u+C} - \frac{1}{2C}\frac{1}{u-C}$$

and so

$$\begin{aligned}
\int_{(1,\infty)} \frac{1}{u^2+t^2-1}\,du &= \int_{(1,\infty)} \frac{u^2-C^2}{}du \\
&= \frac{1}{2C}\lim_{R\to\infty}\left(\int_1^R \frac{1}{u+C}\,du - \int_1^R \frac{1}{u-C}\,du\right) \\
&= \frac{1}{2C}\lim_{R\to\infty}\left(\ln\left(\frac{1+C}{1-C}\right) + \ln\left(\frac{R+C}{R-C}\right)\right) \\
&= \frac{1}{2C}\ln\left(\frac{1+C}{1-C}\right) \\
&= \frac{1}{2\sqrt{1-t^2}}\ln\left(\frac{1+\sqrt{1-t^2}}{1-\sqrt{1-t^2}}\right).
\end{aligned}$$

The first part of our argument shows, in particular,

$$\int_1^\infty f'(t)\,dt = \infty.$$

Since $f(t) = f(1) + \int_1^t f'(s)\,ds$, $t \geqslant 1$, we get $\lim_{t\to\infty} f(t) = \infty$.

Problem 12.32 Solution:

(i) Since

$$\left|\frac{d^m}{dt^m}\,e^{-tX}\right| = \left|X^m e^{-tX}\right| \leqslant X^m$$

m applications of the differentiability lemma, Theorem 12.5, show that $\phi_X^{(m)}(0+)$ exists and that

$$\phi_X^{(m)}(0+) = (-1)^m \int X^m\,d\mathbb{P}.$$

(ii) Using the exponential series we find that

$$\begin{aligned}
e^{-tX} - \sum_{k=0}^m X^k\,\frac{(-1)^k t^k}{k!} &= \sum_{k=m+1}^\infty X^k\,\frac{(-1)^k t^k}{k!} \\
&= t^{m+1}\sum_{j=0}^\infty X^{m+1+j}\,\frac{(-1)^{m+1+j} t^j}{(m+1+j)!}.
\end{aligned}$$

Since the left-hand side has a finite $\mathbb{P}$-integral, so has the right, i.e.

$$\int \left(\sum_{j=0}^\infty X^{m+1+j}\,\frac{(-1)^{m+1+j} t^j}{(m+1+j)!}\right) d\mathbb{P} \quad \text{converges}$$

and we see that

$$\int \Big(e^{-tX} - \sum_{k=0}^{m} X^k \frac{(-1)^k t^k}{k!} \Big) \, d\mathbb{P} = o(t^m)$$

as $t \to 0$.

(iii) We show, by induction in m, that

$$\left| e^{-u} - \sum_{k=0}^{m-1} \frac{(-u)^k}{k!} \right| \leqslant \frac{u^m}{m!} \qquad \forall\, u \geqslant 0. \tag{*}$$

Because of the elementary inequality

$$|e^{-u} - 1| \leqslant u \qquad \forall\, u \geqslant 0$$

the start of the induction $m = 1$ is clear. For the induction step $m \to m+1$ we note that

$$\begin{aligned}
\left| e^{-u} - \sum_{k=0}^{m} \frac{(-u)^k}{k!} \right| &= \left| \int_0^u \Big(e^{-y} - \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} \Big) \, dy \right| \\
&\leqslant \int_0^u \left| e^{-y} - \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} \right| dy \\
&\overset{(*)}{\leqslant} \int_0^u \frac{y^m}{m!} \, dy \\
&= \frac{u^{m+1}}{(m+1)!},
\end{aligned}$$

and the claim follows.

Setting $x = tX$ in (*), we find by integration that

$$\pm\Big(\int e^{-tX} - \sum_{k=0}^{m-1} (-1)^k t^k \frac{\int X^k \, d\mathbb{P}}{k!} \Big) \leqslant \frac{t^m \int X^m \, d\mathbb{P}}{m!}.$$

(iv) If t is in the radius of convergence of the power series, we know that

$$\lim_{m \to \infty} \frac{|t|^m \int X^m \, d\mathbb{P}}{m!} = 0$$

which, when combined with (iii), proves that

$$\phi_X(t) = \lim_{m \to \infty} \sum_{k=0}^{m-1} (-1)^k t^k \frac{\int X^k \, d\mathbb{P}}{k!}.$$

■■

Problem 12.33 Solution:

(i) Wrong, u is NOT continuous on the irrational numbers. To see this, just take a sequence of rationals $q_j \in \mathbb{Q} \cap [0,1]$ approximating $p \in [0,1] \setminus \mathbb{Q}$. Then

$$\lim_j u(q_j) = 1 \neq 0 = u(p) = u(\lim_j q_j).$$

(ii) True. Mind that v is not continuous at 0, but $\{n^{-1}, n \in \mathbb{N}\} \cup \{0\}$ is still countable.

(iii) True. The points where u and v are not 0 (that is: where they are 1) are countable sets, hence measurable and also Lebesgue null sets. This shows that u, v are measurable and almost everywhere 0, hence $\int u\,d\lambda = 0 = \int v\,d\lambda$.

(iv) True. Since $\mathbb{Q} \cap [0,1]$ as well as $[0,1] \setminus \mathbb{Q}$ are dense subsets of $[0,1]$, ALL lower resp. upper Darboux sums are always

$$S_\pi[u] \equiv 0 \ \text{ resp. } \ S^\pi[u] \equiv 1$$

(for any finite partition π of $[0,1]$). Thus upper and lower integrals of u have the value 0 resp. 1 and it follows that u cannot be Riemann integrable.

■■

Problem 12.34 Solution: Note that every function which has finitely many discontinuities is Riemann integrable. Thus, if $\{q_j\}_{j\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$, the functions $u_j(x) := \mathbb{1}_{\{q_1,q_2,\dots,q_j\}}(x)$ are Riemann integrable (with Riemann integral 0) while their increasing limit $u_\infty = \mathbb{1}_{\mathbb{Q}}$ is not Riemann integrable.

■■

Problem 12.35 Solution: Of course we have to assume that u is Borel measurable! By assumption we know that $u_j := u\mathbb{1}_{[0,j]}$ is (properly) Riemann integrable, hence Lebesgue integrable and

$$\int_{[0,j]} u\,d\lambda = \int_{[0,j]} u_j\,d\lambda = (\mathrm{R})\int_0^j u(x)\,dx \xrightarrow[j\to\infty]{} \int_0^\infty u(x)\,dx.$$

The last limit exists because of improper Riemann integrability. Moreover, this limit is an increasing limit, i.e. a 'sup'. Since $0 \leqslant u_j \uparrow u$ we can invoke Beppo Levi's theorem and get

$$\int u\,d\lambda = \sup_j \int u_j\,d\lambda = \int_0^\infty u(x)\,dx < \infty$$

proving Lebesgue integrability.

■■

Problem 12.36 Solution: Observe that $x^2 = k\pi \iff x = \sqrt{k\pi}$, $x \geqslant 0$, $k \in \mathbb{N}_0$. Thus, Since $\sin x^2$ is continuous, it is on every bounded interval Riemann integrable. By a change of variables, $y = x^2$, we get

$$\int_{\sqrt{a}}^{\sqrt{b}} |\sin(x^2)|\,dx = \int_a^b |\sin y|\,\frac{dy}{2\sqrt{y}} = \int_a^b \frac{|\sin y|}{2\sqrt{y}}\,dy$$

which means that for $a = a_k = k\pi$ and $b = b_k = (k+1)\pi = a_{k+1}$ the values $\int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} |\sin(x^2)|\,dx$ are a decreasing sequence with limit 0. Since on $\left[\sqrt{a_k}, \sqrt{a_{k+1}}\right]$ the function $\sin x^2$ has only one sign (and alternates its sign from interval to interval), we can use Leibniz' convergence criterion to see that the series

$$\sum_k \int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} \sin(x^2)\,dx \tag{*}$$

converges, hence the improper integral exists.

The function $\cos x^2$ can be treated similarly. Alternatively, we remark that $\sin x^2 = \cos(x^2 - \pi/2)$.

The functions are not Lebesgue integrable. Either we show that the series (*) does not converge absolutely, or we argue as follows:

$\sin x^2 = \cos(x^2 - \pi/2)$ shows that $\int |\sin x^2|\,dx$ and $\int |\cos x^2|\,dx$ either both converge or diverge. If they would converge (this is equivalent to Lebesgue integrability...) we would find because of $\sin^2 + \cos^2 \equiv 1$ and $|\sin|, |\cos| \leqslant 1$,

$$\begin{aligned}
\infty = \int_0^\infty 1\,dx &= \int_0^\infty \left[(\sin x^2)^2 + (\cos x^2)^2\right] dx \\
&= \int_0^\infty (\sin x^2)^2\,dx + \int_0^\infty (\cos x^2)^2\,dx \\
&\leqslant \int_0^\infty |\sin x^2|\,dx + \int_0^\infty |\cos x^2|\,dx \;<\; \infty,
\end{aligned}$$

which is a contradiction.

Problem 12.37 Solution: Let $r < s$ and, without loss of generality, $a \leqslant b$. A change of variables yields

$$\begin{aligned}
\int_r^s \frac{f(bx) - f(ax)}{x}\,dx &= \int_r^s \frac{f(bx)}{x}\,dx - \int_r^s \frac{f(ax)}{x}\,dx \\
&= \int_{br}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{as} \frac{f(y)}{y}\,dy \\
&= \int_{as}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{br} \frac{f(y)}{y}\,dy
\end{aligned}$$

Using the mean value theorem for integrals, I.12, we get

$$\begin{aligned}
\int_r^s \frac{f(bx) - f(ax)}{x}\,dx &= f(\xi_s) \int_{as}^{bs} \frac{1}{y}\,dy - f(\xi_r) \int_{ar}^{br} \frac{1}{y}\,dy \\
&= f(\xi_s) \ln \tfrac{b}{a} - f(\xi_r) \ln \tfrac{b}{a}.
\end{aligned}$$

Since $\xi_s \in (as, bs)$ and $\xi_r \in (ar, br)$, we find that $\xi_s \xrightarrow[s\to\infty]{} \infty$ and $\xi_r \xrightarrow[r\to 0]{} 0$ which means that

$$\int_r^s \frac{f(bx) - f(ax)}{x}\,dx = \left[f(\xi_s) - f(\xi_r)\right] \ln \tfrac{b}{a} \xrightarrow[r\to 0]{s\to\infty} (M - m) \ln \tfrac{b}{a}.$$

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13 The function spaces $\mathcal{L}^p$. Solutions to Problems 13.1–13.26

Problem 13.1 Solution:

(i) We use Hölder's inequality for $r, s \in (1, \infty)$ and $\frac{1}{r} + \frac{1}{s} = 1$ to get

$$\begin{aligned}
\|u\|_q^q = \int |u|^q \, d\mu &= \int |u|^q \cdot \mathbb{1} \, d\mu \\
&\leqslant \left(\int |u|^{qr} \, d\mu \right)^{1/r} \cdot \left(\int \mathbb{1}^s \, d\mu \right)^{1/s} \\
&= \left(\int |u|^{qr} \, d\mu \right)^{1/r} \cdot (\mu(X))^{1/s}.
\end{aligned}$$

Now let us choose r and s. We take

$$r = \frac{p}{q} > 1 \implies \frac{1}{r} = \frac{q}{p} \qquad \text{and} \qquad \frac{1}{s} = 1 - \frac{1}{r} = 1 - \frac{q}{p},$$

hence

$$\begin{aligned}
\|u\|_q &= \left(\int |u|^p \, d\mu \right)^{q/p \cdot 1/q} \cdot (\mu(X))^{(1-q/p)(1/q)} \\
&= \left(\int |u|^p \, d\mu \right)^{q/p \cdot 1/q} \cdot (\mu(X))^{1/q - 1/p} \\
&= \|u\|_p \cdot (\mu(X))^{1/q-1/p}.
\end{aligned}$$

(ii) If $u \in \mathcal{L}^p$ we know that u is measurable and $\|u\|_p < \infty$. The inequality in (i) then shows that

$$\|u\|_q \leqslant \text{const} \cdot \|u\|_p < \infty,$$

hence $u \in \mathcal{L}^q$. This gives $\mathcal{L}^p \subset \mathcal{L}^q$. The inclusion $\mathcal{L}^q \subset \mathcal{L}^1$ follows by taking $p \rightsquigarrow q$, $q \rightsquigarrow 1$.

Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^p$ be a Cauchy sequence, i.e. $\lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$. Since by the inequality in (i) also

$$\lim_{m,n\to\infty} \|u_n - u_m\|_q \leqslant \mu(X)^{1/q-1/p} \lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$$

we get that $(u_n)_{n \in \mathbb{N}} \subset \mathcal{L}^q$ is also a Cauchy sequence in $\mathcal{L}^q$.

(iii) No, the assertion breaks down completely if the measure μ has infinite mass. Here is an example: μ = Lebesgue measure on $(1, \infty)$. Then the function $f(x) = \frac{1}{x}$ is not integrable over $[1, \infty)$, but $f^2(x) = \frac{1}{x^2}$ is. In other words: $f \notin \mathcal{L}^1(1, \infty)$ but $f \in \mathcal{L}^2(1, \infty)$, hence $\mathcal{L}^2(1, \infty) \not\subset \mathcal{L}^1(1, \infty)$. (Playing around with different exponents shows that the assertion also fails for other $p, q \geqslant 1$....).

Problem 13.2 Solution: This is going to be a bit messy and rather than showing the 'streamlined' solution we indicate how one could find out the numbers oneself. Now let λ be some number in $(0, 1)$ and let α, β be conjugate indices: $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ where $\alpha, \beta \in (1, \infty)$. Then by the Hölder inequality

$$\begin{aligned}
\int |u|^r \, d\mu &= \int |u|^{r\lambda} |u|^{r(1-\lambda)} \, d\mu \\
&\leqslant \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{1}{\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{1}{\beta}} \\
&= \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{r\lambda}{r\lambda\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{r(1-\lambda)}{r(1-\lambda)\beta}}.
\end{aligned}$$

Taking rth roots on both sides yields

$$\begin{aligned}
\|u\|_r &\leqslant \left(\int |u|^{r\lambda\alpha} \, d\mu \right)^{\frac{\lambda}{r\lambda\alpha}} \left(\int |u|^{r(1-\lambda)\beta} \, d\mu \right)^{\frac{(1-\lambda)}{r(1-\lambda)\beta}} \\
&= \|u\|_{r\lambda\alpha}^{\lambda} \|u\|_{r(1-\lambda)\beta}^{1-\lambda}.
\end{aligned}$$

This leads to the following system of equations:

$$p = r\lambda\alpha, \quad q = r(1-\lambda)\beta, \quad 1 = \frac{1}{\alpha} + \frac{1}{\beta}$$

with unknown quantities α, β, λ. Solving it yields

$$\lambda = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}, \quad \alpha = \frac{q-p}{q-r} \quad \beta = \frac{q-p}{r-p}.$$

Problem 13.3 Solution:

(i) If $u, v \in \mathcal{L}^p(\mu)$, then $u + v$ and αu are again in $\mathcal{L}^p(\mu)$; this follows from the homogeneity of the integral and Minkowski's inequality (Corollary 13.4. Using the Cauchy–Schwarz inequality, the product uv is in $\mathcal{L}^p(\mu)$, if $u, v \in \mathcal{L}^{2p}(\mu)$. More generally: if there are conjugate numbers $\alpha, \beta \in [1, \infty]$ (i.e. $\alpha^{-1} + \beta^{-1} = 1$), such that $u \in \mathcal{L}^{\alpha p}$ and $v \in \mathcal{L}^{\beta p}$, then $uv \in \mathcal{L}^p(\mu)$.

(ii) Consider the measure space $((0, 1), \mathscr{B}(0, 1), \lambda)$ and set $u(x) := v(x) := x^{-1/3}$. This gives

$$\int_0^1 |u(x)|^2 \, dx = \int_0^1 x^{-2/3} \, dx = 3\left[x^{1/3}\right]_{x=0}^1 = 3 < \infty,$$

i.e. $u, v \in \mathcal{L}^2(\mu)$. On the other hand, $u \cdot v \notin \mathcal{L}^2(\mu)$ as

$$\int_0^1 |u(x)v(x)|^2 \, dx = \int_0^1 x^{-4/3} \, dx = \lim_{r \to 0} \left[-3x^{-1/3}\right]_{x=r}^1 = \infty.$$

This proves that $\mathcal{L}^2(\mu)$ is not an algebra. Define $\widetilde{u} := u^2$ and $\widetilde{v} := v^2$, we get a similar counterexample which works in $\mathcal{L}^1(\mu)$.

(iii) From Minkowski's inequality we get

$$\begin{aligned} \|u\|_p &= \|(u-v)+v\|_p \leqslant \|u-v\|_p + \|v\|_p \\ \Rightarrow \quad \|u\|_p - \|v\|_p &\leqslant \|u-v\|_p. \end{aligned}$$

If we change the rôles of u and v, we obtain

$$\|v\|_p - \|u\|_p \leqslant \|v-u\|_p = \|u-v\|_p$$

and, therefore,

$$\Big| \|u\|_p - \|v\|_p \Big| = \max\{\|u\|_p - \|v\|_p, \|v\|_p - \|u\|_p\} \leqslant \|u-v\|_p.$$

Problem 13.4 Solution:

(i) We consider the three cases separately.

(a) Every map $u : (\Omega, \{\emptyset, \Omega\}) \to (\mathbb{R}, \{\emptyset, \mathbb{R}\})$ is measurable. *Indeed:* u is measurable if, and only if, $u^{-1}(A) \in \{\emptyset, \Omega\}$ for all $A \in \mathscr{A} = \{\emptyset, \mathbb{R}\}$. Since

$$u^{-1}(\emptyset) = \emptyset \qquad u^{-1}(\mathbb{R}) = \Omega$$

this is indeed true for any map u.

(b) Every measurable map $u : (\Omega, \{\emptyset, \Omega\}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is constant. *Indeed:* Suppose, u is not constant, i.e. there are $\omega_1, \omega_2 \in \Omega$ and $x, y \in \mathbb{R}$, $x \neq y$, such that $u(\omega_1) = x$, $u(\omega_2) = y$. Then $u^{-1}(\{x\}) \notin \{\emptyset, \Omega\}$ as $\omega_1 \in u^{-1}(\{x\})$ (and so $u^{-1}(\{x\}) \neq \emptyset$) and $\omega_2 \notin u^{-1}(\{x\})$ (and so $u^{-1}(\{x\}) \neq \Omega$).

(c) Every measurable map $u : (\Omega, \{\emptyset, \Omega\}) \to (\mathbb{R}, \mathscr{P}(\mathbb{R}))$ is clearly $\{\emptyset, \Omega\}/\mathscr{B}(\mathbb{R})$-measurable. From (b) we know that such functions are constant. On the other hand, constant maps are measurable for any σ-algebra. Therefore, every $\{\emptyset, \Omega\}/\mathscr{P}(\mathbb{R})$-measurable map is constant.

(ii) We determine first the $\sigma(B)$-measurable maps. We claim: every $\sigma(B)/\mathscr{B}(\mathbb{R})$-measurable map is of the form

$$u(\omega) = c_1 \mathbb{1}_B(\omega) + c_2 \mathbb{1}_{B^c}(\omega), \quad \omega \in \Omega, \tag{$\star$}$$

for $c_1, c_2 \in \mathbb{R}$. *Indeed:* If u is given by ($\star$), then

$$u^{-1}(A) = \begin{cases} \Omega, & c_1, c_2 \in A, \\ B, & c_1 \in A, c_2 \notin A, \\ B^c, & c_1 \notin A, c_2 \in A, \\ \emptyset, & c_1, c_2 \notin A \end{cases}$$

for any $A \in \mathscr{B}(\mathbb{R})$. Therefore, u is $\sigma(B)/\mathscr{B}(\mathbb{R})$-measurable. Conversely, assume that the function u is $\sigma(B)/\mathscr{B}(\mathbb{R})$-measurable. Choose any $\omega_1 \in B$, $\omega_2 \in B^c$ and define $c_1 = u(\omega_1)$, $c_2 = u(\omega_2)$. If u were not of the form ($\star$), then there would be some $\omega \in \Omega$ such that $u(\omega) \notin \{c_1, c_2\}$. In this case $A := \{u(\omega)\}$ satisfies $u^{-1}(A) \notin \{\emptyset, \Omega, B, B^c\}$, contradicting the measurability of u.

By definition,

$$\mathcal{L}^p(\Omega, \sigma(B), \mu) = \left\{ u : (\Omega, \sigma(B)) \to (\mathbb{R}, \mathscr{B}(\mathbb{R})) \text{ measurable} : \int |u|^p \, d\mu < \infty \right\}.$$

We have already shown that the $\sigma(B)$-measurable maps are given by ($\star$). Because of the linearity of the integral we see that

$$\int |u|^p \, d\mu = |c_1|^p \mu(B) + |c_2|^p \mu(B^c).$$

Consequently, $u \in \mathcal{L}^p(\Omega, \sigma(B), \mu)$ if, and only if,

- $c_1 = 0$ or $\mu(B) < \infty$
- $c_2 = 0$ or $\mu(B^c) < \infty$.

In particular, every map of the form ($\star$) is in $\mathcal{L}^p(\Omega, \sigma(B), \mu)$ if μ is a finite measure.

Problem 13.5 Solution: Proof by induction in N.

Start $N = 2$: this is just Hölder's inequality.

Hypothesis: the generalized Hölder inequality holds for some $N \geqslant 2$.

Step $N \rightsquigarrow N + 1$:. Let $u_1, \dots, u_N, w$ be $N + 1$ functions and let $p_1, \dots, p_N, q > 1$ be such that $p_1^{-1} + p_2^{-1} + \dots + p_N^{-1} + q^{-1} = 1$. Set $p^{-1} := p_1^{-1} + p_2^{-1} + \dots + p_N^{-1}$. Then, by the ordinary Hölder inequality,

$$\begin{aligned}\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w| \, d\mu &\leqslant \left(\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N|^p \, d\mu \right)^{1/p} \|u\|_q \\ &= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p \, d\mu \right)^{1/p} \|u\|_q\end{aligned}$$

Now use the induction hypothesis which allows us to apply the generalized Hölder inequality for N (!) factors $\lambda_j := p_j/p$, and thus $\sum_{j=1}^N \lambda_j^{-1} = p/p = 1$, to the first factor to get

$$\begin{aligned}\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w| \, d\mu &= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p \, d\mu \right)^{1/p} \|u\|_q \\ &\leqslant \|u\|_{p_1} \cdot \|u\|_{p_2} \cdot \ldots \cdot \|u\|_{p_N} \|u\|_q.\end{aligned}$$

Problem 13.6 Solution: Draw a picture similar to the one used in the proof of Lemma 13.1 (note that the increasing function need not be convex or concave....). Without loss of generality we can assume that $A, B > 0$ are such that $\phi(A) \geqslant B$ which is equivalent to $A \geqslant \psi(B)$ since ϕ and ψ are inverses. Thus,

$$AB = \int_0^B \psi(\eta)\,d\eta + \int_0^{\psi(B)} \phi(\xi)\,d\xi + \int_{\psi(B)}^A B\,d\xi.$$

Using the fact that ϕ increases, we get that

$$\phi(\psi(B)) = B \implies \phi(C) \geqslant B \quad \forall\, C \geqslant \psi(B)$$

and we conclude that

$$\begin{aligned}
AB &= \int_0^B \psi(\eta)\,d\eta + \int_0^{\psi(B)} \phi(\xi)\,d\xi + \int_{\psi(B)}^A B\,d\xi \\
&\leqslant \int_0^B \psi(\eta)\,d\eta + \int_0^{\psi(B)} \phi(\xi)\,d\xi + \int_{\psi(B)}^A \phi(\xi)\,d\xi \\
&= \int_0^B \psi(\eta)\,d\eta + \int_0^A \phi(\xi)\,d\xi \\
&= \Psi(B) + \Phi(A).
\end{aligned}$$

Problem 13.7 Solution: Let us show first of all that $\mathcal{L}^p\text{-}\lim_{k\to\infty} u_k = u$. This follows immediately from $\lim_{k\to\infty} \|u - u_k\|_p = 0$ since the series $\sum_{k=1}^\infty \|u - u_k\|_p$ converges.

Therefore, we can find a subsequence $(u_{k(j)})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} u_{k(j)}(x) = u(x) \quad \text{almost everywhere.}$$

Now we want to show that u is the a.e. limit of the original sequence. For this we mimic the trick from the Riesz–Fischer theorem 13.7 and show that the series

$$\sum_{j=0}^\infty (u_{j+1} - u_j) = \lim_{K\to\infty} \sum_{j=0}^K (u_{j+1} - u_j) = \lim_{K\to\infty} u_K$$

(again we agree on $u_0 := 0$ for notational convenience) makes sense. So let us employ Lemma 13.6 used in the proof of the Riesz–Fischer theorem to get

$$\begin{aligned}
\left\| \sum_{j=0}^\infty (u_{j+1} - u_j) \right\|_p &\leqslant \left\| \sum_{j=0}^\infty |u_{j+1} - u_j| \right\|_p \\
&\leqslant \sum_{j=0}^\infty \|u_{j+1} - u_j\|_p \\
&\leqslant \sum_{j=0}^\infty \big(\|u_{j+1} - u\|_p + \|u - u_j\|_p \big) \\
&< \infty
\end{aligned}$$

where we use Minkowski's inequality, the function u from above and the fact that $\sum_{j=1}^{\infty} \|u_j - u\|_p < \infty$ along with $\|u_1\|_p < \infty$. This shows that $\lim_{K\to\infty} u_K(x) = \sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x))$ exists almost everywhere.

We still have to show that $\lim_{K\to\infty} u_K(x) = u(x)$. For this we remark that a subsequence has necessarily the same limit as the original sequence—whenever both have limits, of course. But then,

$$u(x) = \lim_{j\to\infty} u_{k(j)}(x) = \lim_{k\to\infty} u_k(x) = \sum_{j=0}^{\infty}(u_{j+1}(x) - u_j(x))$$

and the claim follows.

Problem 13.8 Solution: That for every fixed x the sequence

$$u_n(x) := n\mathbb{1}_{(0,1/n)}(x) \xrightarrow[n\to\infty]{} 0$$

is obvious. On the other hand, for any subsequence $(u_{n(j)})_j$ we have

$$\int |u_{n(j)}|^p \, d\lambda = n(j)^p \, \frac{1}{n(j)} = n(j)^{p-1} \xrightarrow[j\to\infty]{} c$$

with $c = 1$ in case $p = 1$ and $c = \infty$ if $p > 1$. This shows that the $\mathcal{L}^p$-limit of this subsequence—let us call it w if it exists at all—cannot be (not even a.e.) $u = 0$.

On the other hand, we know that a sub-subsequence $(\widetilde{u}_{k(j)})_j$ of $(u_{k(j)})_j$ converges pointwise almost everywhere to the $\mathcal{L}^p$-limit:

$$\lim_j \widetilde{u}_{k(j)}(x) = w(x).$$

Since the full sequence $\lim_n u_n(x) = u(x) = 0$ has a limit, this shows that the sub-sub-sequence limit $w(x) = 0$ almost everywhere—a contradiction. Thus, w does not exist in the first place.

Problem 13.9 Solution: Using Minkowski's and Hölder's inequalities we find for all $\epsilon > 0$

$$\begin{aligned}
\|u_k v_k - uv\|_1 &= \|u_k v_k - u_k v + u_k v - uv\| \\
&\leqslant \|u_k \cdot (v_k - v)\| + \|(u_k - u)v\| \\
&\leqslant \|u_k\|_p \|v_k - v\|_q + \|u_k - u\|_p \|v\|_q \\
&\leqslant (M + \|v\|_q)\epsilon
\end{aligned}$$

for all $n \geqslant N_\epsilon$. We use here that the sequence $(\|u_k\|_p)_{k\in\mathbb{N}}$ is bounded. Indeed, by Minkowski's inequality

$$\|u_k\|_p = \|u_k - u\|_p + \|u\|_p \leqslant \epsilon + \|u\|_p =: M.$$

Problem 13.10 Solution: We use the simple identity

$$\begin{aligned}\|u_n - u_m\|_2^2 &= \int (u_n - u_m)^2 \, d\mu \\ &= \int (u_n^2 - 2u_n u_m + u_m) \, d\mu \qquad (*) \\ &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m \, d\mu.\end{aligned}$$

Case 1: $u_n \to u$ in $\mathcal{L}^2$. This means that $(u_n)_{n\in\mathbb{N}}$ is an $\mathcal{L}^2$ Cauchy sequence, i.e. that $\lim_{m,n\to\infty} \|u_n - u_m\|_2^2 = 0$. On the other hand, we get from the lower triangle inequality for norms

$$\lim_{n\to\infty} \big| \|u_n\|_2 - \|u\|_2 \big| \leqslant \lim_{n\to\infty} \|u_n - u\|_2 = 0$$

so that also $\lim_{n\to\infty} \|u_n\|_2^2 = \lim_{m\to\infty} \|u_m\|_2^2 = \|u\|_2^2$. Using (*) we find

$$\begin{aligned}2\int u_n u_m \, d\mu &= \|u_n\|_2^2 + \|u_m\|_2^2 - \|u_n - u_m\|_2^2 \\ &\xrightarrow[n,m\to\infty]{} \|u\|_2^2 + \|u\|_2^2 - 0 \\ &= 2\|u\|_2^2.\end{aligned}$$

Case 2: Assume that $\lim_{n,m\to\infty} \int u_n u_m \, d\mu = c$ for some number $c \in \mathbb{R}$. By the very definition of this double limit, i.e.

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad : \quad \left| \int u_n u_m \, d\mu - c \right| < \epsilon \quad \forall n, m \geqslant N_\epsilon,$$

we see that $\lim_{n\to\infty} \int u_n u_n \, d\mu = c = \lim_{m\to\infty} \int u_m u_m \, d\mu$ hold (with the same c!). Therefore, again by (*), we get

$$\begin{aligned}\|u_n - u_m\|_2^2 &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m \, d\mu \\ &\xrightarrow[n,m\to\infty]{} c + c - 2c \;=\; 0,\end{aligned}$$

i.e. $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2$ and has, by the completeness of this space, a limit.

■■

Problem 13.11 Solution: Use the exponential series to conclude from the positivity of h and $u(x)$ that

$$\exp(hu) = \sum_{j=0}^{\infty} \frac{h^j u^j}{j!} \geqslant \frac{h^N}{N!} u^N.$$

Integrating this gives

$$\frac{h^N}{N!} \int u^N \, d\mu \leqslant \int \exp(hu) \, d\mu < \infty$$

and we find that $u \in \mathcal{L}^N$. Since μ is a finite measure we know from Problem 13.1 that for $N > p$ we have $\mathcal{L}^N \subset \mathcal{L}^p$.

■■

Problem 13.12 Solution:

(i) We have to show that $|u_n(x)|^p := n^{p\alpha}(x+n)^{-p\beta}$ has finite integral—measurability is clear since u_n is continuous. Since $n^{p\alpha}$ is a constant, we have only to show that $(x+n)^{-p\beta}$ is in $\mathcal{L}^1$. Set $\gamma := p\beta > 1$. Then we get from a Beppo Levi and a domination argument

$$\begin{aligned}\int_{(0,\infty)} (x+n)^{-\gamma}\,\lambda(dx) &\leqslant \int_{(0,\infty)} (x+1)^{-\gamma}\,\lambda(dx)\\ &\leqslant \int_{(0,1)} 1\,\lambda(dx) + \int_{(1,\infty)} (x+1)^{-\gamma}\,\lambda(dx)\\ &\leqslant 1 + \lim_{k\to\infty}\int_{(1,k)} x^{-\gamma}\,\lambda(dx).\end{aligned}$$

Now using that Riemann=Lebesgue on intervals where the Riemann integral exists, we get

$$\begin{aligned}\lim_{k\to\infty}\int_{(1,k)} x^{-\gamma}\,\lambda(dx) &= \lim_{k\to\infty}\int_1^k x^{-\gamma}\,dx\\ &= \lim_{k\to\infty}\left[(1-\gamma)^{-1}x^{1-\gamma}\right]_1^k\\ &= (1-\gamma)^{-1}\lim_{k\to\infty}\left(k^{1-\gamma}-1\right)\\ &= (\gamma-1)^{-1} < \infty\end{aligned}$$

which shows that the integral is finite.

(ii) We have to show that $|v_n(x)|^q := n^{q\gamma}e^{-qnx}$ is in $\mathcal{L}^1$—again measurability is inferred from continuity. Since $n^{q\gamma}$ is a constant, it is enough to show that e^{-qnx} is integrable. Set $\delta = qn$. Since

$$\lim_{x\to\infty}(\delta x)^2 e^{-\delta x} = 0 \qquad \text{and} \qquad e^{-\delta x} \leqslant 1 \quad \forall\, x \geqslant 0,$$

and since $e^{-\delta x}$ is continuous on $[0,\infty)$, we conclude that there are constants $C, C(\delta)$ such that

$$\begin{aligned}e^{-\delta x} &\leqslant \min\left\{1, \frac{C}{(\delta x)^2}\right\}\\ &\leqslant C(\delta)\min\left\{1, \frac{1}{x^2}\right\}\\ &= C(\delta)\left(\mathbb{1}_{(0,1)}(x) + \mathbb{1}_{[1,\infty)}\frac{1}{x^2}\right)\end{aligned}$$

but the latter is an integrable function on $(0,\infty)$.

Problem 13.13 Solution: Without loss of generality we may assume that $\alpha \leqslant \beta$. We distinguish between the case $x \in (0,1)$ and $x \in [1,\infty)$. If $x \leqslant 1$, then

$$\frac{1}{x^\alpha} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\alpha + x^\alpha} = \frac{1/2}{x^\alpha + x^\alpha} \quad \forall\, x \leqslant 1;$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((0,1), dx)$ if, and only if, $\alpha < 1$.

Similarly, if $x \geqslant 1$, then

$$\frac{1}{x^\beta} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\beta + x^\beta} = \frac{1/2}{x^\beta + x^\beta} \quad \forall\, x \geqslant 1$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((1,\infty), dx)$ if, and only if, $\beta > 1$.

Thus, $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1(\mathbb{R}, dx)$ if, and only if, both $\alpha < 1$ and $\beta > 1$.

Problem 13.14 Solution: If we use $X = \{1, 2, \dots, n\}$, $x(j) = x_j$, $\mu = \epsilon_1 + \cdots + \epsilon_n$ we have

$$\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \|x\|_{p(\mu)}$$

and it is clear that this is a norm for $p \geqslant 1$ and, in view of Problem 13.19 it is not a norm for $p < 1$ since the triangle (Minkowski) inequality fails. (This could also be shown by a direct counterexample.

Problem 13.15 Solution: Without loss of generality we can restrict ourselves to positive functions—else we would consider positive and negative parts. Separability can obviously considered separately!

Assume that $\mathcal{L}^1_+$ is separable and choose $u \in \mathcal{L}^p_+$. Then $u^p \in \mathcal{L}^1$ and, because of separability, there is a sequence $(f_n)_n \subset \mathcal{D}_1 \subset \mathcal{L}^1$ such that

$$f_n \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p \implies u_n^p \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p$$

if we set $u_n := f_n^{1/p} \in \mathcal{L}^p$. In particular, $u_{n(k)}(x) \to u(x)$ almost everywhere for some subsequence and $\|u_{n(k)}\|_p \xrightarrow[k\to\infty]{} \|u\|_p$. Thus, Riesz's theorem 13.10 applies and proves that

$$\mathcal{L}^p \ni u_{n(k)} \xrightarrow[k\to\infty]{\text{in } \mathcal{L}^p} u.$$

Obviously the separating set $\mathcal{D}_p$ is essentially the same as $\mathcal{D}_1$, and we are done.

The converse is similar (note that we did not make any assumptions on $p \geqslant 1$ or $p < 1$—this is immaterial in the above argument).

Problem 13.16 Solution: We have seen in the lecture that, whenever $\lim_{n\to\infty} \|u - u_n\|_p = 0$, there is a subsequence $u_{n(k)}$ such that $\lim_{k\to\infty} u_{n(k)}(x) = u(x)$ almost everywhere. Since, by assumption, $\lim_{j\to\infty} u_j(x) = w(x)$ a.e., we have also that $\lim_{j\to\infty} u_{n(j)}(x) = w(x)$ a.e., hence $u(x) = w(x)$ almost everywhere.

Problem 13.17 Solution: We remark that $y \mapsto \log y$ is concave. Therefore, we can use Jensen's inequality for concave functions to get for the probability measure $\mu/\mu(X) = \mu(X)^{-1}\mathbb{1}_X\,\mu$

$$\begin{aligned}\int (\log u)\,\frac{d\mu}{\mu(X)} &\leqslant \log\left(\int u\,\frac{d\mu}{\mu(X)}\right)\\ &= \log\left(\frac{\int u\,d\mu}{\mu(X)}\right)\\ &= \log\left(\frac{1}{\mu(X)}\right),\end{aligned}$$

and the claim follows.

Problem 13.18 Solution: As a matter of fact,

$$\int_{(0,1)} u(s)\,ds \cdot \int_{(0,1)} \log u(t)\,dt \leqslant \int_{(0,1)} u(x)\log u(x)\,dx.$$

We begin by proving the hint. $\log x \geqslant 0 \iff x \geqslant 1$. So,

$$\begin{aligned}&\forall y \geqslant 1 : \Big(\log y \leqslant y\log y \iff 1 \leqslant y\Big)\\ \text{and}\quad &\forall y \leqslant 1 : \Big(\log y \leqslant y\log y \iff 1 \geqslant y\Big).\end{aligned}$$

Assume now that $\int_{(0,1)} u(x)\,dx = 1$. Substituting in the above inequality $y = u(x)$ and integrating over $(0,1)$ yields

$$\int_{(0,1)} \log u(x)\,dx \leqslant \int_{(0,1)} u(x)\log u(x)\,dx.$$

Now assume that $\alpha = \int_{(0,1)} u(x)\,dx$. Then $\int_{(0,1)} u(x)/\alpha\,dx = 1$ and the above inequality gives

$$\int_{(0,1)} \log\frac{u(x)}{\alpha}\,dx \leqslant \int_{(0,1)} \frac{u(x)}{\alpha}\log\frac{u(x)}{\alpha}\,dx$$

which is equivalent to

$$\begin{aligned}&\int_{(0,1)} \log u(x)\,dx - \log\alpha\\ &\qquad = \int_{(0,1)} \log u(x)\,dx - \int_{(0,1)} \log\alpha\,dx\\ &\qquad = \int_{(0,1)} \log\frac{u(x)}{\alpha}\,dx\\ &\qquad \leqslant \int_{(0,1)} \frac{u(x)}{\alpha}\log\frac{u(x)}{\alpha}\,dx\\ &\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log\frac{u(x)}{\alpha}\,dx\\ &\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log u(x)\,dx - \frac{1}{\alpha}\int_{(0,1)} u(x)\log\alpha\,dx\\ &\qquad = \frac{1}{\alpha}\int_{(0,1)} u(x)\log u(x)\,dx - \frac{1}{\alpha}\int_{(0,1)} u(x)\,dx\,\log\alpha\end{aligned}$$

$$= \frac{1}{\alpha} \int_{(0,1)} u(x) \log u(x)\, dx - \log \alpha.$$

The claim now follows by adding $\log \alpha$ on both sides and then multiplying by $\alpha = \int_{(0,1)} u(x)\, dx$.

Problem 13.19 Solution:

(i) Let $p \in (0,1)$ and pick the conjugate index $q := p/(p-1) < 0$. Moreover, $s := 1/p \in (1,\infty)$ and the conjugate index t, $\frac{1}{s} + \frac{1}{t} = 1$, is given by

$$t = \frac{s}{s-1} = \frac{\frac{1}{p}}{\frac{1}{p} - 1} = \frac{1}{1-p} \in (1,\infty).$$

Thus, using the normal Hölder inequality for s, t we get

$$\begin{aligned} \int u^p\, d\mu &= \int u^p \frac{w^p}{w^p}\, d\mu \\ &\leqslant \left(\int \left(u^p w^p\right)^s\, d\mu \right)^{1/s} \left(\int w^{-pt}\, d\mu \right)^{1/t} \\ &= \left(\int uw\, d\mu \right)^p \left(\int w^{p/(p-1)}\, d\mu \right)^{1-p}. \end{aligned}$$

Taking pth roots on either side yields

$$\begin{aligned} \left(\int u^p\, d\mu \right)^{1/p} &\leqslant \left(\int uw\, d\mu \right) \left(\int w^{p/(p-1)}\, d\mu \right)^{(1-p)/p} \\ &= \left(\int uw\, d\mu \right) \left(\int w^q\, d\mu \right)^{-1/q} \end{aligned}$$

and the claim follows.

(ii) This 'reversed' Minkowski inequality follows from the 'reversed' Hölder inequality in exactly the same way as Minkowski's inequality follows from Hölder's inequality, cf. Corollary 13.4. To wit:

$$\begin{aligned} \int (u+v)^p\, d\mu &= \int (u+v) \cdot (u+v)^{p-1}\, d\mu \\ &= \int u \cdot (u+v)^{p-1}\, d\mu + \int v \cdot (u+v)^{p-1}\, d\mu \\ &\overset{\text{(i)}}{\geqslant} \|u\|_p \cdot \left\|(u+v)^{p-1}\right\|_q + \|v\|_p \cdot \left\|(u+v)^{p-1}\right\|_q. \end{aligned}$$

Dividing both sides by $\||u+v|^{p-1}\|_q$ proves our claim since

$$\left\|(u+v)^{p-1}\right\|_q = \left(\int (u+v)^{(p-1)q}\, d\mu \right)^{1/q} = \left(\int (u+v)^p\, d\mu \right)^{1-1/p}.$$

Problem 13.20 Solution: By assumption, $|u| \leqslant \|u\|_\infty \leqslant C < \infty$ and $u \not\equiv 0$.

(i) We have

$$M_n = \int |u|^n \, d\mu \leqslant C^n \int d\mu = C^n \mu(X) \in (0, \infty).$$

Note that $M_n > 0$.

(ii) By the Cauchy–Schwarz-Inequality,

$$\begin{aligned} M_n &= \int |u|^n \, d\mu \\ &= \int |u|^{\frac{n+1}{2}} |u|^{\frac{n-1}{2}} \, d\mu \\ &\leqslant \left(\int |u|^{n+1} \, d\mu \right)^{1/2} \left(\int |u|^{n-1} \, d\mu \right)^{1/2} \\ &= \sqrt{M_{n+1} M_{n-1}}. \end{aligned}$$

(iii) The upper estimate follows from

$$M_{n+1} = \int |u|^{n+1} \, d\mu \leqslant \int |u|^n \cdot \|u\|_\infty \, d\mu = \|u\|_\infty M_n.$$

Set $P := \mu/\mu(X)$; the lower estimate is equivalent to

$$\begin{aligned} \left(\int |u|^n \frac{d\mu}{\mu(X)} \right)^{1/n} &\leqslant \frac{\int |u|^{n+1} \frac{d\mu}{\mu(X)}}{\int |u|^n \frac{d\mu}{\mu(X)}} \\ \iff \left(\int |u|^n \, dP \right)^{1+1/n} &\leqslant \int |u|^{n+1} \, dP \\ \iff \left(\int |u|^n \, dP \right)^{(n+1)/n} &\leqslant \int |u|^{n+1} \, dP \end{aligned}$$

and the last inequality follows easily from Jensen's inequality since P is a probability measure:

$$\left(\int |u|^n \, dP \right)^{(n+1)/n} \int |u|^{n \cdot \frac{n+1}{n}} \, dP = \int |u|^{n+1} \, dP.$$

(iv) Following the hint we get

$$\|u\|_n \geqslant \Big(\mu\{u > \|u\|_\infty - \epsilon\} \Big)^{1/n} \big(\|u\|_\infty - \epsilon \big) \xrightarrow[\epsilon \to 0]{n \to \infty} \|u\|_\infty,$$

i.e.

$$\liminf_{n \to \infty} \|u\|_n \geqslant \|u\|_\infty.$$

Combining this with the estimate from (iii), we get

$$\begin{aligned} \|u\|_\infty &\leqslant \liminf_{n \to \infty} \mu(X)^{-1/n} \|u\|_n \\ &\overset{\text{(iii)}}{\leqslant} \liminf_{n \to \infty} \frac{M_{n+1}}{M_n} \\ &\leqslant \limsup_{n \to \infty} \frac{M_{n+1}}{M_n} \\ &\leqslant \|u\|_\infty. \end{aligned}$$

Problem 13.21 Solution: The hint says it all.... Maybe, you have a look at the specimen solution of Problem 13.20, too.

Case 1: $\|u\|_{L^\infty} < \infty$. For $A_\delta := \{u \geqslant \|u\|_\infty - \delta\}$, $\delta > 0$, we gave $\mu(A_\delta) > 0$ and

$$\|u\|_p \geqslant \left(\int_{A_\delta} (\|u\|_\infty - \delta)^p \, d\mu\right)^{\frac{1}{p}} = (\|u\|_\infty - \delta)\mu(A_\delta)^{\frac{1}{p}}.$$

Therefore,

$$\liminf_{p\to\infty} \|u\|_p \geqslant \liminf_{n\to\infty} \left((\|u\|_\infty - \delta)\mu(A_\delta)^{\frac{1}{p}}\right) = \|u\|_\infty - \delta.$$

Since $\delta > 0$ is arbitrary, this shows that $\liminf_{p\to\infty} \|u\|_p \geqslant \|u\|_\infty$.

On the other hand, we have for $p > q$

$$\int |u(x)|^p \, d\mu = \int |u(x)|^{p-q} |u(x)|^q \, d\mu \leqslant \|u\|_\infty^{p-q} \|u\|_q^q.$$

Taking pth roots on both sides of the inequality, we get

$$\limsup_{p\to\infty} \|u\|_p \leqslant \limsup_{p\to\infty} \left(\|u\|_\infty^{\frac{p-q}{p}} \|u\|_q^{\frac{q}{p}}\right) = \|u\|_\infty.$$

This finishes the proof for all $\|u\|_{L^\infty} < \infty$.

Case 2: $\|u\|_{L^\infty} = \infty$. The estimate

$$\limsup_{p\to\infty} \|u\|_p \leqslant \|u\|_\infty$$

is trivially true. The converse inequality follows like this: Define $A_R := \{u \geqslant R\}$, $R > 0$. We have $\mu(A_r) > 0$ (otherwise $\|u\|_{L^\infty} < \infty$!) and, as in the first part of the proof, we find

$$\|u\|_p \geqslant \left(\int_{A_R} R^p \, d\mu\right)^{\frac{1}{p}} = R\mu(A_R)^{\frac{1}{p}}.$$

Thus, $\liminf_{p\to\infty} \|u\|_p \geqslant R$ and since $R > 0$ is arbitrary, the claim follows:

$$\liminf_{p\to\infty} \|u\|_p \geqslant \infty = \|u\|_\infty.$$

Problem 13.22 Solution: We begin with two observations

- If $r \leqslant s \leqslant q$, then $\|u\|_r \leqslant \|u\|_s$. This follows from Jensen's inequality (Theorem 13.13) and the fact that $V(x) := x^{s/r}$, $x \in \mathbb{R}$, is convex (cf. also Problem 13.1). In particular, $\|u\|_r < \infty$ for all $r \in (0, q)$.

- We have
$$\int \log|u|\,d\mu \leqslant \log\|u\|_p \quad \forall\, p \in (0,q). \tag{$\star$}$$
This follows again from Jensen's inequality applied to the convex function $V(x) := -\log x$:
$$-\log\left(\int |u|^p\,d\mu\right) \leqslant \int -\log(|u|^p)\,d\mu - p\int \log|u|\,d\mu;$$
therefore,
$$\log\|u\|_p = \frac{1}{p}\log\left(\int |u|^p\,d\mu\right) \geqslant \int \log|u|\,d\mu.$$

Because of ($\star$) it is enough to show that $\lim_{p\to 0}\|u\|_p \leqslant \exp(\int \ln|u|\,d\mu)$. (Note: by the monotonicity of $\|u\|_p$ as $p \downarrow 0$ we know that the limit $\lim_{p\to 0}\|u\|_p$ exists.) Note that
$$\log a = \inf_{p>0}\frac{a^p-1}{p}, \quad a > 0. \tag{$\star\star$}$$
(Hint: show by differentiation that $p \mapsto \frac{a^p-1}{p}$ is increasing.
Use l'Hospital's rule to show that $\lim_{p\to 0}\frac{a^p-1}{p} = \log a$.) From monotone convergence (mc) we get
$$\begin{aligned}\int \log|u|\,d\mu &\overset{\text{mc}}{=} \inf_{p>0}\int \frac{|u|^p-1}{p}\,d\mu\\ &= \inf_{p>0}\frac{\int |u|^p\,d\mu - 1}{p}\\ &= \inf_{p>0}\frac{\|u\|_p^p - 1}{p} \overset{(\star\star)}{=} \log\|u\|_p\end{aligned}$$
for all $p > 0$. Letting $p \to 0$ finishes the proof.

■■

Problem 13.23 Solution: Without loss of generality we may assume that $f \geqslant 0$. We use the following standard representation of f, see (8.7):
$$f = \sum_{j=0}^N \phi_j \mathbb{1}_{A_j}$$
with $0 = \phi_0 < \phi_1 < \ldots < \phi_N < \infty$ and mutually disjoint sets A_j. Clearly, $\{f \neq 0\} = A_1 \uplus \cdots \uplus A_N$.

Assume first that $f \in \mathcal{E} \cap \mathcal{L}^p(\mu)$. Then
$$\infty > \int f^p\,d\mu = \sum_{j=1}^N \phi_j^p\,\mu(A_j) \geqslant \sum_{j=1}^N \phi_1^p\,\mu(A_j) = \phi_1^p\,\mu(\{f \neq 0\});$$
thus $\mu(\{f \neq 0\}) < \infty$.

Conversely, if $\mu(\{f \neq 0\}) < \infty$, we get
$$\int f^p\,d\mu = \sum_{j=1}^N \phi_j^p\,\mu(A_j) \leqslant \sum_{j=1}^N \phi_N^p\,\mu(A_j) = \phi_N^p\,\mu(\{f \neq 0\}) < \infty.$$

Since this integrability criterion does not depend on $p \geqslant 1$, it is clear that $\mathcal{E}^+ \cap \mathcal{L}^p(\mu) = \mathcal{E}^+ \cap \mathcal{L}^1(\mu)$, and the rest follows since $\mathcal{E} = \mathcal{E}^+ - \mathcal{E}^+$.

Problem 13.24 Solution: (i) $\iff$ (ii) and (iii) $\iff$ (iv), since Λ is concave if, and only if, $V = -\Lambda$ is convex. Moreover, (iii) generalizes (i) and (iv) gives (ii). It is, therefore, enough to verify (iii).

Since u is integrable and takes values in (a, b), we get

$$a = \int a\,\mu(dx) < \int u(x)\,\mu(dx) < \int b\,\mu(dx) = b.$$

This shows that the l.h.S. of the Jensen inequality is well-defined. The rest of the proof is similar to the one of Theorem 13.13: take some affine-linear $\ell(x) = \alpha x + \beta \leqslant V(x)$ – here we only consider $x \in (a, b)$ – and notice that

$$\ell\left(\int u\,d\mu\right) = \alpha \int u\,d\mu + \beta = \int (\alpha u + \beta)\,d\mu \leqslant \int V(u)\,d\mu.$$

Now go to the sup over all affine-linear ℓ below V and the claim follows.

Problem 13.25 Solution:

(i) Note that $\Lambda(x) = x^{1/q}$ is concave—e.g. differentiate twice and show that it is negative—and using Jensen's inequality for positive $f, g \geqslant 0$ yields

$$\begin{aligned}\int fg\,d\mu &= \int g f^{-p/q} \mathbb{1}_{\{f\neq 0\}} f^p\,d\mu \\ &\leqslant \int f^p\,d\mu \left(\frac{\int g^q f^{-p} \mathbb{1}_{\{f\neq 0\}} f^p\,d\mu}{\int f^p\,d\mu}\right)^{1/q} \\ &\leqslant \left(\int f^p\,d\mu\right)^{1-1/q} \left(\int g^q\,d\mu\right)^{1/q}\end{aligned}$$

where we use $\mathbb{1}_{\{f\neq 0\}} \leqslant 1$ in the last step.

Note that $fg \in \mathcal{L}^1$ follows from the fact that $\left(g^q f^{-p} \mathbb{1}_{\{f\neq 0\}}\right) f^p = g^q \in \mathcal{L}^1$.

(ii) The function $\Lambda(x) = (x^{1/p} + 1)^p$ has second derivative

$$\Lambda''(x) = \frac{1-p}{p}\left(1 + x^{-1/p}\right) x^{-1-1/p} \leqslant 0$$

showing that Λ is concave. Using Jensen's inequality gives for $f, g \geqslant 0$

$$\begin{aligned}\int (f+g)^p \mathbb{1}_{\{f\neq 0\}}\,d\mu &= \int \left(\frac{g}{f}\mathbb{1}_{\{f\neq 0\}} + 1\right)^p f^p \mathbb{1}_{\{f\neq 0\}}\,d\mu \\ &\leqslant \int_{\{f\neq 0\}} f^p\,d\mu \left[\left(\frac{\int g^p \mathbb{1}_{\{f\neq 0\}}\,d\mu}{\int_{\{f\neq 0\}} f^p\,d\mu}\right)^{1/p} + 1\right]^p \\ &= \left[\left(\int_{\{f\neq 0\}} g^p\,d\mu\right)^{1/p} + \left(\int_{\{f\neq 0\}} f^p\,d\mu\right)^{1/p}\right]^p.\end{aligned}$$

Adding on both sides $\int_{\{f=0\}}(f+g)^p\,d\mu = \int_{\{f=0\}} g^p\,d\mu$ yields, because of the elementary inequality $A^p + B^p \leqslant (A+B)^p$, $A, B \geqslant 0$, $p \geqslant 1$,

$$\begin{aligned}
&\int (f+g)^p\,d\mu \\
&\leqslant \left[\left(\int_{\{f\neq 0\}} g^p\,d\mu\right)^{1/p} + \left(\int_{\{f\neq 0\}} f^p\,d\mu\right)^{1/p}\right]^p + \left[\int_{\{f=0\}} g^p\,d\mu\right]^{p/p} \\
&\leqslant \left[\left(\int g^p\,d\mu\right)^{1/p} + \left(\int f^p\,d\mu\right)^{1/p}\right]^p.
\end{aligned}$$

Problem 13.26 Solution: Using Hölder's inequality we get

$$|f-a|^p \leqslant (|f|+|a|)^p = (1\cdot|f| + 1\cdot|a|)^p \leqslant 2^{p-1}(|f|^p + |a|^p).$$

Since $\mu(X) < \infty$, this shows that both sides of the asserted integral inequality are finite.

Without loss of generality we may assume that $a > 0$, otherwise we would consider $-f$ instead of f.

Without loss of generality we may assume that $m = \int f\,d\mu = 0$, otherwise we would consider $f - \int f\,d\mu$ instead of f.

Observe that

$$\begin{aligned}
\int_{\{0<f<2a\}} |f|^p\,d\mu &\leqslant (2a)^{p-1}\int_{\{0<f<2a\}} |f|\,d\mu \\
&\leqslant (2a)^{p-1}\int_{\{f>0\}} |f|\,d\mu \\
&= (2a)^{p-1}\int_{\{f<0\}} |f|\,d\mu.
\end{aligned}$$

In the last line we use the fact that

$$\int_{\{f>0\}} |f|\,d\mu = \int f^+\,d\mu \overset{\int f\,d\mu=0}{=} \int f^-\,d\mu = \int_{\{f<0\}} |f|\,d\mu.$$

Thus,

$$\begin{aligned}
\int_{\{0<f<2a\}} |f|^p\,d\mu &\leqslant (2a)^{p-1}\int_{\{f<0\}} |f|\,d\mu \\
&\leqslant 2^{p-1}\int_{\{f<0\}} (a^p \vee |f|^p)\,d\mu \qquad (*) \\
&\leqslant 2^{p-1}\int_{\{f<0\}} |f-a|^p\,d\mu.
\end{aligned}$$

Moreover,

$$\int_{\{f>2a\}} |f|^p\,d\mu \leqslant 2^p \int_{\{f>2a\}} |f-a|^p\,d\mu, \qquad (**)$$

which follows from

$$f > 2a \implies |f-a| = f-a > a.$$

Finally,

$$\int_{\{f\leqslant 0\}} |f|^p \, d\mu \leqslant 2^p \int_{\{f\leqslant 0\}} |f-a|^p \, d\mu. \tag{***}$$

If we combine (*)–(***) we get

$$\begin{aligned}\int |f|^p \, d\mu &= \left\{\int_{\{f>2a\}} + \int_{\{0<f<2a\}} + \int_{\{f\leqslant 0\}}\right\} |f|^p \, d\mu \\ &\leqslant 2^p \int_{\{f>2a\}} |f-a|^p \, d\mu + (2^{p-1}+1) \int_{\{f\leqslant 0\}} |f-a|^p \, d\mu \\ &\leqslant 2^p \int |f-a|^p \, d\mu.\end{aligned}$$

Solution 2 to 13.26: We need the following inequality for $a, b \in \mathbb{R}$ which follows from Hölder's inequality:

$$|a-b|^p \leqslant (|a|+|b|)^p = (1\cdot|a| + 1\cdot|b|)^p \leqslant 2^{p-1}(|a|^p + |b|^p).$$

Set $b = f(x)$. Since $\mu(X) < \infty$, this shows that both sides of the claimed integral inequality are finite.

Assume first that $\mu(X) = 1$. Then we find

$$\begin{aligned}|f(x)-m|^p &\leqslant (|f(x)-a| + |m-a|)^p \\ &\leqslant 2^{p-1}|f(x)-a|^p + 2^{p-1}|m-a|^p \\ &= 2^{p-1}|f(x)-a|^p + 2^{p-1}\left|\int f(y)\,\mu(dy) - a\right|^p \\ &= 2^{p-1}|f(x)-a|^p + 2^{p-1}\left|\int (f(y)-a)\,\mu(dy)\right|^p \\ &\leqslant 2^{p-1}|f(x)-a|^p + 2^{p-1}\int |f(y)-a|^p\,\mu(dy)\end{aligned}$$

by Jensen's inequality. Now we divide by 2^p and integrate both sides with respect to $\mu(dx)$ to get

$$2^{-p}\int |f(x)-m|^p\,\mu(dx) \leqslant \frac{1}{2}\int |f(x)-a|^p\,\mu(dx) + \frac{1}{2}\int |f(y)-a|^p\,\mu(dy)$$

which proves our claim for probability measures.

If μ is a general finite measure we set $g := f - \int f\,d\mu$ and use the previous estimate

$$\int |g|^p \frac{d\mu}{\mu(X)} \leqslant 2^{p-1}\int |g-a| \frac{d\mu}{\mu(X)} \qquad \forall a \in \mathbb{R}.$$

Since a is arbitrary, we see from this

$$\int |f-m|^p \frac{d\mu}{\mu(X)} \leqslant 2^{p-1}\int |f-b| \frac{d\mu}{\mu(X)} \qquad \forall b \in \mathbb{R}.$$

Remark: the same argument shows that we get for any convex function ϕ with the 'doubling property' $\phi(2x) \leqslant c_\phi \phi(x)$ for all x:

$$\int \phi(f-m)\,d\mu \leqslant c_\phi \int \phi(f-a)\,d\mu \qquad \forall a \in \mathbb{R}.$$

■■

14 Product measures and Fubini's theorem. Solutions to Problems 14.1–14.20

Problem 14.1 Solution:

- We have

$$\begin{aligned}(x,y) \in \Big(\bigcup_i A_i\Big) \times B &\iff x \in \bigcup_i A_i \text{ and } y \in B \\ &\iff \exists\, i_0 \,:\, x \in A_{i_0} \text{ and } y \in B \\ &\iff \exists\, i_0 \,:\, (x,y) \in A_{i_0} \times B \\ &\iff (x,y) \in \bigcup_i (A_i \times B).\end{aligned}$$

- We have

$$\begin{aligned}(x,y) \in \Big(\bigcap_i A_i\Big) \times B &\iff x \in \bigcap_i A_i \text{ and } y \in B \\ &\iff \forall\, i \,:\, x \in A_i \text{ and } y \in B \\ &\iff \forall\, i \,:\, (x,y) \in A_i \times B \\ &\iff (x,y) \in \bigcap_i (A_i \times B).\end{aligned}$$

- Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 135 and the fact that inverse maps interchange with all set operations, we get

$$\begin{aligned}(A \times B) \cap (A' \times B') &= \Big[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\Big] \cap \Big[\pi_1^{-1}(A') \cap \pi_2^{-1}(B')\Big] \\ &= \Big[\pi_1^{-1}(A) \cap \pi_1^{-1}(A')\Big] \cap \Big[\pi_2^{-1}(B) \cap \pi_2^{-1}(B')\Big] \\ &= \pi_1^{-1}(A \cap A') \cap \pi_2^{-1}(B \cap B') \\ &= (A \cap A') \times (B \cap B').\end{aligned}$$

- Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 135 and the fact that inverse maps interchange with all set operations, we get

$$\begin{aligned}A^c \times B &= \pi_1^{-1}(A^c) \cap \pi_2^{-1}(B) \\ &= \big[\pi_1^{-1}(A)\big]^c \cap \pi_2^{-1}(B) \\ &= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \big[\pi_1^{-1}(A)\big]^c\end{aligned}$$

$$
\begin{aligned}
&= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \left\{ \left[\pi_1^{-1}(A)\right]^c \cup \left[\pi_2^{-1}(B)\right]^c \right\} \\
&= (X \times B) \cap \left[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\right]^c \\
&= (X \times B) \cap \left[A \times B\right]^c \\
&= (X \times B) \setminus (A \times B).
\end{aligned}
$$

- We have

$$
\begin{aligned}
A \times B \subset A' \times B' &\iff \left[(x,y) \in A \times B \ \Rightarrow\ (x,y) \in A' \times B'\right] \\
&\iff \left[x \in A, y \in B \ \Rightarrow\ x \in A', y \in B'\right] \\
&\iff A \subset A', \quad B \subset B'.
\end{aligned}
$$

Problem 14.2 Solution: Pick two exhausting sequences $(A_k)_k \subset \mathscr{A}$ and $(B_k)_k \subset \mathscr{B}$ such that $\mu(A_k), \nu(B_k) < \infty$ and $A_k \uparrow X$, $B_k \uparrow Y$. Then, because of the continuity of measures,

$$
\begin{aligned}
\mu \times \nu(A \times N) &= \lim_k \mu \times \nu\big((A \times N) \cap (A_k \times B_k)\big) \\
&= \lim_k \mu \times \nu\big((A \cap A_k) \times (N \cap B_k)\big) \\
&= \lim_k \Big[\underbrace{\mu(A \cap A_k)}_{<\infty} \cdot \underbrace{\nu(N \cap B_k)}_{\leqslant \nu(N)=0} \Big] \\
&= 0.
\end{aligned}
$$

Since $A \times N \in \mathscr{A} \times \mathscr{B} \subset \mathscr{A} \otimes \mathscr{B}$, measurability is clear.

Problem 14.3 Solution:

- (a) $\Rightarrow$ (b): If f is $\mu_1 \times \mu_2$-negligible, we can use Tonelli's theorem to infer that

$$
0 = \int_{E_1} \left(\int_{E_2} |f(x_1, x_2)| \, d\mu_2(x_2) \right) d\mu_1(x_1).
$$

  Using Theorem 11.2 we find

$$
\mu_1 \left(\int_{E_2} |f(\cdot, x_2)| \, d\mu_2(x_2) \neq 0 \right) 0.
$$

  This means that $f(x_1, \cdot)$ is for μ_1-almost all x_1 μ_2-negligible.

- (b) $\Rightarrow$ (a): Set

$$
N := \left\{ x_1 \in E_1; \int_{E_2} |f(x_1, x_2)| \, d\mu_2(x_2) \neq 0 \right\}.
$$

  By assumption, $\mu_1(N) = 0$. Therefore,

$$
\int_{E_1} \left(\int_{E_2} |f(x_1, x_2)| \, d\mu_2(x_2) \right) d\mu_1(x_1) = \int_N \left(\int_{E_2} |f(x_1, x_2)| \, d\mu_2(x_2) \right) d\mu_1(x_1)
$$

$$+\int_{E_1\setminus N}\left(\int_{E_2}|f(x_1,x_2)|\,d\mu_2(x_2)\right)\,d\mu_1(x_1).$$

The first integral on the right-hand side is, by Theorem 11.2 equal to 0. The second integral is also 0, due to the definition of the set N. Using Tonelli's theorem we see

$$\int_{E_1\times E_2}|f(x_1,x_2)|\,d\mu_1\times\mu_2(x_1,x_2)=0.$$

- (a) ⇔ (c): Use the symmetry in the variables or argue as in "(a) ⇔ (b)".

Problem 14.4 Solution: Since the two expressions are symmetric in x and y, they must coincide if they converge. Let us, therefore only look at the left hand side.

The inner integral,

$$\int_{(0,\infty)} e^{-xy}\sin x\,\lambda(dx)$$

clearly satisfies

$$\begin{aligned}\int_{(0,\infty)}\left|e^{-xy}\sin x\right|\lambda(dx)&\leqslant\int_{(0,\infty)} e^{-xy}\,\lambda(dx)\\&=\int_0^\infty e^{-xy}\,dx\\&=\left[-\frac{e^{-xy}}{y}\right]_{x=0}^\infty\\&=\frac{1}{x}.\end{aligned}$$

Since the integrand is continuous and has only one sign, we can use Riemann's integral. Thus, the integral exists. To calculate its value we observe that two integrations by parts yield

$$\begin{aligned}\int_0^\infty e^{-xy}\,sinx\,dx&=-e^{-xy}\cos x\Big|_{x=0}^\infty-\int_0^\infty ye^{-xy}\cos x\,dx\\&=1-y\int_0^\infty e^{-xy}\cos x\,dx\\&=1-y\left(e^{-xy}\sin x\Big|_{x=0}^\infty+\int_0^\infty ye^{-xy}\sin x\,dx\right)\\&=1-y^2\int_0^\infty e^{-xy}\sin x\,dx.\end{aligned}$$

And if we solve this equality for the integral expression, we get

$$(1+y^2)\int_0^\infty e^{-xy}\sin x\,dx=1\implies\int_0^\infty e^{-xy}\sin x\,dx=\frac{1}{1+y^2}.$$

Alternative: Since $\sin x=\operatorname{Im}e^{ix}$ we get

$$\int_0^\infty e^{-xy}\,sinx\,dx=\operatorname{Im}\int_0^\infty e^{-(y-i)x}\,dx=\operatorname{Im}\frac{1}{y-i}=\operatorname{Im}\frac{y+i}{y^2+1}=\frac{1}{y^2+1}.$$

Thus the iterated integral exists, since

$$\int_{(0,\infty)} \left|\frac{\sin x}{1+x^2}\right| dx \leqslant \int_{(0,\infty)} \frac{1}{1+x^2}\,dx = \arctan x\Big|_0^\infty = \frac{\pi}{2}.$$

(Here we use again that improper Riemann integrals with positive integrands coincide with Lebesgue integrals.)

In principle, the existence and equality of iterated integrals is not good enough to guarantee the existence of the double integral. For this one needs the existence of the *absolute* iterated integrals—cf. Tonelli's theorem 14.8. In the present case one can see that the absolute iterated integrals exist, though:

On the one hand we find

$$\int_{(0,\infty)} e^{-xy}|\sin(x)|\,\lambda(dx) \leqslant \frac{e^{-xy}}{-y}\Big|_0^\infty = \frac{1}{y}$$

and $\frac{\sin y}{y}$ is, as a bounded continuous function, Lebesgue integrable over $(0,1)$.

On the other hand we can use integration by parts to get

$$\begin{aligned}\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\,dx &= \frac{e^{-xy}}{-y}\sin x\Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi}\frac{e^{-xy}}{-y}\cos x\,dx\\ &= \frac{e^{-xy}}{-y^2}\cos x\Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi}\frac{e^{-xy}}{-y^2}(-1)\sin x\,dx\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{y^2+1}{y^2}\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\,dx &= \frac{e^{-(k+1)\pi y}}{-y^2}(-1)^{k+1} - \frac{e^{-k\pi y}}{-y^2}(-1)^k\\ &= \frac{(-1)^k}{y^2}(e^{-(k+1)\pi y}+e^{-k\pi y}),\end{aligned}$$

i.e. $\int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x\,dx = (-1)^k\frac{1}{y^2+1}(e^{-(k+1)\pi y}+e^{-k\pi y})$.

Now we find a bound for $y\in(1,\infty)$.

$$\begin{aligned}\int_{(0,\infty)} e^{-xy}|\sin(x)|dx &= \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} e^{-xy}\sin x(-1)^k\,dx\\ &= \sum_{k=0}^\infty (-1)^k(-1)^k\frac{1}{y^2+1}(e^{-(k+1)\pi y}+e^{-k\pi y})\\ &\leqslant \frac{2}{y^2+1}\sum_{k=0}^\infty (e^{-\pi y})^k\\ &\overset{y>1}{\leqslant} \frac{2}{y^2+1}\sum_{k=0}^\infty (e^{-\pi})^k\end{aligned}$$

which means that the left hand side is integrable over $(1,\infty)$.

Thus we have

$$\int_{(0,\infty)}\int_{(0,\infty)} |e^{-xy}\sin x\sin y|\,\lambda(dx)\,\lambda(dy)$$

$$\leqslant \int_{(0,1]} \frac{\sin y}{y}\,\lambda(dy) + \int_{(1,\infty)} \frac{2}{y^2+1}\,\lambda(dy) \sum_{k=0}^{\infty} (e^{-\pi})^k$$
$$< \infty.$$

By Fubini's theorem we know that the iterated integrals as well as the double integral exist and their values are identical.

Alternative proof for the absolute convergence of the integral:[1] Let

$$f(x,y) = e^{-xy}\,|\sin x\ \sin y| \geqslant 0 \quad \forall x, y \geqslant 0.$$

By monotone convergence and Tonelli's theorem

$$\begin{aligned}\iint f(x,y)\,dx\,dy &= \lim_{A,B\to\infty} \iint_{(0,A]\times(0,B]} f(x,y)dx\,dy \\ &= \sup_{A,B\geqslant 0} \int_{(0,A]} \int_{(0,B]} f(x,y)\,dy\,dx.\end{aligned}$$

Since the integrands are bounded and continuous, we can use Riemann integrals. Fix $A > 1$ and $B > 1$. Then

$$\int_0^A \int_0^B = \int_0^1 \int_0^1 + \int_0^1 \int_1^B + \int_0^1 \int_1^A + \int_1^A \int_1^B$$

Now we can estimate these expressions separately: since $|\sin t| \leqslant |t|$ we have

$$\begin{aligned}
\int_0^1 \int_0^1 f(x,y)\,dy\,dx &\leqslant \int_0^1 \int_0^1 1\,dx\,dy = 1. \\
\int_0^1 \int_1^B f(x,y)\,dy\,dx &\leqslant \int_1^B \left[\int_0^1 xe^{-xy}\,dx\right] dy \\
&= 1 - \frac{1}{e} + \frac{e^{-B}-1}{B} < 1 - \frac{1}{e}. \\
\int_0^1 \int_1^A f(x,y)\,dx\,dy &\leqslant \int_1^A \left[\int_0^1 ye^{-xy}\,dy\right] dx \\
&= 1 - \frac{1}{e} + \frac{e^{-A}-1}{A} < 1 - \frac{1}{e}. \\
\int_1^A \int_1^B f(x,y)\,dx\,dy &\leqslant \int_1^B \left[\int_1^A xe^{-xy}\,dx\right] dy \\
&= \frac{1}{e} - e^{-A} + \frac{e^{-AB}-e^{-B}}{B} < \frac{1}{e}.
\end{aligned}$$

These estimates now show

$$\int_0^\infty \int_0^\infty e^{-xy}\,|\sin x\ \sin y|\ dx\,dy \leqslant 3 - \frac{1}{e}.$$

■■

[1] This much more elegant proof was communicated to me in July 2012 by Alvaro H. Salas from the Universidad Nacional de Colombia, Department of Mathematics

Problem 14.5 Solution: Note that

$$\frac{d}{dy}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Thus we can compute

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \int_{(0,1)} \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

By symmetry of x and y in the integrals it follows that

$$\int_{(0,1)}\int_{(0,1)} \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx = -\frac{\pi}{4}$$

and therefore the double integral can not exist. Since the existence would imply the equality of the two above integrals. We can see this directly by

$$\begin{aligned}\int_{(0,1)}\int_{(0,1)} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\,dx &\geqslant \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx \\ &= \int_0^1 \frac{x}{x^2+x^2}\,dx \\ &= \frac{1}{2}\int_0^1 \frac{1}{x}\,dx = \infty.\end{aligned}$$

■■

Problem 14.6 Solution: Since the integrand is odd, we have for $y \neq 0$:

$$\int_{(-1,1)} \frac{xy}{(x^2+y^2)^2}\,dx = 0$$

and $\{0\}$ is a null set. Thus the iterated integrals have common value 0. But the double integral does not exist, since for the iterated absolute integrals we get

$$\int_{(-1,1)} \left|\frac{xy}{(x^2+y^2)^2}\right| dx = \frac{1}{|y|}\int_0^{1/|y|} \frac{\xi}{(\xi^2+1)^2}\,d\xi \qquad \geqslant \frac{2}{|y|}\underbrace{\int_0^1 \frac{\xi}{(\xi^2+1)^2}\,d\xi}_{<\infty}.$$

Here we use the substitution $x = \xi|y|$ and the fact that $|y| \leqslant 1$, thus $1/|y| \geqslant 1$. But the outer integral is bounded below by

$$\int_{(-1,1)} \frac{2}{|y|}\,dy \ \text{ which is divergent.}$$

■■

Problem 14.7 Solution: We use the generic notation $f(x, y)$ for any of the integrands.

a) We have

$$\int_0^1 f(x,y)\,dy = \frac{\left|x-\frac{1}{2}\right|}{\left(x-\frac{1}{2}\right)^3}$$

and this function is not integrable (in x) in the interval $(0,1)$. For $0 < y \leqslant \frac{1}{2}$ we have

$$\int_0^1 f(x,y)\,dx = \int_0^{\frac{1}{2}-y} \left(x-\frac{1}{2}\right)^{-3} dx + \int_{\frac{1}{2}+y}^1 \left(x-\frac{1}{2}\right)^{-3} dx = 0.$$

For $\frac{1}{2} \leqslant y \leqslant 1$ this integral is again 0. Therefore,

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx\right) dy = 0.$$

Finally,

$$\int_0^1 |f(x,y)|\,dy = \left|x-\tfrac{1}{2}\right|^{-2} \Rightarrow \int_0^1 \int_0^1 |f(x,y)|\,dx\,dy = \infty.$$

b) We have

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}}\,dy\,dx &= \int_0^1 \left[\frac{1}{x}\frac{x+y}{(x^2+y^2)^{1/2}}\right]_{y=0}^{y=1} dx \\
&= \int_0^1 \left[\frac{x+1}{\sqrt{x^2+1}-1}\right] dx \\
&= \left[\ln \frac{x+\sqrt{x^2+1}}{1+\sqrt{x^2+1}-1}\right]_{x=0}^{x=1} \\
&= \ln 2.
\end{aligned}$$

Bcause of (anti-)symmetry we find

$$\int_0^1 \int_0^1 \frac{x-y}{(x^2+y^2)^{3/2}}\,dx\,dy = -\ln 2.$$

Morevoer,

$$\begin{aligned}
\frac{1}{2}\int_0^1 \int_0^1 \left|\frac{x-y}{(x^2+y^2)^{3/2}}\right| dy\,dx &= \int_0^1 \int_0^x \frac{x-y}{(x^2+y^2)^{3/2}}\,dy\,dx \\
&= \int_0^1 \left[\frac{1}{x}\frac{x-y}{(x^2+y^2)^{1/2}}\right]_{y=0}^{y=x} dx \\
&= (\sqrt{2}-1)\int_0^1 \frac{dx}{x} \\
&= \infty.
\end{aligned}$$

c) Since f is positive, Tonelli's theorem ensures that all three integrals coincide. Let $p \neq 1$. We get

$$\int_0^1 \int_0^1 (1-xy)^{-p}\,dy\,dy = \frac{1}{p-1}\int_0^1 \left((1-x)^{1-p}-1\right) \frac{dx}{x}.$$

This integral is finite if, and only if, $p < 2$. For $p = 1$ we have

$$\int_0^1 \int_0^1 (1-xy)^{-p}\,dy\,dy = -\int_0^1 \ln(1-x)\,\frac{dx}{x} < \infty.$$

Problem 14.8 Solution:

(i) We have $[-n,n] \uparrow \mathbb{R}$ as $n \to \infty$ and $\lambda([-n,n]) = 2n < \infty$. This shows σ-finiteness of λ. Let $(q_j)_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$; set $A_n := \{q_1,\dots,q_n\} \cup (\mathbb{R}\setminus\mathbb{Q})$, then we have $A_n \uparrow \mathbb{R}$ and $\zeta_{\mathbb{Q}}(A_n) = n < \infty$. This shows σ-finiteness of $\zeta_{\mathbb{Q}}$.

We will show that $\zeta_{\mathbb{R}}$ is *not* σ-finite. Assume $\zeta_{\mathbb{R}}$ were σ-finite. Thus, there would be a sequence $A_n \uparrow \mathbb{R}$, $n \in \mathbb{N}$, such that $\zeta_{\mathbb{R}}(A_n) < \infty$. Since $\zeta_{\mathbb{R}}$ is a counting measure, every A_n is countable. Thus, $\mathbb{R}$ is a countable union of countable sets, hence countable – a contradiciton.

(ii) The rationals $\mathbb{Q}$ are a λ null set, hence $\frac{1}{y}\mathbb{Q}$ is for each y a λ null set. We have

$$\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,\lambda(dx) = 0 \quad \forall\, y \in \mathbb{R}.$$

This implies

$$\int_{(0,1)}\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,d\lambda(x)\,d\zeta_{\mathbb{R}}(y) = 0.$$

(iii) Let $x \in (0,1)$. The set $(\frac{1}{x}\mathbb{Q}) \cap (0,1)$ contains infinitely many values, so

$$\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,\zeta_{\mathbb{R}}(dy) = \infty \quad \forall\, x.$$

Therefore, the iterated integral is ∞.

(iv) Let $x \in (0,1)\setminus\mathbb{Q}$. Since $y\cdot x \notin \mathbb{Q}$ for any $y \in \mathbb{Q}$, we have

$$\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,\zeta_{\mathbb{Q}}(dy) = 0 \quad \forall\, x \in (0,1)\setminus\mathbb{Q}.$$

On the other hand, if $x \in \mathbb{Q}\cap(0,1)$, then $y\cdot x \in \mathbb{Q}$ for any $y \in \mathbb{Q}$ and so

$$\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,\zeta_{\mathbb{Q}}(dy) = \infty \quad \forall\, x \in (0,1)\cap\mathbb{Q}.$$

Since $\mathbb{Q}$ is a λ null set, we get

$$\int_{(0,1)}\int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x\cdot y)\,\zeta_{\mathbb{Q}}(dy)\lambda(dx) = \int_{(0,1)} \mathbb{1}_{\mathbb{Q}}(x)\cdot\infty\,d\lambda(x) = 0.$$

(v) The results of (iii),(iv) do not contradict Fubini's or Tonelli's theorem, since these theorems require σ-finiteness of all measures.

Problem 14.9 Solution:

(i) Since the integrand is positive, we can use Tonelli's theorem and work out the integral as an iterated integral

$$I := \int_{[0,\infty)^2} \frac{dx\,dy}{(1+y)(1+x^2y)}$$

$$
\begin{aligned}
&= \int_{[0,\infty)} \frac{1}{1+y} \left(\int_{[0,\infty)} \frac{1}{1+x^2 y}\, dx \right) dy \\
&= \int_{[0,\infty)} \frac{1}{1+y} \frac{\arctan(x\sqrt{y})}{\sqrt{y}} \bigg|_{x=0}^{\infty} dy \\
&= \frac{\pi}{2} \int [0,\infty) \frac{1}{1+y} \frac{1}{\sqrt{y}}\, dy.
\end{aligned}
$$

(Observe that the integrand is continuous, which enables us to use Riemann integrals on bounded intervals. Note that $\int_{[0,\infty)} \cdots = \sup_{n\in\mathbb{N}} \int_{[0,n)} \dots$ because of monotone convergence.) Using the substitution $u = \sqrt{y}$, we get

$$
I = \frac{\pi}{2} \int_{[0,\infty)} \frac{1}{1+u^2}\, du = \pi \arctan(u) \bigg|_{u=0}^{\infty} = \frac{\pi^2}{2}.
$$

(ii) We use partial fractions in (i):

$$
\frac{1}{1+y} \frac{1}{1+x^2 y} = \frac{1}{1-x^2} \frac{1}{1+y} - \frac{x^2}{1-x^2} \frac{1}{1+x^2 y}.
$$

Thus,

$$
\begin{aligned}
I &= \int_{[0,\infty)} \left(\int_{[0,\infty)} \frac{1}{1-x^2} \frac{1}{1+y} - \frac{x^2}{1-x^2} \frac{1}{1+x^2 y}\, dy \right) dx \\
&= \int_{[0,\infty)} \left(\lim_{R\to\infty} \left[\frac{1}{1-x^2} \ln(1+R) - \frac{x^2}{1-x^2} \frac{\ln(1+x^2 R)}{x^2} \right] \right) dx \\
&= \int_{(0,\infty)} \frac{1}{1-x^2} \left(\lim_{R\to\infty} \ln\left(\frac{1+R}{1+x^2 R} \right) \right) dx \\
&= \int_{[0,\infty)} \frac{1}{1-x^2} \ln(x^{-2})\, dx \\
&= 2 \int_{[0,\infty)} \frac{\ln(x)}{x^2-1}\, dx.
\end{aligned}
$$

From (i) we infer that $\int_{[0,\infty)} \frac{\ln x}{x^2-1}\, dx = \frac{I}{2} = \frac{\pi^2}{4}$.

(iii) Using the geometric series we find

$$
\frac{1}{x^2-1} = -\sum_{n\geqslant 0} (x^2)^n = -\sum_{n\geqslant 0} x^{2n}, \qquad |x| < 1,
$$

as well as

$$
\frac{1}{x^2-1} = \frac{1}{x^2} \frac{1}{1-x^{-2}} = \frac{1}{x^2} \sum_{n\geqslant 0} (x^{-2})^n = \sum_{n\geqslant 0} x^{-2(n+1)}, \qquad |x| > 1.
$$

Thus,

$$
\int_{(0,\infty)} \frac{\ln x}{x^2-1}\, dx = -\sum_{n\geqslant 0} \int_{(0,1)} x^{2n} \ln x\, dx + \sum_{n\geqslant 0} \int_{(1,\infty)} x^{-2(n+1)} \ln x\, dx. \qquad (\star)
$$

(In order to swap summation and integration, we use dominated convergence!) Using integration by parts, we find

$$
\int_{(0,1)} x^{2n} \ln x\, dx = \frac{x^{2n+1}}{2n+1} \ln x \bigg|_{x=0}^{1} - \frac{1}{2n+1} \int_{(0,1)} x^{2n}\, dx
$$

$$= -\frac{1}{(2n+1)^2}$$

and, in a similar fashion,

$$\int_{(1,\infty)} x^{-2(n+1)} \ln x \, dx = \frac{x^{-2(n+1)+1}}{-2(n+1)+1} \ln x \Big|_{x=1}^{\infty} - \frac{1}{-2(n+1)+1} \int_{(1,\infty)} x^{-2(n+1)} \, dx$$
$$= \frac{1}{(-2(n+1)+1)^2} = \frac{1}{(2n+1)^2}.$$

Inserting these results into (★), the claim follows from part (ii).

Problem 14.10 Solution:

(i) Since μ is σ-finite, there is an exhausting sequence $(G_n)_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{R})$ such that $\mu(G_n) < \infty$ and $G_n \uparrow \mathbb{R}$. For each $n \in \mathbb{N}$ the set

$$B_k^n := \left\{ x \in G_n; \mu(\{x\}) > \frac{1}{k} \right\}$$

is finite. *Indeed:* Assume there were countably infinitely many $(x_j)_{j\in\mathbb{N}} \subset B_k^n$, $x_j \neq x_i$ for $i \neq j$. Since the sets $\{x_j\}$, $j \in \mathbb{N}$, are disjoint, we conclude that

$$\mu(G_n) \geqslant \mu\left(\sum_{j\in\mathbb{N}} \{x_j\} \right) = \sum_{j\in\mathbb{N}} \mu(\{x_j\}) = \infty.$$

This is a contradiction to $\mu(G_n) < \infty$.

Thus, the set

$$B^n := \{x \in G_n; \mu(\{x\}) > 0\} = \bigcup_{k\in\mathbb{N}} \left\{ x \in G_n; \mu(\{x\}) > \frac{1}{k} \right\}$$

is countable and so is

$$D = \bigcup_{n\in\mathbb{N}} B^n$$

as it is a countable union of countable sets.

(ii) For the diagonal $\mathbb{1}_\Delta(x,y) = \mathbb{1}_{\{y\}}(x)\mathbb{1}_{\mathbb{R}}(y)$ we find from Theorem 14.5:

$$\mu \times \nu(\Delta) = \int_{\mathbb{R}} \left(\int \mathbb{1}_{\{y\}}(x) \, \mu(dx) \right) \nu(dy)$$
$$= \int_{\mathbb{R}} \mu(\{y\}) \mathbb{1}_D(y) \, \nu(dy)$$
$$= \sum_{y\in D} \mu(\{y\})\nu(\{y\}).$$

(In the last step we use that D is countable.)

Problem 14.11 Solution: Note that the diagonal $\Delta \subset \mathbb{R}^2$ is measurable, i.e. the (double) integrals are well-defined. The inner integral on the l.h.S. satisfies

$$\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\lambda(dx) = \lambda(\{y\}) = 0 \quad \forall\, y \in [0,1]$$

so that the left-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\lambda(dx)\,\mu(dy) = \int_{[0,1]} 0\,\mu(dy) = 0.$$

On the other hand, the inner integral on the right-hand side equals

$$\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\mu(dy) = \mu(\{x\}) = 1 \quad \forall\, x \in [0,1]$$

so that the right-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\mu(dy)\,\lambda(dx) = \int_{[0,1]} 1\,\lambda(dx) = 1.$$

This shows that the double integrals are not equal. This does not contradict Tonelli's theorem since μ is not σ-finite.

Problem 14.12 Solution:

(i) Note that, due to the countability of $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ there are no problems with measurability and σ-finiteness (of the counting measure).

Tonelli's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}}$ be a double sequence of positive numbers $a_{jk} \geqslant 0$. Then

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{jk} = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{jk}$$

with the understanding that both sides are either finite or infinite.

Fubini's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}} \subset \mathbb{R}$ be a double sequence of real numbers a_{jk}. If

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} |a_{jk}| \quad\text{or}\quad \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} |a_{jk}|$$

is finite, then all of the following expressions converge absolutely and sum to the same value:

$$\sum_{j\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}|a_{jk}|\right), \quad \sum_{k\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}}|a_{jk}|\right), \quad \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}}|a_{jk}|.$$

(ii) Consider the (obviously σ-finite) measures $\mu_j := \sum_{k\in A_j}\delta_k$ and $\nu = \sum_{j\in\mathbb{N}}\mu_j$. Tonelli's theorem tells us that

$$\begin{aligned}\sum_{j\in\mathbb{N}}\sum_{k\in A_j}|x_k| &= \int_{\mathbb{N}}\int_{\mathbb{N}} |x_k|\,\mu_j(dk)\,\mu(dj)\\ &= \int_{\mathbb{N}}\int_{\mathbb{N}} |x_k|\mathbb{1}_{A_j}(k)\,\mu(dk)\,\mu(dj)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{N}} \int_{\mathbb{N}} |x_k| \mathbb{1}_{A_j}(k)\, \mu(dj)\, \mu(dk) \\
&= \int_{\mathbb{N}} |x_k| \underbrace{\left(\int_{\mathbb{N}} \mathbb{1}_{A_j}(k)\, \mu(dj) \right)}_{=1,\ \text{as the } A_j \text{ are disjoint}} \mu(dk) \\
&= \int_{\mathbb{N}} |x_k|\, \mu(dk) \\
&= \sum_{k\in\mathbb{N}} |x_k|.
\end{aligned}
$$

Problem 14.13 Solution:

(i) Set $U(a,b) := a - b$. Then

$$
U(u(x), y)\mathbb{1}_{[0,\infty)}(y) \geqslant 0 \iff u(x) \geqslant y \geqslant 0
$$

and $U(u(x), y)\mathbb{1}_{[0,\infty)}(y)$ is a combination/sum/product of $\mathscr{B}(\mathbb{R}^2)$ resp. $\mathscr{B}(\mathbb{R})$-measurable functions. Thus $S[u]$ is $\mathscr{B}(\mathbb{R}^2)$-measurable.

(ii) Yes, true, since by Tonelli's theorem

$$
\begin{aligned}
\lambda^2(S[u]) &= \int_{\mathbb{R}^2} \mathbb{1}_{S[u]}(x,y)\, \lambda^2(d(x,y)) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{(x,y)\,:\,u(x)\geqslant y\geqslant 0\}}(x,y)\, \lambda^1(dy)\, \lambda^1(dx) \\
&= \int_{\mathbb{R}} \int_{[0,u(x)]} 1\, \lambda^1(dy)\, \lambda^1(dx) \\
&= \int_{\mathbb{R}} u(x)\, \lambda^1(dx)
\end{aligned}
$$

(iii) Measurability follows from (i) and with the hint. Moreover,

$$
\begin{aligned}
\lambda^2(\Gamma[u]) &= \int_{\mathbb{R}^2} \mathbb{1}_{\Gamma[u]}(x,y)\, \lambda^2(d(x,y)) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{(x,y)\,:\,y=u(x)\}}(x,y)\, \lambda^1(dy)\, \lambda^1(dx) \\
&= \int_{\mathbb{R}} \int_{[u(x),u(x)]} 1\, \lambda^1(dy)\, \lambda^1(dx) \\
&= \int_{\mathbb{R}} \lambda^1(\{u(x)\})\, \lambda^1(dx) \\
&= \int_{\mathbb{R}} 0\, \lambda^1(dx) \\
&= 0.
\end{aligned}
$$

Problem 14.14 Solution: The hint given in the text should be good enough to solve this problem....

Problem 14.15 Solution: Since (i) implies (ii), we will only prove (i) under the assumption that both $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are complete measure spaces. Note that we have to assume σ-finiteness of μ and ν, otherwise the product construction would not work. Pick some set $Z \in \mathscr{P}(X) \setminus \mathscr{A}$ (which is, because of completeness, not a null-set!), and some ν-null set $N \in \mathscr{B}$ and consider $Z \times N$.

We get for some exhausting sequence $(A_k)_k \subset \mathscr{A}$, $A_k \uparrow X$ and $\mu(A_k) < \infty$:

$$\begin{aligned}\mu \times \nu(X \times N) &= \sup_{k\in\mathbb{N}} \mu \times \nu(A_k \times N) \\ &= \sup_{k\in\mathbb{N}} \Big(\underbrace{\mu(A_k)}_{<\infty} \cdot \underbrace{\nu(N)}_{=0} \Big) \\ &= 0;\end{aligned}$$

thus $Z \times N \subset X \times N$ is a subset of a measurable $\mu \times \nu$ null set, hence it should be $\mathscr{A} \otimes \mathscr{B}$-measurable, if the product space were complete. On the other hand, because of Theorem 14.17(iii), if $Z \times N$ is $\mathscr{A} \otimes \mathscr{B}$-measurable, then the section

$$x \mapsto \mathbb{1}_{Z\times N}(x, y) = \mathbb{1}_Z(x)\mathbb{1}_N(y) \stackrel{y\in N}{=} \mathbb{1}_Z(x)$$

is $\mathscr{A}$-measurable which is only possible if $Z \in \mathscr{A}$.

■■

Problem 14.16 Solution:

(i) Let $A \in \mathscr{B}[0, \infty) \otimes \mathscr{P}(\mathbb{N})$, fix $k \in \mathbb{N}$ and consider

$$\mathbb{1}_A(x, k) \text{ and } B_k := \{x : \mathbb{1}_A(x, k) = 1\};$$

because of Theorem 14.17(iii), $B_k \in \mathscr{B}[0, \infty)$. Since

$$\begin{aligned}(x, k) \in A &\iff \mathbb{1}_A(x, k) = 1 \\ &\iff \exists\, k \in \mathbb{N} : \mathbb{1}_A(x, k) = 1 \\ &\iff \exists\, k \in \mathbb{N} : x \in B_k\end{aligned}$$

it is clear that $A = \bigcup_{k\in\mathbb{N}} B_k \times \{k\}$.

(ii) Let $M \in \mathscr{P}(\mathbb{N})$ and set $\zeta := \sum_{j\in\mathbb{N}} \delta_j$; we know that ζ is a (σ-finite) measure on $\mathscr{P}(\mathbb{N})$. Using Tonelli's theorem 14.8 we get

$$\begin{aligned}\pi(B \times M) &:= \sum_{m\in M} \pi(B \times \{m\}) \\ &:= \sum_{m\in M} \int_B e^{-t} \frac{t^m}{m!}\, \mu(dt) \\ &= \int_M \int_B e^{-t} \frac{t^m}{m!}\, \mu(dt)\, \zeta(dm) \\ &= \iint_{B\times M} e^{-t} \frac{t^m}{m!}\, \mu \times \zeta(dt, dm)\end{aligned}$$

which shows that the measure $\pi(dt, dm) := e^{-t} \frac{t^m}{m!} \mu \times \zeta(dt, dm)$ has all the properties required by the exercise.

The uniqueness follows, however, from the uniqueness theorem for measures (Theorem 5.7): the family of 'rectangles' of the form $B \times M \in \mathscr{B}[0,\infty) \times \mathscr{P}(\mathbb{N})$ is a $\cap$-stable generator of the product σ-algebra $\mathscr{B}[0,\infty) \otimes \mathscr{P}(\mathbb{N})$ and contains an exhausting sequence, say, $[0,\infty) \times \{1, 2, \dots k\} \uparrow [0,\infty) \times \mathbb{N}$. But on this generator π is (uniquely) determined by prescribing the values $\pi(B \times \{m\})$.

Problem 14.17 Solution: Assume first that $\lambda \geqslant 0$. The point here is that Corollary 14.15 does not apply to the function $s \mapsto e^{-\lambda s}$ since this function is decreasing and has the value 1 for $s = 0$. Consider therefore $\phi(s) := 1 - e^{-\lambda s}$. This ϕ is admissible in 14.15 and we get

$$\int \phi(T)\, d\mathbb{P} = \int \left(1 - e^{-\lambda T}\right) d\mathbb{P} = \int_0^\infty \lambda e^{-\lambda s}\, \mathbb{P}(T \geqslant s)\, ds.$$

Rearranging this equality then yields

$$\int e^{-\lambda T}\, d\mathbb{P} = 1 - \lambda \int_0^\infty e^{-\lambda s}\, \mathbb{P}(T \geqslant s)\, ds.$$

If $\lambda < 0$ the formula remains valid if we understand it in the sense that either both sides are finite or both sides are infinite. The above argument needs some small changes, though. First, $e^{-\lambda s}$ is now increasing (which is fine) but still takes the value 1 if $s = 0$. So we should change to $\phi(s) := e^{-\lambda s} - 1$. Now the same calculation as above goes through. If one side is finite, so is the other; and if one side is infinite, then the other is infinite, too. The last statement follows from Theorem 14.13 or Corollary 14.15.

Problem 14.18 Solution:

(i) This is similar to Problem 6.1, in particular (i) and (vi).

(ii) Note that

$$\begin{aligned}
\mathbb{1}_B(x,y) &= \mathbb{1}_{(a,b]}(x)\mathbb{1}_{[x,b]}(y) \\
&= \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(a,y]}(x) \\
&= \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(a,b]}(y)\mathbb{1}_{[0,\infty)}(y-x);
\end{aligned}$$

the last expression is, however, a product of (combinations of) measurable functions, thus $\mathbb{1}_B$ is measurable and so is then B.

Without loss of generality we can assume that $a > 0$, all other cases are similar.

Using Tonelli's theorem 14.8 we get

$$\mu \times \nu(B) = \iint \mathbb{1}_B(x,y)\, \mu \times \nu(dx, dy)$$

$$
\begin{aligned}
&= \iint \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(a,y]}(x)\,\mu\times\nu(dx,dy)\\
&= \int_{(a,b]}\int_{(a,y]}\mu(dx)\,\nu(dy)\\
&= \int_{(a,b]}\mu(a,y]\,\nu(dy)\\
&= \int_{(a,b]}\big(\mu(0,y]-\mu(0,a]\big)\,\nu(dy)\\
&= \int_{(a,b]}\mu(0,y]\nu(dy)-\mu(0,a]\int_{(a,b]}\nu(dy)\\
&= \int_{(a,b]}F(y)\,dG(y)-F(a)\big(G(b)-G(a)\big). \qquad (*)
\end{aligned}
$$

We remark at this point already that a very similar calculation (with μ,ν and F,G interchanged and with an open interval rather than a semi-open interval) yields

$$
\begin{aligned}
&\iint \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(y,b]}(x)\,\mu(dx)\,\nu(dy)\\
&\qquad= \int_{(a,b]}G(y-)\,dF(y)-G(a)\big(F(b)-F(a)\big). \qquad (**)
\end{aligned}
$$

(iii) On the one hand we have

$$
\begin{aligned}
\mu\times\nu\big((a,b]\times(a,b]\big) &= \mu(a,b]\nu(a,b]\\
&= \big(F(b)-F(a)\big)\big(G(b)-G(a)\big) \qquad (+)
\end{aligned}
$$

and on the other we find, using Tonelli's theorem at step (T)

$$
\begin{aligned}
&\mu\times\nu\big((a,b]\times(a,b]\big)\\
&\quad= \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)\\
&\quad= \iint \mathbb{1}_{(a,y]}(x)\mathbb{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)+\\
&\qquad\quad+\iint \mathbb{1}_{(y,b]}(x)\mathbb{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)\\
&\quad\overset{T}{=} \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{[x,b]}(y)\,\nu(dy)\,\mu(dx)+\\
&\qquad\quad+\iint \mathbb{1}_{(y,b]}(x)\mathbb{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)\\
&\quad\overset{*,**}{=} \int_{(a,b]}F(y)\,dG(y)-F(a)\big(G(b)-G(a)\big)+\\
&\qquad\quad+\int_{(a,b]}G(y-)\,dF(y)-G(a)\big(F(b)-F(a)\big).
\end{aligned}
$$

Combining this formula with the previous one marked (+) reveals that

$$
F(b)G(b)-F(a)G(a)=\int_{(a,b]}F(y)\,dG(y)+\int_{(a,b]}G(y-)\,dF(y).
$$

Finally, observe that

$$\begin{aligned}\int_{(a,b]} \big(F(y) - F(y-)\big)\, dG(y) &= \int_{(a,b]} \mu(\{y\})\, \nu(dy)\\ &= \sum_{a<y\leqslant b} \mu(\{y\})\nu(\{y\})\\ &= \sum_{a<y\leqslant b} \Delta F(y)\Delta G(y).\end{aligned}$$

(Mind that the sum is at most countable because of Lemma 14.14) from which the claim follows.

(iv) It is clear that uniform approximation allows to interchange limiting and integration procedures so that we *really* do not have to care about this. We show the formula for monomials $t, t^2, t^3, \ldots$ by induction. Write $\phi_n(t) = t^n$, $n \in \mathbb{N}$.

Induction start $n = 1$: in this case $\phi_1(t) = t$, $\phi_1'(t) = 1$ and $\phi(F(s)) - \phi(F(s-)) - \Delta F(s) = 0$, i.e. the formula just becomes

$$F(b) - F(a) = \int_{(a,b]} dF(s)$$

which is obviously true.

Induction assumption: for some n we know that

$$\begin{aligned}\phi_n(F(b)) - \phi_n(F(a)) = \int_{(a,b]} &\phi_n'(F(s-))\, dF(s)\\ &+ \sum_{a<s\leqslant b} \Big[\phi_n(F(s)) - \phi_n(F(s-)) - \phi_n'(F(s-))\Delta F(s)\Big].\end{aligned}$$

Induction step $n \rightsquigarrow n + 1$: Write, for brevity $F = F(s)$ and $F_- = F(s-)$. We have because of (iii) with $G = \phi_n \circ F$ and because of the induction assumption

$$\begin{aligned}&\phi_{n+1}(F(b)) - \phi_{n+1}(F(a))\\ &= F(b)\phi_n(F(b)) - F(a)\phi_n(F(a))\\ &= \int_{(a,b]} F_-^n\, dF + \int_{(a,b]} F_-\, dF^n + \sum \Delta F \Delta F^n\\ &= \int_{(a,b]} F_-^n\, dF + \int_{(a,b]} F_-\, \phi_n'(F_-)\, dF +\\ &\qquad + \sum \Big[F_-\phi_n(F) - F_-\phi_n(F_-) - F_-\phi_n'(F_-)\Delta F\Big] + \sum \Delta F \Delta F^n\\ &= \int_{(a,b]} F_-^n\, dF + \int_{(a,b]} F_-\, nF_-^{n-1}\, dF +\\ &\qquad + \sum \Big[F_-F^n - F_-^{n+1} - F_-nF_-^{n-1}\Delta F + \Delta F \Delta F^n\Big]\\ &= \int_{(a,b]} (n+1)F_-^n\, dF + \sum \Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F \Delta F^n\Big]\\ &= \int_{(a,b]} \phi_{n+1}' \circ F_-\, dF + \sum \Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F \Delta F^n\Big]\end{aligned}$$

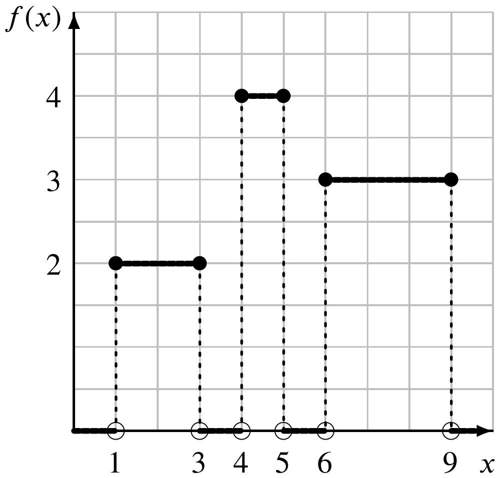
The expression under the sum can be written as

$$
\begin{aligned}
&F_- F^n - F_-^{n+1} - nF_-^n \Delta F + \Delta F \Delta F^n \\
&\quad = (F_- - F)F^n + F^{n+1} - F_-^{n+1} - nF_-^n \Delta F + \Delta F \Delta F^n \\
&\quad = F^{n+1} - F_-^{n+1} + \Delta F\Big(- F^n - nF_-^n + \Delta F^n\Big) \\
&\quad = F^{n+1} - F_-^{n+1} + \Delta F\Big(- F^n - nF_-^n + F^n - F_-^n\Big) \\
&\quad = F^{n+1} - F_-^{n+1} - (n+1)F_-^n \Delta F \\
&\quad = \phi_{n+1}\circ F - \phi_{n+1}\circ F_- - \phi'_{n+1}\circ F_- \Delta F
\end{aligned}
$$

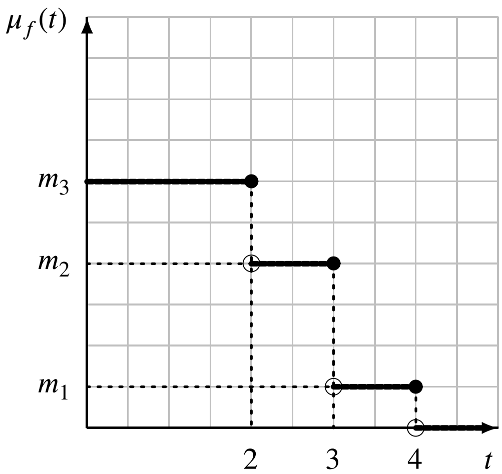
and the induction is complete.

Problem 14.19 Solution:

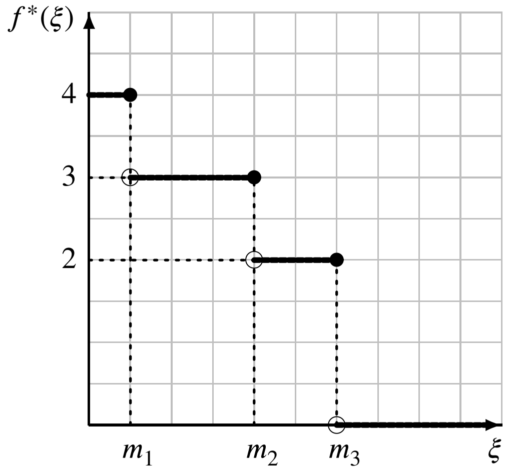
(i) We have the following pictures:



This is the graph of the original function $f(x)$.
Open and full dots indicate the continuity behaviour at the jump points.
x-values are to be measured in μ-length, i.e. x is a point in the measure space $(X, \mathscr{A}, \mu)$.



This is the graph of the associated distribution function $\mu_f(t)$. It is decreasing and left-continuous at the jump points.
t-values are to be measured using Lebesgue measure in $[0, \infty)$.
$m_1 = \mu\big([4, 5]\big)$
$m_2 - m_1 = \mu\big([6, 9]\big)$
$m_3 - m_2 = \mu\big([4, 5]\big)$



This is the graph of the decreasing rearrangement $f^*(\xi)$ of $f(x)$. It is decreasing and right-continuous at the jump points. (Please note that the picture is wrong and actually depicts the left-continuous inverse which is $\inf\{t : \mu_f(t) < \xi\}$ — mind the "$\leqslant$" vs. "$<$" inside the infimum)
ξ-values are to be measured using Lebesgue measure in $[0, \infty)$.
m_1, m_2, m_3 are as in the previous picture.

(ii) The first equality,

$$\int_{\mathbb{R}} |f|^p \, d\mu = p \int_0^\infty t^{p-1} \, \mu_f(t) \, dt,$$

follows immediately from Theorem 14.13 with $u = |f|$ and $\mu_f(t) = \mu(\{|f| \geqslant t\})$.

To show the second equality we have two possibilities. We can...

a) ...show the second equality first for (positive) simple functions and use then a (by now standard...) Beppo Levi/monotone convergence argument to extend the result to all positive measurable functions. Assume that $f(x) = \sum_{j=0}^N a_j \mathbb{1}_{B_j}(x)$ is a positive simple function in standard representation, i.e. $a_0 = 0 < a_1 < \cdots < a_n < \infty$ and the sets $B_j = \{f = a_j\}$ are pairwise disjoint. Then we have

$$\begin{aligned}
\mu(\{f = a_j\}) &= \mu(\{f \geqslant a_j\} \setminus \{f \geqslant a_{j+1}\}) \\
&= \mu(\{f \geqslant a_j\}) - \mu(\{f \geqslant a_{j+1}\}) \\
&= \mu_f(a_j) - \mu_f(a_{j+1}) && (a_{n+1} := \infty, \mu_f(a_{n+1}) = 0) \\
&= \lambda^1\big((\mu_f(a_{j+1}), \mu_f(a_j)]\big) \\
&= \lambda^1(f^* = a_j).
\end{aligned}$$

This proves

$$\int f^p \, d\mu = \sum_{j=0}^n a_j^p \, \mu(B_j) = \sum_{j=0}^n a_j^p \, \lambda^1(f^* = a_j) = \int (f^*)^p \, d\lambda^1$$

and the general case follows from the above-mentioned Beppo Levi argument.

or we can

b) use Theorem 14.13 once again with $u = f^*$ and $\mu = \lambda^1$ provided we know that

$$\mu\big(\{|f| \geqslant t\}\big) = \lambda^1\big(\{f^* \geqslant t\}\big).$$

This, however, follows from

$$\begin{aligned} f^*(\xi) \geqslant t &\iff \inf\{s : \mu_f(s) \leqslant \xi\} \geqslant t \\ &\iff \mu_f(t) \geqslant \xi \qquad\qquad \text{(as } \mu_f \text{ is right cts. \& decreasing)} \\ &\iff \mu\big(\{|f| \geqslant t\}\big) \geqslant \xi \end{aligned}$$

and therefore

$$\lambda^1\big(\{\xi \geqslant 0 : f^*(\xi) \geqslant t\}\big) = \lambda^1\big(\{\xi \geqslant 0 : \mu(|f| \geqslant t) \geqslant \xi\}\big) = \mu(|f| \geqslant t).$$

Problem 14.20 Solution: (By Franzsika Kühn) Fix $t \in \mathbb{R}$. Applying the fundamental theorem of calculus and Fubini's theorem, we find

$$\begin{aligned} F(t+h) - F(t) = \int_X (\phi(t+h,x) - \phi(t,x))\, \mu(dx) &= \int_X \int_t^{t+h} \partial_t \phi(r,x)\, dr\, \mu(dx) \\ &= \int_t^{t+h} \underbrace{\int_X \partial_t \phi(r,x)\, \mu(dx)}_{=: f(r)}\, dr. \end{aligned}$$

for all $h \in \mathbb{R}$. Since f is (by assumption) continuous, this implies

$$\lim_{h\to 0} \frac{1}{h}(F(t+h) - F(t)) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(r)\, dr = f(t) \stackrel{\text{def}}{=} \int_X \partial_t \phi(t,x)\, \mu(dx).$$

---

15 Integrals with respect to image measures. Solutions to Problems 15.1–15.16

Problem 15.1 Solution: The first equality

$$\int u\,d(T(f\mu)) = \int u{\circ}T\ f\,d\mu$$

is just Theorem 15.1 combined with Lemma 10.8 the formula for measures with a density.

The second equality

$$\int u{\circ}T\ f\,d\mu = \int u\,f{\circ}T^{-1}\,dT(\mu)$$

is again Theorem 15.1.

The third equality finally follows again from Lemma 10.8.

■■

Problem 15.2 Solution: Observe that T_ϵ is represented by the $n\times n$ diagonal matrix A with entries ϵ. Since $\det A = \epsilon^n$, the claim follows from Example 15.3(iii).

■■

Problem 15.3 Solution: Let $x, y \in \mathbb{R}$. We have

$$\begin{aligned}
\mathbb{1}_{[0,1]}(x-y)\mathbb{1}_{[0,1]}(y) &= \mathbb{1}_{[-x,-x+1]}(-y)\mathbb{1}_{[0,1]}(y)\\
&= \mathbb{1}_{[x-1,x]}(y)\mathbb{1}_{[0,1]}(y)\\
&= \begin{cases} 0, & x<0 \text{ or } x>2,\\ \mathbb{1}_{[0,x]}(y), & x\in[0,1],\\ \mathbb{1}_{[x-1,1]}, & x\in[1,2]. \end{cases}
\end{aligned} \tag{$*$}$$

This shows that

$$\begin{aligned}
(\mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]})(x) &= \int_{\mathbb{R}} \mathbb{1}_{[0,1]}(x-y)\mathbb{1}_{[0,1]}(y)\,dy\\
&= \begin{cases} 0, & x<0 \ \text{ or } \ x>2,\\ \int_0^x dy = x, & x\in[0,1],\\ \int_{x-1}^1 dy = 2-x, & x\in[1,2], \end{cases}
\end{aligned}$$

$$= x\mathbb{1}_{[0,1]}(x) + (2-x)\mathbb{1}_{[1,2]}(x).$$

Since convolutions are linear and commutative, we get

$$\begin{aligned}
(\mathbb{1}_{[0,1]} * &\mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]})(x) \\
&= \big(\mathbb{1}_{[0,1]} * (\mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]})\big)(x) \\
&= \int \mathbb{1}_{[0,1]}(x-y)(y\mathbb{1}_{[0,1]}(y) + (2-y)\mathbb{1}_{[1,2]}(y))\,dy \\
&= \int y\mathbb{1}_{[0,1]}(x-y)\mathbb{1}_{[0,1]}(y)\,dy + \int (2-y)\mathbb{1}_{[0,1]}(x-y)\mathbb{1}_{[1,2]}(y)\,dy \\
&=: I_1(x) + I_2(x).
\end{aligned}$$

Let us work out the two integrals separately. For the first expression we find using $(*)$

$$\begin{aligned}
I_1(x) &= \begin{cases} 0, & x > 0 \text{ or } x > 2, \\ \int_0^x y = \frac{x^2}{2}, & x \in [0,1], \\ \int_{1-x}^1 y\,dy = \frac{1}{2}(1-(1-x)^2), & x \in [1,2]. \end{cases} \\
&= \frac{x^2}{2}\mathbb{1}_{[0,1]}(x) + \frac{1}{2}(1-(1-x)^2)\mathbb{1}_{[1,2]}(x).
\end{aligned}$$

A similar calculation for the second integral yields

$$\mathbb{1}_{[0,1]}(x-y)\mathbb{1}_{[1,2]}(y) = \mathbb{1}_{[x-1,x]}(y)\mathbb{1}_{[1,2]}(y) = \begin{cases} 0, & x < 1 \text{ or } x > 3, \\ \mathbb{1}_{[1,x]}(y), & x \in [1,2], \\ \mathbb{1}_{[x-1,2]}(y), & x \in [2,3]. \end{cases}$$

This gives

$$\begin{aligned}
I_2(x) &= \begin{cases} 0, & x < 1 \text{ or } x > 3, \\ \int_1^x (2-y)\,dy = 2(x-1) - \frac{1}{2}(x^2-1), & x \in [1,2], \\ \int_{x-1}^2 (2-y)\,dy = 2(3-x) - \frac{1}{2}(4-(1-x)^2), & x \in [2,3] \end{cases} \\
&= \Big(2(x-1) - \frac{1}{2}(x^2-1)\Big)\mathbb{1}_{[1,2]}(x) + \Big(2(1+x) - \frac{1}{2}(4-(1-x)^2)\Big)\mathbb{1}_{[2,3]}(x).
\end{aligned}$$

Finally

$$\begin{aligned}
(\mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]})(x) = \frac{x^2}{2}\mathbb{1}_{[0,1]}(x) &+ \Big(-x^2 + 3x - \frac{3}{2}\Big)\mathbb{1}_{[1,2]}(x) + \\
&\Big(2(3-x) - \frac{1}{2}(4-(1-x)^2)\Big)\mathbb{1}_{[2,3]}(x).
\end{aligned}$$

Problem 15.4 Solution: Observe that the assertion is equivalent to saying

$$(\overline{\operatorname{supp} u + \operatorname{supp} w})^c \subset (\operatorname{supp}(u * w))^c.$$

Assume that $x_0 \in (\overline{\operatorname{supp} u + \operatorname{supp} w})^c$. Since this is an open set, there is some $r > 0$ such that $B_r(x_0) \subset (\overline{\operatorname{supp} u + \operatorname{supp} w})^c$. Pick any $x \in B_r(x_0)$. For all $y \in \operatorname{supp} w$ we find $x - y \notin \operatorname{supp} u$. In particular,

$$u(x-y) \cdot w(y) = 0 \quad \forall\, y \in \operatorname{supp} w.$$

On the other hand, the very definition of the support, gives

$$u(x-y) \cdot w(y) = 0 \quad \forall\, y \notin \operatorname{supp} w.$$

This implies that $u(x-y)w(y) = 0$ for all $y \in \mathbb{R}^n$. From the definition of the convolution we see that $(u * w)(x) = 0$. Since $x \in B_r(x_0)$ is arbitrary, we get $x_0 \notin \operatorname{supp}(u * w)$.

Problem 15.5 Solution:

(i) The measurability of u, w entails that $(x, y) \mapsto u(xy^{-1})w(y)$ is again measurable. From Tonelli's theorem we see the measurability of $x \mapsto u \circledast w(x)$. In order to show commutativity, we use the transformation theorem (Theorem 15.1) for the map $z := \Phi(y) := xy^{-1}$:

$$\begin{aligned} u \circledast w(x) &= \int_{(0,\infty)} u(xy^{-1})w(y)\frac{dy}{y} \\ &= \int_{(0,\infty)} u(z)w(xz^{-1})\frac{dz}{z} \\ &= w \circledast u(x). \end{aligned}$$

Again by Tonelli's theorem

$$\begin{aligned} \int_{(0,\infty)} u \circledast w(x)\, \mu(dx) &= \int_{(0,\infty)} \left(\int_{(0,\infty)} u(xy^{-1})w(y)\frac{dy}{y} \right) \frac{dx}{x} \\ &= \int_{(0,\infty)} \left(\int_{(0,\infty)} u(xy^{-1})\, \frac{dx}{x} \right) w(y)\, \frac{dy}{y}. \end{aligned} \tag{$\star$}$$

Fix $y \in (0, \infty)$ and define $\theta_y := y^{-1}x$. From Theorem 7.10 we know that the image measure $\theta_y(\lambda)(dz)$ of λ is given by $y\lambda(dz)$ gegeben ist, and because of Theorem 15.1 we get

$$\begin{aligned} \int_{(0,\infty)} u(xy^{-1})\, \frac{dx}{x} &= y^{-1} \int_{(0,\infty)} u(xy^{-1})\frac{dx}{xy^{-1}} \\ &= y^{-1} \int_{(0,\infty)} u(z)\frac{\theta_y(\lambda)(dz)}{z} \\ &= \int_{(0,\infty)} u(z)\frac{dz}{z}. \end{aligned} \tag{$\star\star$}$$

If we insert this into ($\star$), we obtain

$$\begin{aligned} \int_{(0,\infty)} u \circledast w(x)\, \mu(dx) &= \int_{(0,\infty)} \left(\int_{(0,\infty)} u(z)\frac{dz}{z} \right) w(y)\frac{dy}{y} \\ &= \int_{(0,\infty)} u\, d\mu \int_{(0,\infty)} w\, d\mu. \end{aligned}$$

(ii) Consider first the case $p = \infty$: As $|u(xy^{-1})| \leqslant \|u\|_{L^\infty(\mu)}$ for μ-a.a. $y \in (0,\infty)$, we get

$$|u \circledast w(x)| \leqslant \int |u(xy^{-1})w(y)|\,\mu(dy) \leqslant \|u\|_{L^\infty} \int |w(y)|\,\mu(dy) = \|u\|_{L^\infty}\|w\|_1.$$

This proves $\|u \circledast w\|_{L^\infty} \leqslant \|u\|_{L^\infty}\|w\|_1$.

Now we take $p \in [1,\infty)$. Note that

$$\nu(dy) := \frac{1}{\|w\|_1}|w(y)|\,\mu(dy)$$

is a probability measure. Jensen's inequality (for $V(x) = x^p$) yields

$$\begin{aligned}
|u \circledast w(x)|^p &\leqslant \left(\int_{(0,\infty)} |u(xy^{-1})||w(y)|\,\mu(dy)\right)^p \\
&= \|w\|_1^p \left(\int_{(0,\infty)} |u(xy^{-1})|\,\nu(dy)\right)^p \\
&\leqslant \|w\|_1^p \int_{(0,\infty)} |u(xy^{-1})|^p\,\nu(dy) \\
&= \|w\|_1^{p-1} \int_{(0,\infty)} |u(xy^{-1})|^p|w(y)|\,\mu(dy),
\end{aligned}$$

and from Tonelli's theorem we get

$$\begin{aligned}
\int |u \circledast w(x)|^p\,d\mu(x) &\leqslant \|w\|_1^{p-1} \int_{(0,\infty)} \left(\int_{(0,\infty)} |u(xy^{-1})|^p|w(y)|\,\mu(dy)\right)\,\mu(dx) \\
&= \|w\|_1^{p-1} \int_{(0,\infty)} \left(\int_{(0,\infty)} |u(xy^{-1})|^p\,\mu(dx)\right)|w(y)|\,\mu(dy).
\end{aligned}$$

Just as in ($\star\star$) we conclude that

$$\begin{aligned}
\int_{(0,\infty)} |u(xy^{-1})|^p\,\mu(dx) &\overset{\text{def}}{=} \int_{(0,\infty)} |u(xy^{-1})|^p\,\frac{dx}{x} = \int_{(0,\infty)} |u(z)|^p\,\frac{dz}{z} \\
&\overset{\text{def}}{=} \int_{(0,\infty)} |u(z)|^p\,\mu(dz).
\end{aligned}$$

If we insert this result into the estimates from above we see

$$\begin{aligned}
\int |u \circledast w(x)|^p\,d\mu(x) &\leqslant \|w\|_1^{p-1} \int_{(0,\infty)} \left(\int_{(0,\infty)} |u(z)|^p\,\mu(dz)\right)|w(y)|\,\mu(dy) \\
&= \|w\|_1^{p-1} \int |u|^p\,d\mu \int |w|\,d\mu \\
&= \|w\|_1\|u\|_p^p.
\end{aligned}$$

Finally, take pth roots:

$$\|u \circledast w\|_p \leqslant \|w\|_1\|u\|_p.$$

Problem 15.6 Solution: We have for any $C \in \mathscr{B}$

$$T(\mu)|_B(C) = T(\mu)(B \cap C)$$

$$
\begin{aligned}
&= \mu\big(T^{-1}(B \cap C)\big) \\
&= \mu\big(T^{-1}(B) \cap T^{-1}(C)\big) \\
&= \mu\big(A \cap T^{-1}(C)\big) \\
&= \mu|_A\big(T^{-1}(C)\big) \\
&= T(\mu|_A)(C).
\end{aligned}
$$

Problem 15.7 Solution: By definition, we find for any Borel set $B \in \mathscr{B}(\mathbb{R}^n)$

$$
\begin{aligned}
\delta_x \star \delta_y(B) &= \iint \mathbb{1}_B(s+t)\,\delta_x(ds)\,\delta_y(dt) \\
&= \int \mathbb{1}_B(x+t)\,\delta_y(dt) \\
&= \mathbb{1}_B(x+y) \\
&= \int \mathbb{1}_B(z)\,\delta_{x+y}(dz)
\end{aligned}
$$

which means that $\delta_x \star \delta_y = \delta_{x+y}$. Note that, by Tonelli's theorem the order of the iterated integrals is irrelevant.

Similarly, since $z + t \in B \iff t \in B - z$, we find

$$
\begin{aligned}
\delta_z \star \mu(B) &= \iint \mathbb{1}_B(s+t)\,\delta_z(ds)\,\mu(dt) \\
&= \int \mathbb{1}_B(z+t)\,\mu(dt) \\
&= \int \mathbb{1}_{B-z}(t)\,\mu(dt) \\
&= \mu(B-z) \\
&= \tau_{-z}(\mu)(B)
\end{aligned}
$$

where $\tau_z(t) := \tau(t-z)$ is the shift operator so that $\tau_{-z}^{-1}(B) = B - z$.

Problem 15.8 Solution: Since $x + y \in B \iff x \in B - y$, we can rewrite formula in 15.4(iii) in the following way:

$$
\begin{aligned}
\mu \star \nu(B) &= \iint \mathbb{1}_B(x+y)\,\mu(dx)\,\nu(dy) \\
&= \int \left[\int \mathbb{1}_{B-y}(x)\,\mu(dx)\right] \nu(dy) \\
&= \int \mu(B-y)\,\nu(dy).
\end{aligned}
$$

Similarly we get

$$
\mu \star \nu(B) = \int \mu(B-y)\,\nu(dy) = \int \nu(B-x)\,\mu(dx).
$$

Thus, if μ has no atoms, i.e. if $\mu(\{z\}) = 0$ for all $z \in \mathbb{R}^n$, we find

$$\mu \star \nu(\{z\}) = \int \mu\big(\{z\} - y\big)\, \nu(dy) = \int \mu\big(\underbrace{\{z - y\}}_{=0}\big)\, \nu(dy) = 0.$$

Problem 15.9 Solution: Because of Tonelli's theorem we can iterate the very definition of 'convolution' of two measures, Definition 15.4(iii), and get

$$\mu_1 \star \cdots \star \mu_n(B) = \int \cdots \int \mathbb{1}_B(x_1 + \cdots + x_n)\, \mu_1(dx_1) \cdots \mu_n(dx_n)$$

so that the formula derived at the end of Remark 15.5(ii), page 156, applies and yields

$$\begin{aligned}
&\int |\omega|\, \mathbb{P}^{\star n}(d\omega) \\
&\quad= \int \cdots \int |\omega_1 + \omega_2 + \cdots + \omega_n|\, \mathbb{P}(d\omega_1)\, \mathbb{P}(d\omega_2) \cdots \mathbb{P}(d\omega_n) \\
&\quad\overset{*}{\leqslant} \int \cdots \int \Big(|\omega_1| + |\omega_2| + \cdots + |\omega_n|\Big)\, \mathbb{P}(d\omega_1)\, \mathbb{P}(d\omega_2) \cdots \mathbb{P}(d\omega_n) \\
&\quad= \sum_{j=1}^n \int \cdots \int |\omega_j|\, \mathbb{P}(d\omega_1)\, \mathbb{P}(d\omega_2) \cdots \mathbb{P}(d\omega_n) \\
&\quad= \sum_{j=1}^n \int |\omega_j|\, \mathbb{P}(d\omega_j) \cdot \prod_{k \neq j} \int \mathbb{P}(d\omega_k) \\
&\quad= \sum_{j=1}^n \int |\omega_j|\, \mathbb{P}(d\omega_j) \\
&\quad= n \int |\omega_1|\, \mathbb{P}(d\omega_1)
\end{aligned}$$

where we use the symmetry of the iterated integrals in the integrating measures as well as the fact that $\mathbb{P}(\mathbb{R}^n) = \int \mathbb{P}(d\omega_k) = 1$. Note that we could have $+\infty$ on either side, i.e. the integrability condition is only important for the second assertion.

The equality $\int \omega\, \mathbb{P}^{\star n}(d\omega) = n \int \omega\, \mathbb{P}(d\omega)$ follows with same calculation (note that we do not get an inequality as there is no need for the triangle inequality at point (*) above). The integrability condition is now needed since the integrands are no longer positive. Note that, since $\omega \in \mathbb{R}^n$, the above equality is an equality between vectors in $\mathbb{R}^n$; this is no problem, just read the equality coordinate-by-coordinate.

Problem 15.10 Solution: Since the convolution $p \mapsto u \star p$ is linear, it is enough to consider monomials of the form $p(x) = x^k$. Thus, by the binomial formula,

$$\begin{aligned}
u \star p(x) &= \int u(x - y)\, y^k\, dy \\
&= \int u(y)\, (x - y)^k\, dy
\end{aligned}$$

$$= \sum_{j=0}^{k} \binom{k}{j} x^j \int u(y)\, y^{k-j}\, dy.$$

Since $\operatorname{supp} u$ is compact, there is some $r > 0$ such that $\operatorname{supp} u \subset B_r(0)$ and we get for any $m \in \mathbb{N}_0$, and in particular for $m = k - j$ or $m = k$, that

$$\begin{aligned}\left| \int u(y) y^m \, dy \right| &\leqslant \int_{\operatorname{supp} u} \|u\|_\infty |y|^m \, dy \\ &\leqslant \int_{B_r(0)} \|u\|_\infty r^m \, dy \\ &= 2r \cdot r^m \cdot \|u\|_\infty\end{aligned}$$

which is clearly finite. This shows that $u \star p$ exists and that it is a polynomial.

Problem 15.11 Solution: That the convolution $u \star w$ is bounded and continuous follows from Theorem 15.8.

Monotonicity follows from the monotonicity of the integral: if $x \leqslant z$, then

$$u \star w(x) = \int \underbrace{u(y)}_{\geqslant 0} \cdot \underbrace{w(x-y)}_{\leqslant w(z-y)} \, dy \leqslant \int u(y) \cdot w(z-y)\, dy = u \star w(y).$$

Problem 15.12 Solution: (This solution is written for $u \in C_c(\mathbb{R}^n)$ and $w \in C^\infty(\mathbb{R}^n)$).

Let $\partial_i = \partial / \partial x_i$ denote the partial derivative in direction x_i where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Since

$$w \in C^\infty \implies \partial_i w \in C^\infty,$$

it is enough to show $\partial_i (u \star w) = u \star \partial_i w$ and to iterate this equality. In particular, we find $\partial^\alpha (u \star w) = u \star \partial^\alpha w$ where

$$\partial^\alpha = \frac{\partial^{\alpha_1 + \cdots \alpha_n}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}, \quad \alpha \in \mathbb{N}_0^n.$$

Since u has compact support and since the derivative is a local operation (i.e., we need to know a function only in a neighbourhood of the point where we differentiate), and since we have for any $r > 0$

$$\sup_{y \in \operatorname{supp} u} \sup_{x \in B_r(0)} \left| \frac{\partial}{\partial x_i} w(x-y) \right| \leqslant c(r),$$

we can use the differentiability lemma for parameter-dependent integrals, Theorem 12.5 to find for any $x \in B_{r/2}(0)$, say,

$$\begin{aligned}\frac{\partial}{\partial x_i} \int u(y) w(x-y)\, dy &= \int u(y) \frac{\partial}{\partial x_i} w(x-y)\, dy \\ &= \int u(y) \big(\tfrac{\partial}{\partial x_i} w\big)(x-y)\, dy\end{aligned}$$

$$= u \star \partial_i w(x).$$

Problem 15.13 Solution: Let χ_t be a Friedrichs mollifier. From Lemma 15.10 we know

$$u \in C_c(\mathbb{R}^n) \implies u * \chi_t \in C_c^\infty(\mathbb{R}^n).$$

Since $u \in C_c(\mathbb{R}^n)$ is uniformly continuous, we find that

$$\lim_{t\to 0} \sup_x |u(x) - u(x - tz)| = 0$$

and since $\int \chi_t(y)\,dy = \int \chi_t(x-y)\,dy = 1$ we get

$$\begin{aligned}
|u(x) - u * \chi_t(x)| &= \left| \int (u(x) - u(y))\chi_t(x-y)\,dy \right| \\
&\leqslant \int |u(x) - u(y)| t^{-n} \chi\left(\tfrac{x-y}{t}\right) dy \\
&= \int |u(x) - u(x - tz)| \chi(z)\,dz \\
&\leqslant \int \sup_x |u(x) - u(x-tz)| \chi(z)\,dz \\
&\xrightarrow[t\to 0]{\text{dom. conv.}} 0.
\end{aligned}$$

In the last step we use the integrable dominating function $2\|u\|_\infty \chi(u)$.

Problem 15.14 Solution: The measurability considerations are just the same as in Theorem 15.6, so we skip this part.

By assumption,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r};$$

We can rewrite this as

$$\frac{1}{r} + \underbrace{\left[\frac{1}{p} - \frac{1}{r}\right]}_{=1-\frac{1}{q}\in[0,1)} + \underbrace{\left[\frac{1}{q} - \frac{1}{r}\right]}_{=1-\frac{1}{p}\in[0,1)} = 1. \tag{*}$$

Now write the integrand appearing in the definition of $u \star w(x)$ in the form

$$|u(x-y)w(y)| = \left[|u(x-y)|^{p/r} |w(y)|^{q/r}\right] \cdot \left[|u(x-y)|^{1-p/r}\right] \cdot \left[|w(y)|^{1-q/r}\right]$$

and apply the generalized Hölder inequality (cf. Problem 13.5) with the exponents from (*):

$$|u \star w(x)| \leqslant \int |u(x-y)w(y)|\,dy$$

$$\leqslant \left[\int |u(x-y)|^p|w(y)|^q\,dy\right]^{\frac{1}{r}}\left[\int |u(x-y)|^p\,dy\right]^{\frac{1}{p}-\frac{1}{r}}\left[\int |w(y)|^q\,dy\right]^{\frac{1}{q}-\frac{1}{r}}.$$

Raising this inequality to the rth power we get, because of the translation invariance of Lebesgue measure,

$$\begin{aligned}|u\star w(x)|^r &\leqslant \left[\int |u(x-y)|^p|w(y)|^q\,dy\right]\|u\|_p^{r-p}\cdot\|w\|_q^{r-q}\\ &= |u|^p\star|w|^q(x)\cdot\|u\|_p^{r-p}\cdot\|w\|_q^{r-q}.\end{aligned}$$

Now we integrate this inequality over x and use Theorem 15.6 for $p=1$ and the integral

$$\int |u|^p\star|w|^q(x)\,dx = \||u|^p\star|w|^q\|_1 \leqslant \|u\|_p^p\cdot\|w\|_q^q.$$

Thus,

$$\|u\star w\|_r^r = \int |u\star w(x)|^r\,dx \leqslant \|u\|_p^p\cdot\|w\|_q^q\cdot\|u\|_p^{r-p}\cdot\|w\|_q^{r-q} = \|u\|_p^r\cdot\|w\|_q^r$$

and the claim follows.

Problem 15.15 Solution: For $N=1$ the inequality is trivial, for $N=2$ it is in line with Problem 15.14 with $p=q$.

Let us, first of all, give a *heuristic derivation* of this result which explains how one arrives at the particular form for the value of $p=p(r,N)$. We may assume that $N\geqslant 2$. Set $F_j := f_j\star\ldots\star f_N$ for $j=1,2,\ldots N-1$. Then

$$\begin{aligned}&\|f_1\star\cdots\star f_N\|_r\\ &\quad\leqslant \|f_1\|_p\|F_2\|_{q_2} = \|f_1\|_p\|f_2\star F_3\|_{q_2}\\ &\qquad\text{by Pr. 15.14 where } \tfrac{1}{r}+1 = \tfrac{1}{p}+\tfrac{1}{q_2} = \left(\tfrac{1}{p}-1\right)+\tfrac{1}{q_2}+1\\ &\quad\leqslant \|f_1\|_p\|f_2\|_p\|F_3\|_{q_3} = \|f_1\|_p\|f_2\|_p\|f_3\star F_4\|_{q_3}\\ &\qquad\text{by Pr. 15.14 where } \tfrac{1}{r}+1 = \left(\tfrac{1}{p}-1\right)+\underbrace{\tfrac{1}{p}+\tfrac{1}{q_3}}_{=\frac{1}{q_2}+1} = 2\left(\tfrac{1}{p}-1\right)+1+\tfrac{1}{q_3}\end{aligned}$$

and repeating this procedure $N-2$ times we arrive at

$$\begin{aligned}\|f_1\star\cdots\star f_N\|_r &\leqslant \|f_1\|_p\cdots\|f_{N-2}\|_p\cdot\|f_{N-1}\star f_N\|_{q_{N-1}}\\ &\leqslant \|f_1\|_p\cdots\|f_{N-2}\|_p\cdot\|f_{N-1}\|_p\cdot\|f_N\|_{q_N}\end{aligned}$$

with the condition

$$\frac{1}{r}+1 = (N-2)\Big(\frac{1}{p}-1\Big)+1+\frac{1}{q_{N-1}} = (N-2)\Big(\frac{1}{p}-1\Big)+\frac{1}{p}+\frac{1}{q_N}$$

and since we need $q_N = p$ we get

$$\frac{1}{r} + 1 = (N-2)\Big(\frac{1}{p} - 1\Big) + \frac{2}{p} = \frac{N}{p} - N + 2$$

and rearranging this identity yields

$$p = \frac{Nr}{(N-1)r+1}.$$

If you do not like this derivation of if you got lost counting the repetitions, here's the *formal proof* using induction—but with the drawback that one needs a good educated guess what $p = p(N,r)$ should look like. The start of the induction $N = 2$ is done in Problem 15.14 (starting at $N = 1$ won't help much as we need Young's inequality for $N = 2$ anyway...).

The induction hypothesis is, of course,

$$\|f_1 \star \cdots \star f_M\|_t \leqslant \prod_{j=1}^{M} \|f_j\|_\tau \quad \text{for all} \quad M = 1,2,\dots,N-1$$

where $t > 0$ is arbitrary and $\tau = \frac{Mt}{(M-1)t+1}$.

The induction step uses Young's inequality:

$$\|f_1 \star f_2 \star \cdots \star f_N\|_r \leqslant \|f_1\|_p \cdot \|f_2 \star \cdots \star f_N\|_q$$

where $p = \frac{Nr}{(N-1)r+1}$ and q is given by

$$\frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{q} = \frac{(N-1)r+1}{Nr} + \frac{1}{q} = 1 + \frac{1}{q} - \frac{1}{N} + \frac{1}{Nr}$$

so that

$$q = \frac{Nr}{N+r-1}.$$

Using the induction hypothesis we now get

$$\|f_1 \star \cdots \star f_N\|_r \leqslant \|f_1\|_p \cdot \|f_2 \star \cdots \star f_N\|_q \leqslant \|f_1\|_p \cdot \big(\|f_2\|_s \cdots \|f_N\|_s\big)$$

where s is, because of the induction assumption, given by

$$\begin{aligned}
s &= \frac{(N-1)q}{(N-2)q+1} \\
&= \frac{(N-1)\frac{Nr}{N+r-1}}{(N-2)\frac{Nr}{N+r-1}+1} \\
&= \frac{(N-1)Nr}{(N-2)Nr+N+r-1} \\
&= \frac{(N-1)Nr}{N^2r-2Nr+r+(N-1)} \\
&= \frac{(N-1)Nr}{(N-1)^2r+(N-1)} \\
&= \frac{Nr}{(N-1)r+1} = p
\end{aligned}$$

and we are done.

Problem 15.16 Solution: Note that $v(x) = \frac{d}{dx}(1-\cos x)\mathbb{1}_{[0,2\pi)}(x) = \mathbb{1}_{(0,2\pi)}(x)\sin x$. Thus,

(i)

$$u \star v(x) = \int_0^{2\pi} 1_{\mathbb{R}}(x-y)\sin y\,dy = \int_0^{2\pi} \sin y\,dy = 0 \quad \forall x.$$

(ii) Since all functions u, v, w, ϕ are continuous, we can use the usual rules for the (Riemann) integral and get, using integration by parts and the fundamental theorem of integral calculus,

$$\begin{aligned} v \star w(x) &= \int \tfrac{d}{dx}\phi(x-y) \int_{-\infty}^{x} \phi(t)\,dt\,dx \\ &= \int \big(-\tfrac{d}{dy}\phi(x-y)\big) \int_{-\infty}^{y} \phi(t)\,dt\,dx \\ &= \int \phi(x-y)\,\tfrac{d}{dy} \int_{-\infty}^{y} \phi(t)\,dt\,dx \\ &= \int \phi(x-y)\,\phi(y)\,dy \\ &= \phi \star \phi(x). \end{aligned}$$

If $x \in (0, 4\pi)$, then $x - y \in (0, 2\pi)$ for some suitable $y = y_=$ and even for all y from an interval $(y_0 - \epsilon, y_0 + \epsilon) \subset (0, 2\pi)$. Since ϕ is positive with support $[0, 2\pi]$, the positivity follows.

(iii) Obviously,

$$(u \star v) \star w \overset{(i)}{=} 0 \star w = 0$$

while

$$\begin{aligned} u \star (v \star w)(x) &= \int 1_{\mathbb{R}}(x-y) v \star w(y)\,dy \\ &= \int v \star w(y)\,dy \\ &= \int \phi \star \phi(y)\,dy \\ &> 0. \end{aligned}$$

Note that w is not an (pth power, $p < \infty$) integrable function so that we cannot use Fubini's theorem to prove associativity of the convolution.

■■

16 Jacobi's transformation theorem. Solutions to Problems 16.1–16.12

Problem 16.1 Solution: Since F and F_i are F_σ-sets, we get

$$F = \bigcup_{k\in\mathbb{N}} C_k, \qquad F_i = \bigcup_{k\in\mathbb{N}} C_k^i$$

for closed sets C_k resp. C_k^i. Since complements of closed sets are open, we find, using the rules for (countable) unions and intersections that

(i) $$\bigcap_{i=1}^n F_i = \bigcap_{i=1}^n \bigcup_{k\in\mathbb{N}} C_k^i = \bigcup_{k\in\mathbb{N}} \underbrace{\bigcap_{i=1}^n C_k^i}_{\text{closed set}}.$$

(ii) $$\bigcup_{i\in\mathbb{N}} F_i = \bigcup_{i\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} C_k^i = \underbrace{\bigcup_{(i,k)\in\mathbb{N}\times\mathbb{N}} C_k^i}_{\text{countable union!}}.$$

Moreover, $\bigcap_{i\in\mathbb{N}} F_i^c = \bigcap_{i\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} [C_k^i]^c = \underbrace{\bigcap_{(i,k)\in\mathbb{N}\times\mathbb{N}} [C_k^i]^c}_{\text{countable intersection!}}.$

(iii) $$F = \bigcup_{k\in\mathbb{N}} C_k \Rightarrow F^c = \left[\bigcup_{k\in\mathbb{N}} C_k\right]^c = \bigcap_{k\in\mathbb{N}} \underbrace{C_k^c}_{\text{open}}.$$

(iv) Set $c_1 := C$ and $C_i = \emptyset$, $i \geqslant 2$. Then $C = \bigcup_{i\in\mathbb{N}} C_i$ is an F_σ-set.

■■

Problem 16.2 Solution: Write $\lambda = \lambda^n$ and $\mathscr{B} = \mathscr{B}(\mathbb{R}^n)$. Fix $B \in \mathscr{B}$. According to Lemma 16.12 there are sets $F \in F_\sigma$ and $G \in G_\delta$ such that

$$F \subset B \subset G \text{ and } \lambda(F) = \lambda(B) = \lambda(G).$$

Since for closed sets C_j and open sets U_j we have $F = \bigcup C_j$ and $G = \bigcap U_j$ we get for some $\epsilon > 0$ and suitable $M = M_\epsilon \in \mathbb{N}$, $N = N_\epsilon \in \mathbb{N}$ that

$$C_1 \cup \dots \cup C_N \subset B \subset U_1 \cap \dots \cap U_M$$

and

$$\left|\lambda(U_1 \cap \dots \cap U_M) - \lambda(B)\right| \leqslant \epsilon, \tag{*}$$

$$\left|\lambda(B) - \lambda(C_1 \cup \dots \cup C_N)\right| \leqslant \epsilon. \tag{**}$$

Since finite unions of closed sets are closed and finite intersections of open sets are open, (*) proves outer regularity while (**) proves inner regularity (w.r.t. close sets).

To see inner regularity with compact sets, we note that the closed set $C' := C_1 \cup \dots \cup C_N$ is approximated by the following compact sets

$$K_\ell := \overline{B_\ell(0)} \cap C' \uparrow C' \quad \text{as} \quad \ell \to \infty$$

and, because of the continuity of measures, we get for suitably large $L = L_\epsilon \in \mathbb{N}$ that

$$\left|\lambda(K_L) - \lambda(C_1 \cup \dots \cup C_N)\right| \leqslant \epsilon$$

which can be combined with (**) to give

$$\left|\lambda(K_L) - \lambda(B)\right| \leqslant 2\epsilon.$$

This shows inner regularity for the compact sets.

■■

Problem 16.3 Solution: Notation (for brevity): Write $\lambda = \lambda^n$, $\bar\lambda = \overline{\lambda^n}$, $\mathscr{B} = \mathscr{B}(\mathbb{R}^n)$ and $\mathscr{B}^* = \mathscr{B}^*(\mathbb{R}^n)$. By definition, $B^* = B \cup N^*$ where N^* is a subset of a $\mathscr{B}$-measurable null set N. (We indicate $\mathscr{B}^*$-sets by an asterisk, C (with and without ornaments and indices C' ...) is always a closed set and U etc. is always an open set.

Solution 1: Following the hint we get (with the notation of Problem 11.6)

$$\begin{aligned}
\lambda(B) = \bar\lambda(B^*) &= \lambda^*(B^*) \\
&= \inf_{\mathscr{B}\ni A\supset B^*} \lambda(A) && \text{(by 11.6)} \\
&= \inf_{\mathscr{B}\ni A\supset B^*} \inf_{U\supset A} \lambda(U) && \text{(by 16.2)} \\
&\leqslant \inf_{U'\supset B\cup N} \inf_{U\supset U'} \lambda(U) && \text{(as } B *\subset B\cup N) \\
&= \inf_{U'\supset B^*} \lambda(U') && \text{(by 16.2)} \\
&= \lambda(B\cup N) && \text{(by 16.2)} \\
&\leqslant \lambda(B) + \lambda(N) \\
&= \lambda(B).
\end{aligned}$$

Inner regularity (for closed sets) follows similarly,

$$\begin{aligned}
\lambda(B) = \bar\lambda(B^*) &= \lambda_*(B^*) \\
&= \sup_{\mathscr{B}\ni A\subset B^*} \lambda(A) && \text{(by 11.6)} \\
&= \sup_{\mathscr{B}\ni A\subset B^*} \sup_{C\subset A} \lambda(C) && \text{(by 16.2)}
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \sup_{C'\subset B^*} \sup_{C\subset C'} \lambda(C) \\
&= \sup_{C'\subset B^*} \lambda(C') && \text{(by 16.2)} \\
&\geqslant \sup_{C'\subset B} \lambda(C') && (\text{as } B \subset B^*) \\
&= \lambda(B), && \text{(by 16.2)}
\end{aligned}
$$

and inner regularity for compact sets is the same calculation.

There is a more elementary ....

Solution 2: (without Problem 11.6). Using the definition of the completion we get

$$
\begin{aligned}
\bar{\lambda}(B^*) = \lambda(B) &= \sup_{C'\subset B} \lambda(C') \\
&\leqslant \sup_{C\subset B^*} \lambda(C) \\
&\leqslant \sup_{C''\subset B\cup N} \lambda(C'') \\
&= \lambda(B\cup N) \\
&= \lambda(B)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\bar{\lambda}(B^*) = \lambda(B) &= \inf_{U'\supset B} \lambda(U') \\
&\leqslant \inf_{U\supset B^*} \lambda(U) \\
&\leqslant \inf_{U''\supset B\cup N} \lambda(U'') \\
&= \lambda(B\cup N) \\
&= \lambda(B).
\end{aligned}
$$

Problem 16.4 Solution:

(i) Using the result of Problem 7.12 we write $x, y \in C$ as triadic numbers:

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} = 0.x_1x_2x_3\dots \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{y_i}{3^i} = 0.y_1y_2y_3\dots$$

where $x_i, y_i \in \{0,2\}$. In order to enforce uniqueness, we only want to have *truly* infinite sums, i.e. we use $0.002222\dots$ instead of $0.01000\dots$ etc.

Obviously, every $z \in C - C$ is of the form $z = x - y$ with $x, y \in C$ and so $z = 0.z_1z_2z_3\dots$ with $z_i = x_i - y_i \in \{-2, 0, 2\}$. Thus,

$$\frac{1}{2}(z+1) = \frac{1}{2}\left(\sum_{i=1}^{\infty} \frac{x_i - y_i}{3^i} + \sum_{i=1}^{\infty} \frac{2}{3^i}\right) = \frac{1}{2}\sum_{i=1}^{\infty} \frac{x_i - y_i + 2}{3^i} = \sum_{i=1}^{\infty} \frac{w_i}{3^i}.$$

By construction, $w_i = \frac{1}{2}(x_i - y_i + 2) \in \frac{1}{2}\{0, 2, 4\} = \{0, 1, 2\}$, i.e. the numbers $\frac{1}{2}(z+1)$ make up the whole interval $[0, 1]$.

This shows that $C - C = [-1, 1]$.

(ii) Let $\alpha(x, y) = x - y$ as in the hint. This is a Lipschitz (Hölder-1) continuous map from $\mathbb{R}^2 \to \mathbb{R}$ and it has the following property: $C \times C \mapsto \alpha(C, C) = [-1, 1]$. But $C \times C$ is a Lebesgue null set in $\mathbb{R}^2$ while $\lambda^1[-1, 1] = 2$. This situation cannot occur in Corollary 16.14.

Problem 16.5 Solution:

(i) Obviously, $\mathscr{G} \subset \mathscr{B}[0, \infty)$. On the other hand, $\sigma(\mathscr{G})$ contains all open intervals of the form

$$(\alpha, \beta) = \bigcup_{n \in \mathbb{N}} \big[\alpha - \tfrac{1}{n}, \infty\big) \setminus \big[\beta, \infty\big), \qquad 0 \leqslant \alpha < \beta < \infty \tag{*}$$

and all intervals of the form

$$[0, \beta) = [0, \infty) \setminus [\beta, \infty), \qquad \beta > 0. \tag{**}$$

Thus,

$$\sigma(\mathscr{G}) \supset \mathscr{O}(\mathbb{R}) \cap [0, \infty)$$

since any open set $U \in \mathscr{O}(\mathbb{R})$ is a countable union of open intervals,

$$U = \bigcup_{\substack{\alpha < \beta, \alpha, \beta \in \mathbb{Q} \\ (\alpha, \beta) \subset U}} (\alpha, \beta),$$

so that $U \cap [0, \infty) \in \mathscr{O} \cap [0, \infty)$ is indeed a countable union of sets of the form (*) and (**). Thus,

$$\mathscr{B}[0, \infty) = \sigma(\mathscr{O} \cap [0, \infty)) \subset \sigma(\mathscr{G}) \subset \mathscr{B}[0, \infty).$$

(ii) That μ is a measure follows from Lemma 10.8 (for a proof, see the online section ‘additional material’). Since

$$\rho(B) = \mu(T_{1/5}^{-1}(B)) = T_{1/5}(\rho)(B)$$

where $T_{1/5}(x) = \frac{1}{5} \cdot x$, ρ is an image measure, hence a measure.

Since

$$\rho[a, \infty) = \mu[5a, \infty) \leqslant \mu[a, \infty) \qquad \forall\, a \geqslant 0,$$

we have $\rho\big|_{\mathscr{G}} \leqslant \mu\big|_{\mathscr{G}}$. On the other hand,

$$\rho\big[\tfrac{3}{5}, \tfrac{4}{5}\big) = \mu[3, 4) = 1 > 0 = \mu\big[\tfrac{3}{5}, \tfrac{4}{5}\big).$$

This does not contradict Lemma 16.6 since $\mathscr{G}$ is not a semi-ring.

Problem 16.6 Solution: We want to show that

a) $\lambda^n(x+B) = \lambda^n(B)$, $B \in \mathscr{B}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ (Theorem 5.8(i));

b) $\lambda^n(t \cdot B) = t^n \lambda^n(B)$, $B \in \mathscr{B}(\mathbb{R}^n)$, $t \geqslant 0$ (Problem 5.9);

c) $A(\lambda^n) = |\det A^{-1}| \cdot \lambda^n$, $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0$ (Theorem 7.10).

From Theorem 16.4 we know that for any C^1-diffeomorphism ϕ the formula

$$\lambda^n(\phi(B)) = \int_B |\det D\phi| \, d\lambda^n$$

holds. Thus a), b), c) follow upon setting

a) $\phi(y) = x + y \implies D\phi \equiv 1 \implies |\det D\phi| \equiv 1$;

b) $\phi(y) = t \cdot y \implies D\phi \equiv t \cdot \mathrm{id} \implies |\det D\phi| \equiv t^n$;

c) $\phi(y) = A^{-1}y \implies D\phi(y) \equiv A^{-1} \implies |\det D\phi| \equiv |\det A|^{-1}$.

Problem 16.7 Solution:

(i) The map $\Phi : \mathbb{R} \ni x \mapsto (x, f(x))$ is obviously bijective and differentiable with derivative $D\Phi(x) = (1, f'(x))$ so that $|D\Phi(x)|^2 = 1 + (f'(x))^2$. The inverse of Φ is given by $\Phi^{-1} : (x, f(x)) \mapsto x$ which is clearly differentiable.

(ii) Since $|D\Phi(x)| = \sqrt{1+(f'(x))^2}$ is positive and measurable, it is a density function and $\mu := |D\Phi(x)| \cdot \lambda$ is a measure, cf. Lemma 10.8, while $\sigma = \Phi(\mu)$ is an image measure in the sense of Definition 7.7.

(iii) This is Theorem 15.1 and/or Problem 15.1.

(iv) The normal is, by definition, orthogonal to the gradient: $D\Phi(x) = (1, f'(x))$; obviously $|n(x)| = 1$ and

$$n(x) \cdot D\Phi(x) = \frac{\begin{pmatrix} -f'(x) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}}{\sqrt{1+(f'(x))^2}} = 0.$$

Further,

$$\widetilde{\Phi}(x,r) = \begin{pmatrix} x - \frac{rf'(x)}{\sqrt{1+[f'(x)]^2}} \\ f(x) + \frac{r}{\sqrt{1+[f'(x)]^2}} \end{pmatrix},$$

so that

$$\begin{aligned} D\widetilde{\Phi}(x,r) &= \left(\frac{\partial \widetilde{\Phi}(x,r)}{\partial(x,r)} \right) \\ &= \begin{pmatrix} 1 - r\frac{\partial}{\partial x}\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} & f'(x) + r\frac{\partial}{\partial x}\frac{1}{\sqrt{1+[f'(x)]^2}} \\ -\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} & \frac{1}{\sqrt{1+[f'(x)]^2}} \end{pmatrix} \end{aligned}$$

For brevity we write f, f', f'' instead of $f(x), f'(x), f''(x)$. Now

$$\frac{\partial}{\partial x}\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} = \frac{f''\sqrt{1+[f']^2} - f'\,\frac{f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}$$

and

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{1+[f'(x)]^2}} = \frac{-\frac{f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}.$$

Thus, $\det D\widetilde{\Phi}(x,r)$ becomes

$$\begin{aligned}
&\frac{1}{\sqrt{1+[f']^2}}\left(1 - \frac{r\,f''\sqrt{1+[f']^2} - \frac{r\,[f']^2 f''}{\sqrt{1+[f']^2}}}{1+[f']^2}\right)\\
&\qquad + \frac{f'}{\sqrt{1+[f']^2}}\left(f' - \frac{\frac{r\,f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}\right)\\
&= \frac{1}{\sqrt{1+[f']^2}} - \frac{r\,f'' - \frac{r\,[f']^2 f''}{1+[f']^2}}{1+[f']^2} + \frac{[f']^2}{\sqrt{1+[f']^2}} - \frac{\frac{r\,[f']^2 f''}{1+[f']^2}}{1+[f']^2}\\
&= \frac{1+[f']^2}{\sqrt{1+[f']^2}} - \frac{r\,f''}{1+[f']^2}\\
&= \sqrt{1+[f']^2} - \frac{r\,f''}{1+[f']^2}
\end{aligned}$$

If x is from a compact set, say $[c,d]$, we can, because of the continuity of f, f' and f'', achieve that for sufficiently small values of $|r| < \epsilon$ we get that $\det D\widetilde{\Phi} > 0$, i.e. $\widetilde{\Phi}$ is a local C^1-diffeomorphism.

(v) The set is a 'tubular' neighbourhood of radius r around the graph Γ_f for $x \in [c,d]$. Measurability follows, since $\widetilde{\Phi}$ is a diffeomorphism, from the fact that the set $C(r)$ is the image of the cartesian product of measurable sets.

(vi) Because of part (iv) we have, for fixed x and sufficiently small values of r, that the determinant is positive so that

$$\begin{aligned}
&\lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\left|\det D\widetilde{\Phi}(x,s)\right|\lambda^1(ds)\\
&= \lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\left|\sqrt{1+(f'(x))^2} - \frac{s\,f''(x)}{1+(f'(x))^2}\right|\lambda^1(ds)\\
&= \lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\left(\sqrt{1+(f'(x))^2} - \frac{s\,f''(x)}{1+(f'(x))^2}\right)\lambda^1(ds)\\
&= \lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\sqrt{1+(f'(x))^2}\,\lambda^1(ds)\\
&\qquad - \lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\frac{s\,f''(x)}{1+(f'(x))^2}\,\lambda^1(ds)\\
&= \sqrt{1+(f'(x))^2} - \frac{f''(x)}{1+(f'(x))^2}\lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)} s\,\lambda^1(ds)
\end{aligned}$$

$$= \sqrt{1+(f'(x))^2}$$
$$= \left|\det D\widetilde{\Phi}(x,0)\right|.$$

(vii) We have

$$\begin{aligned}
&\frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{C(r)}(x,y)\,\lambda^2(dx,dy)\\
&= \frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{\widetilde{\Phi}(\Phi^{-1}(C)\times(-r,r))}(x,y)\,\lambda^2(dx,dy)\\
&= \frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{\Phi^{-1}(C)\times(-r,r)}(z,s)\left|\det D\widetilde{\Phi}(z,s)\right|\lambda^2(dz,ds) && \text{(Thm 16.4)}\\
&= \int_{\mathbb{R}} \mathbb{1}_{\Phi^{-1}(C)}(z)\,\underbrace{\left[\frac{1}{2r}\int_{(-r,r)}\left|\det D\widetilde{\Phi}(z,s)\right|\lambda^1(ds)\right]}_{\xrightarrow[r\downarrow 0]{}\left|\det D\widetilde{\Phi}(z,0)\right|}\,\lambda^1(dz). && \text{(Tonelli)}
\end{aligned}$$

Since $\Phi^{-1}(C)$ is a bounded subset of $\mathbb{R}$, we can use the result of part (vii) and dominated convergence and the proof is finished.

(viii) This follows from (i)–(iii) and the fact that

$$\left|\det D\widetilde{\Phi}(x,0)\right| = \sqrt{1+(f'(x))^2}$$

and the geometrical meaning of the weighted area $\frac{1}{2r}\,\lambda^2(C(r))$—recall that $C(r)$ was a tubular neighbourhood of the graph.

Problem 16.8 Solution:

(i) $|\det D\Phi(x)|$ is positive and measurable, hence a density and, by Lemma 10.8, $|\det D\Phi|\cdot\lambda^d$ is a measure. Therefore, $\Phi(|\det D\Phi|\cdot\lambda^d)$ is an image measure in the sense of Definition 7.7.

Using the rules for densities and integrals w.r.t. image measures we get (cf. e.g. Theorem 15.1 and/or Problem 15.1)

$$\int_M u\,d\lambda_M = \int_M u\,d\Phi\big(|\det D\Phi|\cdot\lambda^d\big) = \int_{\Phi^{-1}(M)} u\circ\Phi\cdot|\det D\Phi|\,d\lambda^d.$$

(ii) This is the formula from part (i) with $\Phi = \theta_r$; observe that $\theta_r(\mathbb{R}^n) = \mathbb{R}^n$.

(iii) The equality

$$\int u\,d\lambda^n = \int_{(0,\infty)}\int_{\{\|x\|=1\}} u(r\,x)\,r^{n-1}\,\sigma(dx)\,\lambda^1(dr)$$

is just Theorem 16.22. The equality

$$\begin{aligned}
&\int_{(0,\infty)}\int_{\{\|x\|=r\}} u(x)\,\sigma(dx)\,\lambda^1(dr)\\
&\qquad= \int_{(0,\infty)}\int_{\{\|x\|=1\}} u(r\,x)\,r^{n-1}\,\sigma(dx)\,\lambda^1(dr)
\end{aligned}$$

follows from part (ii).

Problem 16.9 Solution: We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_{(0,\infty)} y^{-1/2}\, e^{-y}\, \lambda(dy).$$

Using the change of variables $y = \phi(x) = x^2$, we get $D\phi(x) = 2x$ and

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_{(0,\infty)} e^{-x^2}\, \lambda(dx) = 2\int_{(-\infty,\infty)} e^{-x^2}\, \lambda(dx) \overset{16.16}{=} \sqrt{\pi}.$$

Problem 16.10 Solution: Write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Then

$$\begin{aligned}
D\Phi(r,\theta,\omega) &= \begin{pmatrix} \frac{\partial\Phi_1}{\partial r} & \frac{\partial\Phi_1}{\partial\theta} & \frac{\partial\Phi_1}{\partial\omega} \\ \frac{\partial\Phi_2}{\partial r} & \frac{\partial\Phi_2}{\partial\theta} & \frac{\partial\Phi_2}{\partial\omega} \\ \frac{\partial\Phi_3}{\partial r} & \frac{\partial\Phi_3}{\partial\theta} & \frac{\partial\Phi_3}{\partial\omega} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\omega & -r\sin\theta\cos\omega & -r\cos\theta\sin\omega \\ \sin\theta\cos\omega & r\cos\theta\cos\omega & -r\sin\theta\sin\omega \\ \sin\omega & 0 & r\cos\omega \end{pmatrix}
\end{aligned}$$

Developing according to the bottom row we calculate for the determinant

$$\begin{aligned}
&\det D\Phi(r,\theta,\omega) \\
&= \sin\omega \det\begin{pmatrix} -r\sin\theta\cos\omega & -r\cos\theta\sin\omega \\ r\cos\theta\cos\omega & -r\sin\theta\sin\omega \end{pmatrix} \\
&\qquad + r\cos\omega \det\begin{pmatrix} \cos\theta\cos\omega & -r\sin\theta\cos\omega \\ \sin\theta\cos\omega & r\cos\theta\cos\omega \end{pmatrix} \\
&= \sin\omega\Big(r^2\sin^2\theta\cos\omega\sin\omega + r^2\cos^2\theta\cos\omega\sin\omega\Big) \\
&\qquad + r\cos\omega\Big(r\cos^2\theta\cos^2\omega + r\sin^2\theta\cos^2\omega\Big) \\
&= r^2\sin^2\omega\cos\omega + r^2\cos\omega\cos^2\omega \\
&= r^2\cos\omega
\end{aligned}$$

where we use repeatedly the elementary relation $\sin^2\phi + \cos^2\phi = 1$.

Thus,

$$\begin{aligned}
&\iiint_{\mathbb{R}^3} u(x,y,z)\, d\lambda^3(x,y,z) \\
&\quad = \iiint_{\Phi^{-1}(\mathbb{R}^3)} u\circ\Phi(r,\theta,\omega)\, |\det D\Phi(r,\theta,\omega)|\, d\lambda^3(r,\theta,\omega) \\
&\quad = \int_0^\infty \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} U(r\cos\theta\cos\omega, r\sin\theta\cos\omega, r\sin\omega)\, r^2\cos\omega\, dr\, d\theta\, d\omega.
\end{aligned}$$

Problem 16.11 Solution:

(i) We change in

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$$

variables according to $u^2 = t$, and get

$$\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1}\,du.$$

Using Tonelli's theorem we find

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\left(\int_0^\infty e^{-u^2} u^{2x-1}\,du\right)\left(\int_0^\infty e^{-v^2} v^{2y-1}\,dv\right)\\ &= 4\int_{(0,\infty)^2} e^{-u^2-v^2} u^{2x-1} v^{2y-1}\,d(u,v).\end{aligned}$$

(ii) We have to show that $B(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y)$. Using polar coordinates in (i) we see

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_{r=0}^\infty \int_{\phi=0}^{2\pi} e^{-r^2} r^{2x+2y-1} (\cos\phi)^{2x-1} (\sin\phi)^{2y-1}\,d\phi\,dr\\ &= 4\left(\int_{r=0}^\infty e^{-r^2} r^{2x+2y-1}\,dr\right)\left(\int_{\phi=0}^{\pi/2} (\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\,d\phi\right). \qquad (\star)\end{aligned}$$

Setting $s := r^2$ we see

$$\int_{r=0}^\infty e^{-r^2} r^{2x+2y-1}\,dr = \frac{1}{2}\int_{s=0}^\infty e^{-s} s^{(x+y)-1}\,ds = \frac{1}{2}\Gamma(x+y).$$

Change variables in the second integral of $(\star)$ according to $t = \cos^2\phi$ and use $\sin^2\phi + \cos^2\phi = 1$. This yields

$$\int_{\phi=0}^{\pi/2} (\cos\phi)^{2x-1}(\sin\phi)^{2y-1}\,d\phi = \frac{1}{2}\int_0^1 t^{2x-1}(1-t)^{2y-1}\,dt = \frac{1}{2}B(x,y).$$

■■

Problem 16.12 Solution: We introduce planar polar coordinates as in Example 16.15:

$$(x,y) = (r\cos\theta, r\sin\theta), \quad r > 0,\ \theta \in [0, 2\pi).$$

Thus,

$$\begin{aligned}&\iint_{\|x\|^2+\|y\|^2<1} x^m\, y^n\, d\lambda^2(x,y)\\ &= \int_0^1 \int_0^{2\pi} r^{n+m+1}\, \cos^m\theta\, \sin^n\theta\, dr\, d\theta\\ &= \left(\int_0^1 r^{n+m+1}\,dr\right)\left(\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta\right) \qquad (*)\\ &= \frac{r^{m+n+2}}{m+n+2}\Bigg|_{r=0}^{r=1}\left(\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta\right)\\ &= \frac{1}{m+n+2}\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta.\end{aligned}$$

Consider the integral

$$\frac{1}{m+n+2}\int_0^{2\pi}\cos^m\theta\,\sin^n\theta\,d\theta;$$

Since sine and cosine are periodic and since we integrate over a whole period, we can also write

$$\frac{1}{m+n+2}\int_{-\pi}^{\pi}\cos^m\theta\,\sin^n\theta\,d\theta;$$

If n is odd, $\sin^n\theta$ is odd while $\cos^m\theta$ is always even. Thus, the integral equals, for odd n, zero.

Since the l.h.s. of the expression (*) is symmetric in m and n, so is the r.h.s. and we get

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\,y^n\,d\lambda^2(x,y)=0$$

whenever m or n or both are odd.

If both m and n are even, we get

$$\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\,y^n\,d\lambda^2(x,y)=\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ \pm x>0,\ \pm y>0}} x^m\,y^n\,d\lambda^2(x,y)$$

for any choice of signs, thus

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\,y^n\,d\lambda^2(x,y)=4\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\,y^n\,d\lambda^2(x,y).$$

Introducing planar polar coordinates yields, as seen above, for even m and n,

$$\begin{aligned}
4\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\,y^n\,d\lambda^2(x,y) &= \frac{4}{m+n+2}\int_0^{\pi/2}\cos^m\theta\,\sin^n\theta\,d\theta\\
&= \frac{4}{m+n+2}\int_0^1 (1-t^2)^{\frac{m-1}{2}}(t^2)^{\frac{n-1}{2}}\,t\,dt
\end{aligned}$$

where we use the substitution $t=\sin\theta$ and $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-t^2}$. A further substitution $s=t^2$ yields

$$\begin{aligned}
&= \frac{2}{m+n+2}\int_0^1 (1-s)^{\frac{m-1}{2}}s^{\frac{n-1}{2}}\,ds\\
&= \frac{2}{m+n+2}\int_0^1 (1-s)^{\frac{m+1}{2}-1}s^{\frac{n+1}{2}-1}\,ds\\
&= \frac{2}{m+n+2}\,B\big(\tfrac{m+1}{2},\tfrac{n+1}{2}\big)
\end{aligned}$$

which is Euler's Beta function. There is a well-known relation between the Euler Beta- and Gamma functions:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{*}$$

so that, finally,

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\, y^n\, d\lambda^2(x,y) = \begin{cases} 0 & m \text{ or } n \text{ odd;} \\ \frac{2}{m+n+2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+m+2}{2}\right)} & \text{else} \\ \quad = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+m+4}{2}\right)} & \end{cases}$$

where we also use the rule that $x\Gamma(x) = \Gamma(x+1)$.

Let us briefly sketch the proof of (*): our calculation shows that

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta;$$

multiplying this formula with $r^{2x+2y-1}\,e^{-r^2}$, integrating w.r.t. r over $(0,\infty)$ and changing variables according to $s = r^2$ yields on the one hand

$$\begin{aligned}\int_0^\infty B(x,y)\, r^{2x+2y-1}\, e^{-r^2}\, dr &= \frac{1}{2}\int_0^\infty B(x,y)\, s^{x+y-1}\, e^{-s}\, ds \\ &= \frac{1}{2}\, B(x,y)\, \Gamma(x+y)\end{aligned}$$

while, on the other hand, we get by switching from polar to cartesian coordinates,

$$\begin{aligned}&\int_0^\infty B(x,y)\, r^{2x+2y-1}\, e^{-r^2}\, dr \\ &\quad = 2\int_0^\infty \int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, r^{2x+2y-1}\, e^{-r^2}\, dr\, d\theta \\ &\quad = 2\int_0^\infty \int_0^{\pi/2} (r\sin\theta)^{2x-1}(r\cos\theta)^{2y-1}\, e^{-r^2}\, r\, dr\, d\theta \\ &\quad = 2\iint_{(0,\infty)\times(0,\infty)} \xi^{2x-1}\, \eta^{2y-1}\, e^{-\xi^2-\eta^2}\, d\xi\, d\eta \\ &\quad = 2\int_{(0,\infty)} \xi^{2x-1}\, e^{-\xi^2}\, d\xi \int_{(0,\infty)} \eta^{2y-1}\, e^{-\eta^2}\, d\eta \\ &\quad = \frac{1}{2}\int_{(0,\infty)} s^{x-1}\, e^{-s}\, ds \int_{(0,\infty)} t^{y-1}\, e^{-t}\, dt \\ &\quad = \frac{1}{2}\Gamma(x)\Gamma(y)\end{aligned}$$

with the obvious applications of Tonelli's theorem and, in the penultimate equality, the obvious substitutions.

■■

17 Dense and determining sets. Solutions to Problems 17.1–17.9

Problem 17.1 Solution: Let $f \in \mathcal{L}^p(\mu)$ and fix $\epsilon > 0$. It is enough to show that there is some $h \in \mathcal{C}$ such that $\|f - h\|_p \leqslant \epsilon$. Since $\mathcal{D}$ is dense in $\mathcal{L}^p(\mu)$, there exists some $g \in \mathcal{D}$ satisfying $\|f - g\|_p \leqslant \epsilon/2$. On the other hand, as $\mathcal{C}$ is dense in $\mathcal{D}$, there is some $h \in \mathcal{D}$ such that $\|g - h\|_p \leqslant \epsilon/2$. Now the triangle inequality gives

$$\|f - h\|_p \leqslant \|f - g\|_p + \|g - h\|_p \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Problem 17.2 Solution:

(i) Continuity follows from the continuity of the function $x \mapsto d(x, A)$, cf. (17.1). Clearly, $0 \leqslant u_k \leqslant 1$ and $u_k|_K = 1$ and $u|U_k^c = 0$. Since $U_K \downarrow K$, we get $u_k \downarrow \mathbb{1}_K$. Since $\overline{U}_k$ is closed and bounded, it is clear that $\overline{U}_k$ is compact, i.e. $\operatorname{supp} u_k$ is compact.

(ii) This follows from (i) and monotone convergence.

(iii) We have $\mu(K) = \nu(K)$ for all compact sets $K \subset \mathbb{R}^n$ and the compact sets generate the Borel σ-algebra. In particular, this holds for $[-k, k]^n \uparrow \mathbb{R}^n$, so that the conditions for the uniqueness theorem for measures (Theorem 5.7) are satisfied. We conclude that $\mu = \nu$.

(iv) Since each x has a compact neighbourhood, we can choose k so large that $\overline{B_{1/k}(x)}$ becomes compact. In particular, $K \subset \bigcup_{x \in K} B_{1/k(x)}(x)$ is an open cover. We can choose each $k(x)$ so large, that $B_{1/k(x)}(x)$ has a compact closure. Since K is compact, we find finitely many x_i such that $K \subset \bigcup_i B_{1/k(x_i)}(x_i) = U_k$ where $k := \max_i k_i$. In particular, $\overline{U}_k \subset \bigcup_i \overline{B_{1/(x_i)}(x_i)}$ is compact. This produces a sequence of $U_k \downarrow K$. The rest follows almost literally as in the previous steps.

Problem 17.3 Solution:

(i) We have to show that $\|\tau_h f\|_p^p = \|f\|_p$ for all $p \in \mathcal{L}^p(dx)$. This is an immediate consequence of the invariance of Lebesgue measure under translations:

$$\|\tau_h f\|_p^p = \int_{\mathbb{R}} |f(x-h)|^p \, dx = \int_{\mathbb{R}} |f(y)|^p \, dy = \|f\|_p^p.$$

(ii) We show the assertion first for $f \in C_c(\mathbb{R})$. If $f \in C_c(\mathbb{R})$, then $K := \operatorname{supp} f$ is compact. Pick $R > 0$ in such a way that $K + B_1(0) \subset \overline{B_R(0)}$. Since $\lim_{h\to 0} f(x-h) = f(x)$ and

$$|f(x-h) - f(x)| \leqslant 2\|f\|_\infty \mathbb{1}_{\overline{B_R(0)}}(x) \in \mathcal{L}^p(dx)$$

for any $h < 1$, we can use dominated convergence to get

$$\|\tau_h f - f\|_p^p = \int |f(x-h) - f(x)|^p \, dx \xrightarrow[h\to 0]{} 0.$$

Now take $f \in \mathcal{L}^p(dx)$. Since $C_c(\mathbb{R})$ is dense in $\mathcal{L}^p(dx)$, cf. Theorem 17.8, there is a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ such that $\|f_n - f\|_p \to 0$. From part (i) we get

$$\begin{aligned}\|\tau_h f - f\|_p &\leqslant \underbrace{\|\tau_h(f - f_n)\|_p}_{\leqslant \|f_n - f\|_p} + \|\tau_h f_n - f_n\|_p + \|f_n - f\|_p \\ &\xrightarrow[h\to 0]{} 2\|f_n - f\|_p \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

This finishes the proof of the first assertion. The second claim follows in a similar way. Consider first $f \in C_c(\mathbb{R})$ and $K := \operatorname{supp} f$. Since K is compact, there is some $R > 0$ with $(h + K) \cap K = \emptyset$[1] for all $h > R$. If $h > R$, then

$$|f(x-h) - f(x)|^p = |f(x-h)|^p \mathbb{1}_K(x+h) + |f(x)|^p \mathbb{1}_K(x)$$

and so

$$\begin{aligned}\|\tau_h f - f\|_p^p &= \int_{K+h} |f(x-h)|^p \, dx + \int_K |f(x)|^p \, dx \\ &= \int_K |f(y)|^p \, dy + \int_K |f(x)|^p \, dx \\ &= 2\|f\|_p^p.\end{aligned}$$

This proves the assertion for $f \in C_c(\mathbb{R})$, and the general case follows via density as in the first part of (ii).

Problem 17.4 Solution:

(i) Continuity is an immediate consequence of the dominated convergence theorem: assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to $x \in \mathbb{R}$. Since $\mathbb{1}_{[x_n - h, x_n + h]} \to \mathbb{1}_{[x-h,x+h]}$ a.e. and $f \in \mathcal{L}^1(dx)$, we see that $M_h f(x_n) \to M_h f(x)$ as $n \to \infty$.

Contractivity of M_h follows from

$$\begin{aligned}\int |M_h f(x)| \, dx &= \frac{1}{2h} \int \left| \int_{x-h}^{x+h} f(t) \, dt \right| dx \\ &\leqslant \frac{1}{2h} \int_{-h}^{h} \underbrace{\int |f(x+t)| \, dx}_{\int |f(y)|\, dy = \|f\|_1} \, dt \leqslant \|f\|_1\end{aligned}$$

(use Tonelli's theorem to interchange the order of integrations).

[1] We use the notation $h + K := \{h + x; x \in K\}$.

(ii) Assume first that $f \in C_c(\mathbb{R})$. Because of the continuity of the function f we find

$$|M_h f(x) - f(x)| \leqslant \frac{1}{2h} \int_{-h}^{h} |f(x+t) - f(x)| \, dx \leqslant \sup_{t \in [-h,h]} |f(x+t) - f(x)| \xrightarrow[h \to 0]{} 0$$

for all $x \in \mathbb{R}$. Since the support of f, $K := \operatorname{supp} f$, is compact, there is some $R > 0$ such that $K + B_1(0) \subseteq \overline{B_R(0)}$. For $h < 1$ we get $M_h f(x) = 0 = f(x)$ if $x \notin \overline{B_R(0)}$. Since $|M_h f(x)| \leqslant |f(x)|$ for $x \in \mathbb{R}$, we get

$$|M_h f(x) - f(x)| = |M_h f(x) - f(x)| \mathbb{1}_{\overline{B_R(0)}}(x) \leqslant 2\|f\|_\infty \mathbb{1}_{\overline{B_R(0)}}(x) \in \mathcal{L}^1(dx).$$

An application of the dominated convergence theorem reveals

$$\|M_h f - f\|_1 = \int |M_h f(x) - f(x)| \, dx \xrightarrow{h \to 0} 0,$$

i.e. the claim is true for any $f \in C_c(\mathbb{R})$. Now we take a general $f \in \mathcal{L}^1(dx)$. Because of Theorem 17.8 there is a sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(\mathbb{R})$ such that $\|f_n - f\|_1 \to 0$. Therefore,

$$\begin{aligned} \|M_h f - f\|_1 &\leqslant \underbrace{\|M_h(f - f_n)\|_1}_{=\|f_n - f\|_1} + \|M_h f_n - f_n\|_1 + \|f_n - f\|_1 \\ &\xrightarrow[h \to 0]{} 2\|f_n - f\|_1 \xrightarrow[n \to \infty]{} 0. \end{aligned}$$

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Problem 17.5 Solution:

(i) Let $A \in \mathscr{B}(X)$ such that $f := \mathbb{1}_A \in \mathcal{L}^p(\mu)$. Clearly, $\mu(A) < \infty$ and because of the outer regularity of μ there is an open set $U \subset X$ such that $A \subset U$ and $\mu(U) < \infty$. Literally as in the proof of Lemma 17.3 we can construct some $\phi_\epsilon \in C_{\mathrm{Lip}}(X) \cap \mathcal{L}^p(\mu)$ with $\|f - \phi_\epsilon\|_p \leqslant \epsilon$ (just replace in the proof $C_b(X)$ with $C_{\mathrm{Lip}}(X)$).

(ii) If $f \in \mathcal{L}^p(\mu)$, then the Sombrero lemma shows that there is a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ satisfying $0 \leqslant f_n \leqslant f$, $f_n \uparrow f$. Using the monotone convergence theorem, we see $\int (f - f_n)^p \, d\mu \downarrow 0$; in particular, there is some $n \in \mathbb{N}$ such that $\|f_n - f\|_p \leqslant \epsilon$. Using linearity and the result of part (i), we get some $\phi_\epsilon \in C_{\mathrm{Lip}}(X)$ such that $\|f_n - \phi_\epsilon\|_p \leqslant \epsilon$. Therefore,

$$\|f - \phi_\epsilon\|_p \leqslant \|f - f_n\|_p + \|f_n - \phi_\epsilon\|_p \leqslant 2\epsilon.$$

(iii) We use the decomposition $f = f^+ - f^-$. Since $f^+, f^- \in \mathcal{L}^p(\mu)$, part (ii) furnishes functions $\phi, \psi \in C_{\mathrm{Lip}}(X) \cap \mathcal{L}^p(\mu)$ such that $\|f^+ - \phi\|_p \leqslant \epsilon$ and $\|f^- - \psi\|_p \leqslant \epsilon$. Consequently,

$$\|f - (\phi - \psi)\|_p \leqslant \|f^+ - \phi\|_p + \|f^- - \psi\|_p \leqslant 2\epsilon.$$

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Problem 17.6 Solution: A set $U \subset X$ is said to be *relatively compact* if it closure $\overline{U}$ is compact.

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a countable dense subset of X. By assumption, each x_n has a relatively compact open neighbourhood: $x_n \in V_n$ and $\overline{V}_n$ is compact. Since $B_{1/k}(x_n) \subset V_n$ for sufficiently large values of $k \geqslant k_0(x_n)$, we see that the balls $B_{1/k}(x_n)$, $k \geqslant k_0(x_n)$, are also relatively compact. Thus,

$$\{B_{1/k}(x_n) \,:\, n \in \mathbb{N}, k \geqslant k_0(x_n)\} =: \{U_n; n \in \mathbb{N}\}$$

is a sequence of relatively compact, open sets. For any open set $U \subset X$ we find

$$U = \bigcup_{\substack{n\in\mathbb{N}\\ U_n\subset U}} U_n.$$

(The inclusion '$\supset$' is obvious. In order to see '$\subset$' we observe that for any $x \in U$ there is some $r > 0$ with $B_r(x) \subset U$. Since $(x_n)_{n\in\mathbb{N}}$ is dense, we may choose $n \in \mathbb{N}$ and $k \geqslant k_0(x_n)$ such that $B_{1/k}(x_n) \subset B_r(x) \subset U$.)

(ii) The sets $K_n := \overline{U}_1 \cup \cdots \cup \overline{U}_n$ are compact and increase towards X.

(iii) Assume that $U \subset X$ is an open set such that $\mu(U) < \infty$ and let $(U_n)_{n\in\mathbb{N}}$ be the sequence from part (i). Because of (i), there is a subsequence $(U_{n(k)})_{k\in\mathbb{N}} \subset (U_n)_{n\in\mathbb{N}}$ such that $U = \bigcup_k U_{n(k)}$. Set $W_n := \bigcup_{k=1}^n U_{n(k)}$ and observe that $W_n \in \mathcal{D}$. Since $W_n \uparrow U$, Beppo Levi's theorem shows that

$$\|\mathbb{1}_{W_n} - \mathbb{1}_U\|_p \xrightarrow[n\to\infty]{} 0.$$

This tells us that $\mathbb{1}_U \in \overline{\mathcal{D}}$.

(iv) First we show that μ is outer regular. Set

$$G_n := \bigcup_{k=1}^n U_k.$$

Obviously, the G_n are open sets, $G_n \uparrow X$ and $\mu(G_n) < \infty$ – here we use that the U_k are relatively compact and that μ is finite on compact sets. This means that the assumptions of Theorem H.3 are satisfied, and we see that μ is outer regular.

Let $B \in \mathscr{B}(X)$, $\mu(B) < \infty$ and fix $\epsilon > 0$. Since μ is outer regular, there is a sequence of open sets $(U_n)_{n\in\mathbb{N}}$ such that $U_n \supset B$ and $\mu(U_n) < \infty$. By monotone convergence, $\|\mathbb{1}_{U_n} - \mathbb{1}_B\|_p \to 0$ as $n \to \infty$. Pick $n \in \mathbb{N}$ such that $\|\mathbb{1}_{U_n} - \mathbb{1}_B\|_p \leqslant \epsilon$. Because of (iii), there is some $D \in \mathcal{D}$ with $\|\mathbb{1}_{U_n} - \mathbb{1}_D\|_p \leqslant \epsilon$. Consequently,

$$\|\mathbb{1}_B - \mathbb{1}_D\|_p \leqslant \|I_B - \mathbb{1}_{U_n}\|_p + \|\mathbb{1}_{U_n} - \mathbb{1}_D\|_p \leqslant 2\epsilon.$$

(v) By definition, $\overline{\mathcal{D}} \subset \mathcal{L}^p(\mu)$, i.e. it is enough to show that for every $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$ there is some $D \in \mathcal{D}$ such that $\|f - \mathbb{1}_D\|_p \leqslant \epsilon$. Using the Sombrero lemma (Corollary 8.9) and the dominated convergence theorem we can construct a sequence of simple functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$ such that $\|f - f_n\|_p \to 0$. If n is sufficiently large, we have $\|f - f_n\|_p \leqslant \epsilon$. Since f_n is of the form

$$f_n(x) = \sum_{j=1}^{N} c_j \mathbb{1}_{B_j}(x)$$

where $c_j \in \mathbb{R}$, $B_j \in \mathscr{B}(X)$, $j = 1, \dots, N$, we can use part (iv) to get $D \in \mathcal{D}$ with $\|f_n - \mathbb{1}_D\|_p \leqslant \epsilon$. With the triangle inequality we see that $\|f - \mathbb{1}_D\|_p \leqslant 2\epsilon$. The separability of $\mathcal{L}^p(\mu)$ now follows from the fact that $\mathcal{D}$ is a countable set.

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Problem 17.7 Solution:

(i) Assume first that A is an open set. Without loss of generality $A \neq \emptyset$. Fix $\epsilon > 0$. Since

$$\left\{x \in A : d(x, A^c) < \tfrac{1}{n}\right\} \downarrow \emptyset \quad \text{as } n \to \infty$$

the continuity of measures furnishes some $N \in \mathbb{N}$ such that

$$\mu\left\{d(\cdot, A^c) < \tfrac{1}{n}\right\} < \epsilon \quad \forall n \geqslant N.$$

Define $\phi_n(x) := \min\{nd(x, A^c), 1\}$. Clearly, $\phi_n \in C_b(X)$ and $\|\phi_\epsilon\|_\infty \leqslant 1 = \|\mathbb{1}_A\|_\infty$. Since $0 \leqslant \phi_n \leqslant \mathbb{1}_A \in \mathcal{L}^p$ we even have $\phi_n \in \mathcal{L}^p(\mu)$. Moreover,

$$\{\mathbb{1}_A \neq \phi_n\} \subset \left\{d(\cdot, A^c) < \tfrac{1}{n}\right\};$$

therefore, $\mu\{\mathbb{1}_A \neq \phi_n\} \leqslant \epsilon$ for all $n \geqslant N$. Using dominated convergence gives $\|\mathbb{1}_A - \phi_n\|_p \xrightarrow[n\to\infty]{} 0$. If $n \geqslant N$ is large enough, we get $\|\mathbb{1}_A - \phi_n\|_p \leqslant \epsilon$. For such n, the functions ϕ_n satisfy all requirements of the theorem.

In order to show the claim for any Borel set $A \in \mathscr{B}(X)$, we proceed as in the proof of Lemma 17.3: let $U \subset X$, $\mu(U) < \infty$, and define

$$\mathscr{D} := \{A \in \mathscr{B}(U) : \forall \epsilon > 0 \quad \exists \phi_\epsilon \in C_b(X) \cap \mathcal{L}^p(\mu) \text{ satisfying the assertion for } f = \mathbb{1}_A\}.$$

As in the proof of Lemma 17.3 we see that $\mathscr{D}$ is a Dynkin system. By construction, the open sets are contained in $\mathscr{D}$, and so $\mathscr{B}(U) \subset \mathscr{D}$.

If $A \in \mathscr{B}(X)$ is an arbitrary Borel set with $\mathbb{1}_A \in \mathcal{L}^p(\mu)$, we have $\mu(A) < \infty$. Since μ is outer regular, there exists an open set $U \subset X$ such that $A \subset U$ and $\mu(U) < \infty$. Since $A \in \mathscr{B}(U) \subset \mathscr{D}$, the claim follows.

(ii) Let $f \in \mathcal{L}^p(\mu)$, $0 \leqslant f \leqslant 1$, and fix $\epsilon > 0$. Without loss of generality we may assume that $\|f\|_\infty = 1$, otherwise we would use $f/\|f\|_\infty$. The (proof of the) Sombrero lemma (Theorem 8.8) shows that

$$f_n := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\left\{\frac{k}{2^n} \leqslant f < \frac{k+1}{2^n}\right\}} + n\mathbb{1}_{\{f>n\}} \overset{0 \leqslant f \leqslant 1}{=} \sum_{k=0}^{2^n-1} \frac{k}{2^n} \mathbb{1}_{\left\{\frac{k}{2^n} \leqslant f < \frac{k+1}{2^n}\right\}}, \quad n \in \mathbb{N},$$

monotonically converges to f. With $f_0 := 0$ we get

$$f = \lim_{n\to\infty}(f_n - f_0) = \lim_{n\to\infty}\sum_{j=1}^{n}(f_j - f_{j-1}) = \sum_{j\geqslant 1}(f_j - f_{j-1}) = \sum_{j\geqslant 1}\frac{1}{2^j}\phi_j$$

for $\phi_j := 2^j(f_j - f_{j-1})$. We claim that

$$\phi_j(x) \in \{0,1\} \quad \forall x \in \left\{f_{j-1} = \frac{k}{2^{j-1}}\right\}.. \tag{$\star\star$}$$

Indeed: By definition, f_j attains on $\left\{f_{j-1} = \frac{k}{2^{j-1}}\right\} = \left\{\frac{k}{2^{j-1}} \leqslant f < \frac{k+1}{2^{j-1}}\right\}$ only the values $\frac{2k}{2^j}$ and $\frac{2k+1}{2^j}$. In the first case, we have $\phi_j = 0$, in the latter $\phi_j = 1$. Thus, $\phi_j(x) = 1$ happens if, and only if,

$$x \in \left\{f_j = \frac{2k+1}{2^j}\right\} = \left\{\frac{2k+1}{2^j} \leqslant f < \frac{2k+2}{2^j}\right\}.$$

Therefore, we can write $A_j := \{\phi_j = 1\}$ in the following form

$$A_j = \bigcup_{k=0}^{2^{n-1}-1}\left\{\frac{2k+1}{2^j} \leqslant f < \frac{2k+2}{2^j}\right\}.$$

Since $\phi_j = \mathbb{1}_{A_j}$, we get

$$f = \sum_{j\geqslant 1}\frac{1}{2^j}\mathbb{1}_{A_j}.$$

Observe that $\mathbb{1}_{A_j} \leqslant 2^j f \in \mathcal{L}^p(\mu)$. Because of part (i), there is for every $j \geqslant 1$ a function $\phi_{j,\epsilon} \in C_b(X) \cap \mathcal{L}^p(\mu)$ such that

$$\|\phi_{j,\epsilon} - \phi_j\|_p \leqslant \frac{\epsilon}{2^j},\ \mu\{\phi_{j,\epsilon} \neq \phi_j\} \leqslant \frac{\epsilon}{2^j} \quad \text{and} \quad \|\phi_{j,\epsilon}\|_\infty \leqslant \|\phi_j\|_\infty \leqslant 1.$$

The function $\phi_\epsilon := \sum_{j\geqslant 1}\frac{\phi_{j,\epsilon}}{2^j}$ enjoys all required properties:

- ϕ_ϵ is continuous (since it is the uniform limit of continuous functions):

$$\left\|\phi_\epsilon - \sum_{j=1}^{n}\frac{\phi_{j,\epsilon}}{2^j}\right\|_\infty \leqslant \sum_{j=n+1}^{\infty}\frac{1}{2^j}\|\phi_{j,\epsilon}\|_\infty \leqslant \sum_{j=n+1}^{\infty}\frac{1}{2^j}\xrightarrow[n\to\infty]{} 0.$$

- $\|\phi_\epsilon\|_\infty \leqslant \sum_{j\geqslant 1}\frac{\|\phi_{j,\epsilon}\|_\infty}{2^j} \leqslant \sum_{j\geqslant 1}\frac{1}{2^j} = 1 = \|f\|_\infty$.
- $\|\phi_\epsilon - f\|_p \leqslant \sum_{j\geqslant 1}\frac{1}{2^j}\|\phi_{j,\epsilon} - \phi_j\|_p \leqslant \epsilon\sum_{j\geqslant 1}\frac{1}{2^j} \leqslant \epsilon$. In particular, $\phi_\epsilon \in \mathcal{L}^p(\mu)$.
- $\mu\{\phi_\epsilon \neq f\} \leqslant \sum_{j\geqslant 1}\mu\{\phi_{j,\epsilon} \neq \phi_j\} \leqslant \sum_{j\geqslant 1}\epsilon 2^{-j} = \epsilon$.

(iii) Observe, first of all, that the theorem holds for all $g \in \mathcal{L}^p(\mu)$ with $0 \leqslant g \leqslant \|g\|_\infty < \infty$; for this, apply part (ii) to $g/\|g\|_\infty$. Without loss of generality we may assume for such g that $\phi_\epsilon \geqslant 0$; otherwise we would consider $\widetilde{\phi}_\epsilon := \phi_\epsilon \vee 0$.

Let $f \in \mathcal{L}^p(\mu)$ and $\|f\|_\infty < \infty$. We write $f = f^+ - f^-$ and, because of the preceding remark, there are functions $\phi_\epsilon, \psi_\epsilon \in C_b(X) \cap \mathcal{L}^p(\mu)$, $\phi_\epsilon \geqslant 0$, $\psi_\epsilon \geqslant 0$, such that

$$\|\phi_\epsilon\|_\infty \leqslant \|f^+\|_\infty, \quad \mu\left\{f^+ \neq \phi_\epsilon\right\} \leqslant \epsilon \quad \text{and} \quad \|f^+ - \phi_\epsilon\|_p \leqslant \epsilon$$

and

$$\|\psi_\epsilon\|_\infty \leqslant \|f^-\|_\infty, \quad \mu\left\{f^- \neq \psi_\epsilon\right\} \leqslant \epsilon \quad \text{and} \quad \|f^- - \psi_\epsilon\|_p \leqslant \epsilon.$$

For $\Phi_\epsilon := \phi_\epsilon - \psi_\epsilon \in C_b(X) \cap \mathcal{L}^p(\mu)$ we find

$$\mu\{\Phi_\epsilon \neq f\} \leqslant \mu\{\phi_\epsilon \neq f^+\} + \mu\{\psi_\epsilon \neq f^-\} \leqslant 2\epsilon$$

as well as

$$\|\Phi_\epsilon\|_\infty \leqslant \max\{\|f^+\|_\infty, \|f^-\|_\infty\} = \|f\|_\infty$$

(this step requires that $\phi_\epsilon \geqslant 0$ and $\psi_\epsilon \geqslant 0$). The triangle inequality yields

$$\|f - \Phi_\epsilon\|_p \leqslant \|f^+ - \phi_\epsilon\|_p + \|f^- - \psi_\epsilon\|_p \leqslant 2\epsilon.$$

Consequently, Φ_ϵ satisfies the conditions of the theorem for f.

(iv) Fix $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$. Using the Markov inequality we get

$$\mu\{|f| \geqslant R\} \leqslant \frac{1}{R^p} \int |f|^p \, d\mu.$$

In particular, we can pick a sufficiently large $R > 0$ such that $\mu\{|f| \geqslant R\} \leqslant \epsilon$. Using monotone convergence, we see

$$\int_{\{|f|>R\}} |f|^p \, d\mu < \epsilon$$

if $R > 0$ is large. Setting $f_R := (-R) \vee f \wedge R$, we can use (iii) to construct a function $\phi_\epsilon \in C_b(X) \cap \mathcal{L}^p(\mu)$ with

$$\|\phi_\epsilon\|_\infty \leqslant \|f_R\|_\infty, \quad \mu\left\{f_R \neq \phi_\epsilon\right\} \leqslant \frac{\epsilon}{R^p} \quad \text{and} \quad \|f_R - \phi_\epsilon\|_p \leqslant \epsilon.$$

Obviously, $\|\phi_\epsilon\|_\infty \leqslant \|f\|_\infty$. Moreover,

$$\begin{aligned}
&\|\phi_\epsilon - f\|_p^p \\
&\quad = \int_{\{|f| \leqslant R\}} |\phi_\epsilon - f|^p \, d\mu + \underbrace{\int_{\substack{\{|f|>R\} \\ \cap\{\phi_\epsilon = f_R\}}} |\phi_\epsilon - f|^p \, d\mu}_{=: I_1} + \underbrace{\int_{\substack{\{|f|>R\} \\ \cap\{\phi_\epsilon \neq f_R\}}} |\phi_\epsilon - f|^p \, d\mu}_{=: I_2} \\
&\quad \leqslant \|\phi_\epsilon - f_R\|_p^p + I_1 + I_2.
\end{aligned}$$

Let us estimate I_1 and I_2 separately. Since $f_R|_{\{|f|>R\}} = R$, we get

$$\begin{aligned}
I_1 &= \int_{\{f>R\}\cap\{\phi_\epsilon = f_R\}} (f-R)^p \, d\mu + \int_{\{f<-R\}\cap\{\phi_\epsilon = f_R\}} (-R-f)^p \, d\mu \\
&\leqslant \int_{\{f>R\}\cap\{\phi_\epsilon = f_R\}} \underbrace{f^p}_{|f|^p} \, d\mu + \int_{\{f<-R\}\cap\{\phi_\epsilon = f_R\}} \underbrace{(-f)^p}_{|f|^p} \, d\mu \\
&\leqslant \int_{\{|f|>R\}} |f|^p \, d\mu < \epsilon.
\end{aligned}$$

With the elementary estimate

$$|a+b|^p \leqslant C(p)(a^p+b^p) \quad \forall a,b \geqslant 0,\ p \geqslant 1 \tag{$\sharp$}$$

(in fact, $C(p)=2^{p-1}$) we get

$$\begin{aligned}
I_2 &\leqslant C(p)\int_{\{|f|>R\}\cap\{\phi_\epsilon\neq f_R\}} |\phi_\epsilon|^p\,d\mu + C(p)\int_{\{|f|>R\}\cap\{\phi_\epsilon\neq f_R\}} |f|^p\,d\mu\\
&\leqslant C(p)\|\phi_\epsilon\|_\infty^p\,\mu\{\phi_\epsilon\neq f_R\} + C(p)\int_{\{|f|>R\}} |f|^p\,d\mu\\
&\leqslant C(p)R^p\frac{\epsilon}{R^p} + C(p)\epsilon.
\end{aligned}$$

Therefore,

$$\|\phi_\epsilon - f\|_p^p \leqslant \epsilon^p + \epsilon + 2C(p)\epsilon.$$

Since $\epsilon>0$ is arbitrary, $\|\phi_\epsilon - f\|_p$ is as small as we want it to be. Finally,

$$\mu\{f\neq\phi_\epsilon\} \leqslant \mu\{f_R\neq\phi_\epsilon\} + \mu\{|f|\geqslant R\} \leqslant 2\epsilon.$$

This shows that ϕ_ϵ enjoys all required properties.

Remark: ($\sharp$) follows from Hölder's inequality

$$\left|\sum_{j=1}^n x_j\cdot y_j\right| \leqslant \left(\sum_{j=1}^n |x_j|^p\right)^{\frac1p}\cdot\left(\sum_{j=1}^n |y_j|^q\right)^{\frac1q}$$

for $x,y\in\mathbb{R}^n$ and conjugate indices $p,q\geqslant 1$. If we take, in particular, $d=2$, $x=(a,b)$, $y=(1,1)$, then

$$|a\cdot 1 + b\cdot 1| \leqslant (|a|^p+|b|^p)^{\frac1p}\cdot 2^{\frac1q}.$$

Raising both sides to the pth power proves the estimate.

Problem 17.8 Solution: We see immediately that $\int_a^b p(x)f(x)\,dx = 0$ for all polynomials p. Fix $g\in C[a,b]$ and $\epsilon>0$. By Weierstraß' theorem, there is some polynomial p such that $\|g-p\|_\infty\leqslant\epsilon$. Therefore,

$$\begin{aligned}
\left|\int_a^b g(x)f(x)\,dx\right| &= \left|\int_a^b (g(x)-p(x))f(x)\,dx + \underbrace{\int_a^b p(x)f(x)\,dx}_{=0}\right|\\
&\leqslant \int_a^b \underbrace{|p(x)-g(x)|}_{\leqslant\epsilon}\,|f(x)|\,dx\\
&\leqslant \epsilon\int_a^b |f(x)|\,dx.
\end{aligned}$$

From this we conclude that

$$\int_a^b g(x)f(x)\,dx = 0 \quad \forall g\in C[a,b].$$

Define measures $\mu^{\pm}$ by $\mu^{\pm}(dx) := \mathbb{1}_{[a,b]}(x)\mathbb{1}_{\{\pm f>0\}}(x)\,dx$. Then $\int g\,d\mu^+ = \int g\,d\mu^-$ for all $g \in C[a,b]$. According to Theorem 17.12, $C[a,b]$ is a determining set, and so $\mu^+ = \mu^-$. This is only possible if $\mu = 0$, hence $f = 0$ Lebesgue a.e.

Problem 17.9 Solution:

(i) First of all, we note that it is enough to know that the polynomials are uniformly dense in the set $C[-1,1]$. This follows immediately from the observation that any function in $C[0,1]$ can be mapped onto $C[a,b]$ using the affine transform $a + t(b-a)$, $t \in [0,1]$ – and vice versa. Fix $u \in C[-1,1]$ and define a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ by

$$p_n(x) := \frac{1}{c_n}\left(\frac{x^2}{16} - 1\right)^n, \qquad x \in \mathbb{R},$$

where $c_n := \int_{-4}^{4}(x^2/16-1)^n\,dx$. Since $u \in C[-1,1]$, there is some $\widetilde{u} \in C(\mathbb{R})$ such that $\widetilde{u}(x) = 0$ for $|x| > 2$ and $\widetilde{u}(x) = u(x)$ for $x \in [-1,1]$. Define $\widetilde{p}_n(x) := p_n(x)\mathbb{1}_{[-4,4]}(x)$ and

$$u_n(x) := \widetilde{u} \star \widetilde{p}_n(x) = \int \widetilde{u}(x-y)\widetilde{p}_n(y)\,dy, \qquad x \in \mathbb{R}.$$

We find

$$u_n(x) = \int \widetilde{u}(x-y)p_n(y)\,dy \quad \forall x \in [-2,2],$$

since

$$|x| \leqslant 2 \implies \widetilde{u}(x-y) = 0 \quad \forall |y| > 2.$$

Using the fact that

$$u_n(x) = \int \widetilde{u}(y)p_n(x-y)\,dy, \quad x \in [-2,2]$$

we see that $u_n|_{[-2,2]}$ is a polynomial. Let us show that $u_n \to \widetilde{u}$ converges uniformly – and since $\widetilde{u}|_{[-1,1]} = u$, the claim follows. Using that $\widetilde{p}_n \geqslant 0$ and $\int \widetilde{p}_n\,dx = 1$ we get

$$\begin{aligned}
\left|u_n(x) - \widetilde{u}(x)\right| &= \left|\int (\widetilde{u}(x-y) - \widetilde{u}(x))\widetilde{p}_n(y)\,dy\right| \\
&\leqslant \int_{\left[-\frac{1}{R},\frac{1}{R}\right]} |\widetilde{u}(x-y) - \widetilde{u}(x)|\widetilde{p}_n(y)\,dy \\
&\quad + \int_{\mathbb{R}\setminus\left[-\frac{1}{R},\frac{1}{R}\right]} |\widetilde{u}(x-y) - \widetilde{u}(x)|\widetilde{p}_n(y)\,dy \\
&=: I_1(x) + I_2(x)
\end{aligned}$$

for all $R > 0$. Let us bound I_1 and I_2 separately. Since $\widetilde{u}(x) = 0$ for $|x| > 2$, the function $\widetilde{u}$ is uniformly continuous and we get

$$I_1(x) \leqslant \sup_{y\in\left[-\frac{1}{R},\frac{1}{R}\right]} |\widetilde{u}(x-y) - \widetilde{u}(x)| \int_{\left[-\frac{1}{R},\frac{1}{R}\right]} \widetilde{p}_n(y)\,dy$$

$$\leqslant \sup_{y\in\left[-\frac{1}{R},\frac{1}{R}\right]} |\widetilde{u}(x-y)-\widetilde{u}(x)|$$

$$\xrightarrow[R\to\infty]{} 0$$

uniformly for all x. Because of the boundedness of $\widetilde{u}$ we see that

$$I_2(x) \leqslant 2\|\widetilde{u}\|_\infty \int_{\mathbb{R}\setminus\left[-\frac{1}{R},\frac{1}{R}\right]} \widetilde{p}_n(y)\,dy.$$

Since $\widetilde{p}_n(y) \downarrow 0$ for all $y \neq 0$, we can use the monotone convergence theorem to conclude that $I_2 \xrightarrow[n\to\infty]{} 0$ uniformly in x. This proves the claim.

(ii) Fix $u \in C_c[0,\infty)$. Since u has compact support, $u(x) = 0$ for large x; in particular, $u\circ(-\log)(x) = 0$ if x is small. Therefore,

$$\begin{cases} u\circ(-\log)(x), & x \in (0,1] \\ 0, & x = 0, \end{cases}$$

defines a continuous function on $[0,1]$. According to (i), there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ with $p_n \to u\circ(-\log)$ uniformly.

(iii) For $p(x) := x^n$ we obviously have $p(e^{-t}) = e^{-nt} = \epsilon_n(t)$ and, by assumption,

$$\int p(e^{-t})\,\mu(dt) = \int \epsilon_n(t)\,\mu(dt) = \int \epsilon_n(t)\,\nu(dt) = \int p(e^{-t})\,\nu(dt). \qquad (\star)$$

Using the linearity of the integral, this equality extends to arbitrary polynomials p. Assume that $u \in C_c[0,\infty)$ and $(p_n)_{n\in\mathbb{N}}$ as in (ii). Since p_n converges uniformly to $u\circ(-\log)$, we can interchange integration and limit to get

$$\begin{aligned}
\int u\,d\mu &= \int (u\circ(-\log))(e^{-t})\,\mu(dt) \\
&= \lim_{n\to\infty} \int p_n(e^{-t})\,\mu(dt) \\
&\overset{(\star)}{=} \lim_{n\to\infty} \int p_n(e^{-t})\,\nu(dt) \\
&= \int (u\circ(-\log))(e^{-t})\,\nu(dt) \\
&= \int u\,d\nu.
\end{aligned}$$

18 Hausdorff measure. Solutions to Problems 18.1–18.7

Problem 18.1 Solution: This is clear from the monotonicity of the infimum and the fact that there are more $\mathscr{P}$-δ-covers than $\mathscr{C}$-δ-covers, i.e. we have

$$\overline{\mathcal{H}}^{\phi}_{\delta,\mathscr{P}}(A) \leqslant \overline{\mathcal{H}}^{\phi}_{\delta,\mathscr{C}}(A).$$

■■

Problem 18.2 Solution: From the proof of Corollary 18.10 we know, using the monotonicity of measures

$$\begin{aligned}\overline{\mathcal{H}}^{\phi}(A) = \mathcal{H}^{\phi}(G) &= \lim_{k\to\infty} \mathcal{H}^{\phi}(U^k) \\ &\overset{U^k \supset A}{\geqslant} \inf\left\{\mathcal{H}^{\phi}(U) : U \supset A,\ U \text{ open}\right\} \overset{U \supset A}{\geqslant} \overline{\mathcal{H}}^{\phi}(A).\end{aligned}$$

When using the monotonicity we must make sure that $\mathcal{H}^{\phi}(U^k) < \infty$ – this we can enforce by $U^k \rightsquigarrow U^k \cap U$ (where U is the open set with finite Hausdorff measure).

For counting measure this is clearly violated: Any open set $U \supset A := \{a\}$ has infinitely many points! Nevertheless A is itself a G_δ-set.

■■

Problem 18.3 Solution: By Corollary 18.10 there are open sets U_i such that $H := \bigcap_i U_i \supset B$ and $\mathcal{H}^{\phi}(H \setminus B) = 0$ or $\mathcal{H}^{\phi}(H) = \mathcal{H}^{\phi}(B)$. Now we can write each U_i as an F_σ-set:

$$U_i = \bigcup_{B_r(x)\subset U_i, x\in U_i} \overline{B_{r/2}(x)}$$

is indeed a countable union of closed sets, since $U_i \subset X$ contains a countable dense subset. So we have

$$U_i = \bigcup_k F_{ik} \quad \text{for closed sets } F_{ik}.$$

Without loss of generality we may assume that the sets F_{ik} increase in k, otherwise we would consider $F_{i1} \cup \dots \cup F_{ik}$. By the continuity of measure (here we require the measurability of B!) we have

$$\lim_{k\to\infty} \mathcal{H}^{\phi}(B \cap F_{ik}) = \mathcal{H}^{\phi}(B \cap U_i) = \mathcal{H}^{\phi}(B).$$

In particular, for every $\epsilon > 0$ there is some $k(i)$ with

$$\mathcal{H}^\phi(B \setminus F_{ik(i)}) \leqslant \epsilon/2^i, \quad i \in \mathbb{N}.$$

Consider the closed set $F = \bigcap_i F_{ik(i)}$ and observe that

$$\mathcal{H}^\phi(F) \geqslant \mathcal{H}^\phi(F \cap B) \geqslant \mathcal{H}^\phi(B) - \sum_i \mathcal{H}^\phi(B \setminus F_{ik(i)}) \geqslant \mathcal{H}^\phi(B) - \sum_i \frac{\epsilon}{2^i} = \mathcal{H}^\phi(B) - \epsilon.$$

Since $F \subset \bigcap_i U_i$, we get

$$\mathcal{H}^\phi(F \setminus B) \leqslant \mathcal{H}^\phi\left(\bigcap_i U_i \setminus B\right) = \mathcal{H}^\phi(H \setminus B) = 0.$$

By Corollary 18.10, the set $F \setminus B$ is contained in a G_δ-set $G = \bigcap_i V_i$ (where the V_i are open sets) such that $\mathcal{H}^\phi(G) = 0 = \mathcal{H}^\phi(F \setminus B)$. Thus,

$$F \setminus G = F \cap \bigcup_i V_i^c = \bigcup_i \underbrace{F \cap V_i^c}_{\text{closed}}$$

is an F_σ-set inside B – we have $F \setminus G \subset F \setminus (F \setminus B) \subset B$ – and

$$\mathcal{H}^\phi(F \setminus G) \geqslant \mathcal{H}^\phi(F) - \mathcal{H}^\phi(G) \geqslant \mathcal{H}^\phi(B) - \epsilon.$$

Now consider $\epsilon = \frac{1}{n}$ and take unions of the thus obtained F_σ-sets. But, clearly, countable unions of F_σ-sets are still F_σ.

■■

Problem 18.4 Solution: Fix $A \subset \mathbb{R}^n$. We have to show that for any $Q \subset \mathbb{R}^n$ the equality

$$\#Q = \#(Q \cap A) + \#(Q \setminus A)$$

holds. We distinguish between two cases.

Case 1: $\#Q = \infty$. Then at least one of the terms $\#(Q \cap A)$, $\#(Q \setminus A)$ on the right-hand side must be infinite, so the equality is clear.

Case 2: $\#Q < \infty$. Then both sets $(Q \cap A)$, $(Q \setminus A)$ are finite and, as such, they are metrically separated. Therefore we can use the fact that $\overline{\mathcal{H}}^0(A) = \#(A)$ is a metric outer measure (Theorem 18.5) to get equality.

■■

Problem 18.5 Solution: Use Lemma 18.17 to see $0 \leqslant \dim_{\mathcal{H}} B \leqslant \dim_{\mathcal{H}} \mathbb{R}^n$ as $B \subset \mathbb{R}^n$. From Example 18.18 we know that $\dim_{\mathcal{H}} \mathbb{R}^n = n$.

If B contains an open set U (or a set of non-zero Lebesgue measure), we see $\mathcal{H}^n(B) \geqslant \mathcal{H}^n(U) > 0$; intersect with a large open ball K to make sure that $\mathcal{H}^n(B \cap K) < \infty$ and $U \cap K \subset B \cap K$. This shows $n = \dim_{\mathcal{H}}(B \cap K) \leqslant \dim_{\mathcal{H}}(B) \leqslant n$.

■■

Problem 18.6 Solution: By self-similarity, we see for the Sierpinski triangle of generation i, S^{i-1} and its follow-up stage $S^i = S^i_1 \cup S^i_2 \cup S^i_3$ that the S^i_k's are scaled versions of S with a factor $\frac{1}{2}$. So,

$$\mathcal{H}^s(S^{i-1}) = \mathcal{H}^s(S^i_1) + \mathcal{H}^s(S^i_2) + \mathcal{H}^s(S^i_3) = 3 \cdot 2^{-s}\mathcal{H}^s(S^{i-1})$$

and dividing by $\mathcal{H}^s(S^{i-1})$ and solving the equality $1 = 3 \cdot 2^{-s} \iff 2^s = 3 \iff s = \log 3/\log 2$

Koch's snowflake S has in each subsequent generation stage 4 new parts, each scaled by $1/3$, so

$$\mathcal{H}^s(S) = \mathcal{H}^s(S_1) + \mathcal{H}^s(S_2) + \mathcal{H}^s(S_3) + \mathcal{H}^s(S_4) = 4 \cdot 3^{-s}\mathcal{H}^s(S)$$

and dividing by $\mathcal{H}^s(S)$ and solving the equality $1 = 4 \cdot 3^{-s} \iff 3^s = 4 \iff s = \log 4/\log 3$.

■■

Problem 18.7 Solution: Let $(S_i)_{i\in\mathbb{N}}$ be an ϵ-cover of A. Then we have

$$\begin{aligned}\sum_{i=1}^{\infty} \phi(\operatorname{diam} U_i) &= \sum_{i=1}^{\infty} \frac{\phi(\operatorname{diam} U_i)}{\psi(\operatorname{diam} U_i)}\psi(\operatorname{diam} U_i)\\ &\leqslant \sum_{i=1}^{\infty} \sup_{x\leqslant\epsilon} \frac{\phi(x)}{\psi(x)}\psi(\operatorname{diam} U_i)\\ &= \sup_{x\leqslant\epsilon} \frac{\phi(x)}{\psi(x)} \sum_{i=1}^{\infty} \psi(\operatorname{diam} U_i).\end{aligned}$$

Taking the inf over all admissible ϵ-covers shows

$$\overline{\mathcal{H}}^{\phi}_{\epsilon}(A) \leqslant \sup_{x\leqslant\epsilon} \frac{\phi(x)}{\psi(x)}\overline{\mathcal{H}}^{\psi}_{\epsilon}(A) \leqslant \sup_{x\leqslant\epsilon} \frac{\phi(x)}{\psi(x)}\overline{\mathcal{H}}^{\psi}(A).$$

Letting $\epsilon \to 0$ yields

$$\overline{\mathcal{H}}^{\phi}(A) = \lim_{\epsilon\to 0}\overline{\mathcal{H}}^{\phi}_{\epsilon}(A) \leqslant \lim_{\epsilon\to 0}\sup_{x\leqslant\epsilon} \frac{\phi(x)}{\psi(x)}\overline{\mathcal{H}}^{\psi}(A) = \limsup_{x\to 0} \frac{\phi(x)}{\psi(x)}\overline{\mathcal{H}}^{\psi}(A) = 0.$$

■■

19 The Fourier transform. Solutions to Problems 19.1–19.9

Problem 19.1 Solution:

(a) By definition,

$$\begin{aligned}
\widehat{\mathbb{1}_{[-1,1]}}(\xi) &= \frac{1}{2\pi}\int \mathbb{1}_{[-1,1]}(x)e^{-i\,x\xi}\,dx \\
&= \frac{1}{2\pi}\left[-\frac{e^{-i\,x\xi}}{i\,\xi}\right]_{x=-1}^{1} \\
&= \frac{1}{2\pi}\frac{1}{i\,\xi}\left(e^{i\,\xi}-e^{-i\,\xi}\right) \\
&= \frac{1}{\pi}\frac{\sin\xi}{\xi}
\end{aligned}$$

for $\xi \neq 0$. Here we use that $\sin\xi = \operatorname{Im} e^{i\xi} = \frac{1}{2i}(e^{i\xi} - e^{-i\,\xi})$. For $\xi = 0$ we have

$$\widehat{\mathbb{1}_{[-1,1]}}(0) = \frac{1}{2\pi}\int \mathbb{1}_{[-1,1]}(x)\,dx = \frac{1}{\pi}.$$

(Note that $\frac{\sin\xi}{\xi} \to 1$ as $\xi \to 0$, i.e. the Fourier transform is continuous at $\xi = 0$ – as one would expect.)

(b) The convolution theorem, Theorem 19.11, shows that $\widehat{f * g} = (2\pi)\hat{f}\cdot\hat{g}$. Because of part (a) we get

$$\mathscr{F}(\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-1,1]})(\xi) = (2\pi)\left(\frac{1}{\pi}\frac{\sin\xi}{\xi}\right)^2 = \frac{2}{\pi}\frac{\sin^2\xi}{\xi^2}.$$

(c) We get from the definition that

$$\begin{aligned}
\mathscr{F}(e^{-(\cdot)}\mathbb{1}_{[0,\infty)}(\cdot))(\xi) &= \frac{1}{2\pi}\int_0^\infty e^{-x}e^{-i\,x\xi}\,dx \\
&= \frac{1}{2\pi}\int_0^\infty e^{-x(1+i\,\xi)}\,dx \\
&= -\frac{1}{2\pi}\frac{1}{1+i\,\xi}\left[e^{-x(1+i\,\xi)}\right]_{x=0}^{\infty} \\
&= \frac{1}{2\pi}\frac{1}{1+i\,\xi}.
\end{aligned}$$

(d) Obviously, we have

$$\int e^{-i\,x\xi}e^{-|x|} = \int_{(-\infty,0)} e^{-i\,x\xi}e^{x}\,dx + \int_{(0,\infty)} e^{-i\,x\xi}e^{-x}\,dx$$

$$= \int_{(0,\infty)} e^{i y\xi} e^{-y}\,dy + \int_{(0,\infty)} e^{-i x\xi} e^{-x}\,dx.$$

Thus,

$$\begin{aligned}
\mathscr{F}(e^{-|\cdot|})(\xi) &= \mathscr{F}(e^{-\cdot}\mathbb{1}_{[0,\infty)})(-\xi) + \mathscr{F}(e^{-\cdot}\mathbb{1}_{[0,\infty)})(\xi)\\
&\overset{(c)}{=} \frac{1}{2\pi}\left(\frac{1}{1-i\,\xi} + \frac{1}{1+i\,\xi}\right)\\
&= \frac{1}{\pi}\frac{1}{1+\xi^2}.
\end{aligned}$$

(e) From (d) and $\mathscr{F}\circ\mathscr{F}u(x) = (2\pi)^{-1}u(-x)$ (cf. Corollary 19.24) we find

$$\mathscr{F}\left(\frac{1}{1+x^2}\right)(\xi) \overset{(d)}{=} \pi\cdot\mathscr{F}\circ\mathscr{F}(e^{-|\cdot|})(\xi) = \frac{1}{2}e^{-|-\xi|} = \frac{1}{2}e^{-|\xi|}.$$

(f) Note that

$$\begin{aligned}
\int_{[-1,1]}(1-|x|)e^{-i x\xi}\,dx &= \int_{[-1,1]} e^{-i x\xi}\,dx + \int_{[-1,0]} xe^{-i x\xi}\,dx - \int_{[0,1]} xe^{-i x\xi}\,dx\\
&= \int_{[-1,1]} e^{-i x\xi}\,dx + \int_{[0,1]} (-y)e^{i y\xi}\,dy - \int_{[0,1]} xe^{-i x\xi}\\
&= \int_{[-1,1]} e^{-i x\xi}\,dx - \int_{[0,1]} x\,\underbrace{(e^{i x\xi} + e^{-i x\xi})}_{2\cos(x\xi)}\,dx.
\end{aligned}$$

The first expression is as in part (a). For the second integral we use integration by parts:

$$\begin{aligned}
\int_0^1 x\cos(x\xi)\,dx &= \left[x\frac{\sin(x\xi)}{\xi}\right]_{x=0}^1 - \frac{1}{\xi}\int_0^1 \sin(x\xi)\,dx\\
&= \frac{\sin(\xi)}{\xi} - \frac{1}{\xi}\left[\frac{\cos(x\xi)}{\xi}\right]_{x=0}^1\\
&= \frac{\sin(\xi)}{\xi} - \frac{\cos(\xi)}{\xi^2} + \frac{1}{\xi^2}.
\end{aligned}$$

Thus,

$$\mathscr{F}(\mathbb{1}_{[-1,1]}(1-|\cdot|))(\xi) = \frac{1}{\pi}\frac{\sin\xi}{\xi} - \frac{1}{\pi}\left(\frac{\sin\xi}{\xi} - \frac{\cos\xi}{\xi^2} + \frac{1}{\xi^2}\right) = \frac{1}{\pi}\frac{1-\cos\xi}{\xi^2}.$$

(g) By definition,

$$\mathscr{F}\left(\sum_{k=0}^\infty \frac{t^k}{k!}e^{-t}\delta_k\right)(\xi) = \frac{1}{2\pi}\int e^{-i x\xi}\sum_{k=0}^\infty \frac{t^k}{k!}e^{-t}\delta_k(dx) = \frac{1}{2\pi}\sum_{k=0}^\infty \frac{t^k}{k!}e^{-t}e^{-i k\xi}.$$

Since $e^{-i k\xi} = (e^{-i\xi})^k$, we conclude that

$$\mathscr{F}\left(\sum_{k=0}^\infty \frac{t^k}{k!}e^{-t}\delta_k\right)(\xi) = \frac{1}{2\pi}\sum_{k=0}^\infty \frac{(te^{-i\xi})^k}{k!}e^{-t} = \frac{1}{2\pi}e^{-t}e^{te^{-i\xi}} = \frac{1}{2\pi}e^{t(e^{-i\xi}-1)}.$$

(h) The same calculation as in (g) yields

$$
\begin{aligned}
\mathscr{F}\left(\sum_{n=0}^{k}\binom{n}{k}p^k q^{n-k}\delta_k\right)(\xi) &= \frac{1}{2\pi}\int e^{-i\,x\xi}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\delta_k(dx)\\
&= \frac{1}{2\pi}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}e^{-i\,\xi k}\\
&= \frac{1}{2\pi}\sum_{k=0}^{n}\binom{n}{k}(pe^{-i\,\xi})^k q^{n-k}\\
&= \frac{1}{2\pi}\left(pe^{-i\xi}+q\right)^n.
\end{aligned}
$$

In the final step we use the binomial theorem.

Problem 19.2 Solution: Observe that for complex numbers $u, v \in \mathbb{C}$

$$
\begin{aligned}
|u+v|^2 &= (u+v)\overline{(u+v)}\\
&= (u+v)(\bar{u}+\bar{v})\\
&= u\bar{u}+u\bar{v}+v\bar{u}+v\bar{v}\\
&= |u|^2+2\operatorname{Re}u\bar{v}+|v|^2
\end{aligned}
$$

and so, setting $v \rightsquigarrow -v$

$$|u-v|^2 = |u|^2-2\operatorname{Re}u\bar{v}+|v|^2$$

and so, setting $v \rightsquigarrow iv$

$$|u+iv|^2 = |u|^2+2\operatorname{Im}u\bar{v}-|v|^2$$

and so, setting $v \rightsquigarrow -iv$

$$|u-iv|^2 = |u|^2-2\operatorname{Im}u\bar{v}-|v|^2$$

And this gives

$$|u+v|^2-|u-v|^2+i|u+iv|^2-i|u-iv|^2 = 4\operatorname{Re}u\bar{v}+4i\operatorname{Im}u\bar{v} = 4u\bar{v}.$$

Thus, we have the following ‘polarization’ formula

$$
\begin{aligned}
\int u\bar{v}\,dx &= \frac{1}{4}\left[\int |u+v|^2\,dx-\int |u-v|^2\,dx+i\int |u+iv|^2\,dx-i\int |u-iv|^2\,dx\right]\\
&= \frac{1}{4}\left[\|u+v\|_2^2-\|u-v\|_2^2+i\|u+iv\|_2^2-i\|u-iv\|_2^2\right]
\end{aligned}
$$

and now the claim follows directly from the statement of Plancherel’s theorem.

Alternative solution: Mimic the proof of Theorem 19.20: We have $u, v, \widehat{u}, \widehat{v} \in L^2(\lambda^n)$ (as a result of Theorem 19.20), and so $u\cdot\bar{v}$ and $\widehat{u}\cdot\overline{\widehat{v}}$ are integrable. Therefore,

$$\int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi \quad = \quad (2\pi)^{-n}\int \widehat{u}(\xi)\overline{\widecheck{v}}(\xi)\,d\xi$$

$$\overset{19.12}{=} (2\pi)^{-n} \int u(x)\mathscr{F}\left[\check{\overline{v}}\right](x)\,dx$$
$$\overset{19.9}{=} (2\pi)^{-n} \int u(x)\overline{v(x)}\,dx.$$

Problem 19.3 Solution: Assume that $\widetilde{\mu} = \mu$. We have

$$\begin{aligned}
\chi(\xi) &= \int e^{-ix\xi}\,\mu(dx)\\
&\overset{\widetilde{\mu}=\mu}{=} \int e^{-ix\xi}\,\widetilde{\mu}(dx)\\
&\overset{15.1}{=} \int e^{-i(-x)\xi}\,\mu(dx)\\
&= \int \overline{e^{-ix\xi}}\,\mu(dx)\\
&= \overline{\int e^{-ix\xi}\,\mu(dx)}\\
&= \overline{\chi(\xi)}.
\end{aligned}$$

Therefore, χ is real-valued. On the other hand, the above calculation shows that

$$\overline{\chi(\xi)} = \int e^{-ix\xi}\widetilde{\mu}(dx).$$

This means that $\chi = \overline{\chi}$ entails $\mathscr{F}\mu = \mathscr{F}\widetilde{\mu}$, and so $\mu = \widetilde{\mu}$ because of the injectivity of the Fourier transform.

Problem 19.4 Solution: From linear algebra we know that a symmetric positive definite matrix has a unique symmetric positive square root, i.e. there is some $B \in \mathbb{R}^{n\times n}$ which is symmetric and positive definite such that $B^2 = A$. Since $\det(B^2) = (\det B)^2$, we see that $\det B = \sqrt{\det A} > 0$. Now we change coordinates according to $y := Bx$

$$\begin{aligned}
\int e^{-i\langle x,\xi\rangle}e^{-\langle x,Ax\rangle}\,dx &= \int e^{-i\langle x,\xi\rangle}e^{-\langle Bx,Bx\rangle}\,dx\\
&= \frac{1}{\det B}\int e^{-i\langle B^{-1}y,\xi\rangle}e^{-|y|^2}\,dy\\
&= \frac{1}{\sqrt{\det A}}\int e^{-i\langle y,B^{-1}\xi\rangle}e^{-|y|^2}\,dy.
\end{aligned}$$

If we set

$$g_{1/2}(x) := \frac{1}{\pi^{n/2}}\exp\left(-|x|^2\right),$$

cf. Example 19.2(iii), then the calculation from above gives

$$\mathscr{F}(e^{-\langle\cdot,A\cdot\rangle})(\xi) = \frac{\pi^{n/2}}{\sqrt{\det A}}\mathscr{F}(g_{1/2})\left(B^{-1}\xi\right).$$

Example 19.2(iii) now shows

$$\mathscr{F}(e^{-\langle\cdot,A\cdot\rangle})(\xi) = \frac{\pi^{n/2}}{\sqrt{\det A}}\frac{1}{(2\pi)^n}\exp\left(-\frac{|B^{-1}\xi|^2}{4}\right).$$

Finally, since $B^{-1} = (B^{-1})^\top$,

$$|B^{-1}\xi|^2 = \langle B^{-1}\xi, B^{-1}\xi\rangle = \langle \xi, \underbrace{(B^{-1}B^{-1})}_{A^{-1}}\xi\rangle,$$

we infer that

$$\mathscr{F}(e^{-\langle\cdot,A\cdot\rangle})(\xi) = \frac{1}{\sqrt{\det A}}\frac{1}{2^{n/2}}\frac{1}{(2\pi)^{n/2}}\exp\left(-\frac{\langle\xi, A^{-1}\xi\rangle}{4}\right).$$

Problem 19.5 Solution: $g_t(x) = (2\pi t)^{-1/2}e^{-x^2/2t}$ and $\widehat{g}_t(\xi) = (2\pi)^{-1}e^{-t\xi^2/2}$. By Plancherel's theorem (Theorem 19.20, plus polarization) or by Problem 19.2 we see that

$$\begin{aligned}
\int \widehat{u}(\xi)e^{-t|\xi|^2/2}\,d\xi &= (2\pi)\int \widehat{u}(\xi)\widehat{g}_t(\xi)\,d\xi \\
&= \int u(x)g_t(x)\,dx \\
&= \int u(x)(2\pi t)^{-1/2}e^{-x^2/2t}\,dx \\
&= (2\pi)^{-1/2}\int u(ty)e^{-y^2/2}\,dy \\
&\leqslant c\|u\|_\infty.
\end{aligned}$$

(In fact, $c = 1$, see Example 14.11). Now let $t \uparrow 0$ using monotone convergence and use that, by assumption, $\widehat{u} \geqslant 0$.

The same argument holds for L^2-functions since $g_t \in L^2$.

Problem 19.6 Solution: We follow the hint and find using Fubini's theorem

$$\begin{aligned}
&2\left(\frac{R}{2}\right)^n\int_{-1/R}^{1/R}\cdots\int_{-1/R}^{1/R}\int_{\mathbb{R}^n}(1-e^{i\langle x,\xi\rangle})\,\mu(dx)\,d\xi_1\ldots d\xi_d \\
&\quad= 2\int_{\mathbb{R}^n}\left(\frac{R}{2}\right)^n\int_{-1/R}^{1/R}\cdots\int_{-1/R}^{1/R}(1-e^{i\langle x,\xi\rangle})\,d\xi_1\ldots d\xi_d\,\mu(dx) \\
&\quad= 2\int_{\mathbb{R}^n}\frac{R}{2}\int_{-1/R}^{1/R}\cdots\frac{R}{2}\int_{-1/R}^{1/R}(1-e^{i\langle x,\xi\rangle})\,d\xi_1\ldots d\xi_d\,\mu(dx) \\
&\quad= 2\int_{\mathbb{R}^n}\left(1-\frac{R}{2}\int_{-1/R}^{1/R}\cdots\frac{R}{2}\int_{-1/R}^{1/R}e^{i\langle x,\xi\rangle}\,d\xi_1\ldots d\xi_d\right)\mu(dx) \\
&\quad= 2\int_{\mathbb{R}^n}\left(1-\prod_{n=1}^n\frac{R}{2}\int_{-1/R}^{1/R}e^{ix_n\xi_n}\,d\xi_n\right)\mu(dx)
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{\mathbb{R}^n}\left(1-\prod_{n=1}^n \frac{R}{2}\left[\frac{e^{ix_n\xi_n}}{ix_n}\right]_{\xi_n=-1/R}^{\xi_n=1/R}\right)\mu(dx)\\
&= 2\int_{\mathbb{R}^n}\left(1-\prod_{n=1}^n \frac{e^{ix_n/R}-e^{-ix_n/R}}{2ix_n/R}\right)\mu(dx)\\
&= 2\int_{\mathbb{R}^n}\left(1-\prod_{n=1}^n \frac{\sin(x_n/R)}{x_n/R}\right)\mu(dx)\\
&\geqslant 2\int_{\mathbb{R}^n\setminus[-2R,2R]^n}\left(1-\prod_{n=1}^n \frac{\sin(x_n/R)}{x_n/R}\right)\mu(dx).
\end{aligned}
$$

In the last step we use that the integrand is positive since $|\sin y/y|\leqslant 1$. Observe that

$$
x\in\mathbb{R}^n\setminus[-2R,2R]^n \iff \exists n=1,\dots,n : |x_n|>2R
$$

and so

$$
\prod_{n=1}^n \frac{\sin(x_n/R)}{x_n/R}\leqslant \frac{1}{2}
$$

hence

$$
\begin{aligned}
2\left(\frac{R}{2}\right)^n \int_{-1/R}^{1/R}\cdots\int_{-1/R}^{1/R}\int_{\mathbb{R}^n}&(1-e^{i\langle x,\xi\rangle})\,\mu(dx)\,d\xi_1\dots d\xi_d\\
&\geqslant 2\int_{\mathbb{R}^n\setminus[-2R,2R]^n}\left(1-\prod_{n=1}^n \frac{\sin(x_n/R)}{x_n/R}\right)\mu(dx)\\
&\geqslant 2\int_{\mathbb{R}^n\setminus[-2R,2R]^n}\left(1-\frac{1}{2}\right)\mu(dx)\\
&\geqslant \int_{\mathbb{R}^n\setminus[-2R,2R]^n}\mu(dx).
\end{aligned}
$$

Remark. A similar inequality exists for the Fourier transform (instead of the inverse Fourier transform). This has the form

$$
\mu\left(\mathbb{R}^n\setminus[-2R,2R]^n\right)\leqslant 2(\pi R)^n\int_{[-1/R,1/R]^n}\left(\widehat{\mu}(0)-\operatorname{Re}\widehat{\mu}(\xi)\right)\,d\xi.
$$

Problem 19.7 Solution:

(i) Let $\xi_1,\dots,\xi_n\in\mathbb{R}^n$ and $\lambda_1,\dots,\lambda_n\in\mathbb{C}$. From the definition of the Fourier transform we get

$$
\begin{aligned}
\sum_{i,k=1}^n \phi(\xi_j-\xi_k)\lambda_j\bar\lambda_k &= \frac{1}{(2\pi)^n}\sum_{j,k=1}^n \lambda_j\bar\lambda_k\int e^{-i\,x(\xi_j-\xi_k)}\,d\mu(x)\\
&= \frac{1}{(2\pi)^n}\sum_{j,k=1}^n \lambda_j\bar\lambda_k\int e^{-i\,x\xi_j}\overline{e^{-i\,x\xi_k}}\,d\mu(x)\\
&= \frac{1}{(2\pi)^n}\int\left(\sum_{j=1}^n \lambda_j e^{-i\,x\xi_j}\right)\overline{\left(\sum_{k=1}^n \lambda_k e^{-i\,x\xi_k}\right)}\,d\mu(x)
\end{aligned}
$$

$$= \frac{1}{(2\pi)^n} \int \left| \sum_{j=1}^{n} \lambda_j e^{-i x \xi_j} \right|^2 d\mu(x) \geqslant 0.$$

Note that this already implies that $\phi(-\xi) = \overline{\phi(\xi)}$. The argument is as follows: If we have for a matrix (a_{jk}) that $\sum_{jk} a_{jk} \lambda_j \bar{\lambda}_j \geqslant 0$, then

$$0 \leqslant \sum_{jk} a_{jk} \lambda_j \bar{\lambda}_k = \overline{\sum_{jk} a_{jk} \lambda_j \bar{\lambda}_k} = \sum_{jk} \overline{a_{jk}} \bar{\lambda}_j \lambda_k = \sum_{kj} \overline{a_{kj}} \bar{\lambda}_k \lambda_j$$

which means that $a_{jk} = \overline{a_{kj}}$. Apply this to the matrix $a_{jk} = \phi(\xi_j - \xi_k)$ with $m = 2$ and $\xi_1 = \xi$ and $\xi_2 = 0$ to infer that $\phi(\xi) = \overline{\phi(-\xi)}$.

(ii) We want to use the differentiability lemma for parameter-dependent integrals. For this we define

$$u(\xi, x) := \frac{1}{(2\pi)^n} e^{-i x \xi}.$$

Since μ is a finite measure and $|u(x,\xi)| \leqslant (2\pi)^{-n}$, we find $u(\xi,\cdot) \in L^1(\mu)$. Moreover,

$$\begin{aligned} |\partial_{\xi_j} u(\xi, x)| &= (2\pi)^{-d} |x_j| \leqslant (2\pi)^{-d} |x| \\ &\leqslant (2\pi)^{-d} \left(\mathbb{1}_{[-1,1]}(x) + |x|^m \mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x) \right) =: w(x) \in L^1(\mu) \end{aligned}$$

is an integrable majorant. With Theorem 12.5 we find

$$\partial \xi_j \phi(\xi) = \partial_{\xi_j} \int u(\xi, x) \, \mu(dx) = \frac{1}{(2\pi)^n} \int (-i x_j) e^{-i x \xi} \mu(dx).$$

Iterating this argument, we see that $\partial^\alpha \phi$ exists for any $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leqslant m$.

(iii) We follow the hint and consider first the case $d = 1$ and $n = 1$. We can rewrite the expression $\phi(2h) - 2\phi(0) + \phi(-2h)$ using Fourier transforms:

$$\begin{aligned} \phi(2h) - 2\phi(0) + \phi(-2h) &= \frac{1}{2\pi} \int (e^{-i 2hx} - 2 + e^{i 2hx}) \, \mu(dx) \\ &= \frac{1}{\pi} \int (\cos(2hx) - 1) \mu(dx). \end{aligned}$$

L'Hospital's theorem applies and gives

$$\frac{1 - \cos(2y)}{4y^2} \xrightarrow{y \to 0} \frac{1}{2}.$$

Now we can use Fatou's lemma

$$\begin{aligned} \int x^2 \frac{1}{2} \, \mu(dx) &= \int x^2 \lim_{h \to 0} \frac{1 - \cos(2hx)}{4(hx)^2} \, \mu(dx) \\ &\leqslant \liminf_{h \to 0} \frac{1}{4h^2} \int (1 - \cos(2hx)) \, \mu(dx) \\ &= -\pi \liminf_{h \to 0} \frac{1}{4h^2} \big(\phi(2h) - 2\phi(0) + \phi(-2h) \big) \\ &= -\pi \phi''(0) < \infty. \end{aligned}$$

If $n \geqslant 1$, we use induction. Assume that $\phi \in C^{2n}(\mathbb{R})$ and that the assertion has been proved for $n-1$. Since $\phi \in C^{2n}(\mathbb{R}) \Rightarrow \phi \in C^{2(n-1)}$, we see by the induction assumption that $\int |x|^{2(n-1)}\,d\mu(x) < \infty$. Thus, $\nu(dx) := x^{2(n-1)}\mu(dx)$ is a measure and

$$\begin{aligned}\widehat{\nu}(\xi) &= \frac{1}{2\pi}\int x^{2(n-1)}e^{-ix\xi}\,d\mu(x)\\ &= \frac{1}{2\pi}\frac{1}{(-i)^{2(n-1)}}\frac{d^{2(n-1)}}{d\xi^{2(n-1)}}\int e^{-ix\xi}\,d\mu(x).\end{aligned}$$

Consequently, we see that $\hat{\nu} \in C^2(\mathbb{R})$. The first part of the proof ($n = 1$) gives

$$\int |x|^{2n}\,d\mu(x) = \int |x|^2\,d\nu(x) < \infty.$$

If $d \geqslant 1$, then we set $\pi_j(x) := x_j$, $x \in \mathbb{R}^n$, $j \in \{1, \dots, d\}$. Apply the case $d = 1$ to the measures $\pi_j(\mu)$.

(iv) Assume that $z \in \mathbb{C}^n$. If $K := \operatorname{supp}\mu$ is compact, then we get, because of the continuity of e^{-izx}, that $M := \sup_{x\in K}|e^{-izx}| < \infty$. From

$$\int u\,d\mu = \int_{\operatorname{supp}\mu} u\,d\mu \quad \text{for any } u \geqslant 0$$

we conclude that

$$\int |e^{-izx}|\,d\mu(x) \leqslant M\mu(\mathbb{R}^n) < \infty,$$

i.e.

$$\phi(z) = \frac{1}{(2\pi)^n}\int e^{-izx}\,d\mu(x)$$

is well-defined. Setting

$$u_n(x) := \frac{1}{(2\pi)^n}\sum_{k=0}^n \frac{(-izx)^k}{k!}, \quad x \in \mathbb{R}^n,$$

we get

$$|u_n(x)| \leqslant \frac{1}{(2\pi)^n}\sum_{k=0}^n \frac{|zx|^k}{k!} \leqslant \frac{1}{(2\pi)^n}e^{|zx|} \leqslant \frac{1}{(2\pi)^n}\sup_{x\in K} e^{|zx|} < \infty.$$

Since μ is a finite measure, we can use the dominated convergence theorem to get

$$\begin{aligned}\phi(z) &= \int \lim_{n\to\infty} u_n(x)\,\mu(dx)\\ &= \lim_{n\to\infty}\int u_n(x)\,d\mu(x)\\ &= \frac{1}{(2\pi)^n}\sum_{k=0}^\infty \frac{1}{k!}\int (zx)^k\,d\mu(x).\end{aligned}$$

This proves that ϕ is analytic.

Problem 19.8 Solution: Note that $e^{ix/n} \xrightarrow{n\to\infty} 1$ for all $x \in \mathbb{R}$. On the other hand, we gather from $\int_B e^{ix/n}\,dx = 0$ that $\mathbb{1}_B e^{i\cdot/n} \in \mathcal{L}^1(dx)$. As $|e^{ix/n}| = 1$, we get $\lambda^1(B) < \infty$. By dominated convergence

$$0 = \lim_{n\to\infty} \int_B e^{ix/n}\,dx = \int_B \underbrace{\lim_{n\to\infty} e^{ix/n}}_{1}\,dx = \lambda^1(B).$$

Alternative solution: Set $f(x) := \mathbb{1}_B(x)$; by assumption, $\hat{f}(1/n) = 0$. Since the Fourier transform is continous, cf. 19.3, we get

$$\hat{f}(0) = \lim_{n\to\infty} \hat{f}\left(\frac{1}{n}\right) = 0.$$

On the other hand, $\hat{f}(0) = (2\pi)^{-1}\lambda^1(B)$.

Problem 19.9 Solution:

(i) $\Leftarrow$: Since $\mu(\mathbb{R} \setminus \frac{2\pi}{\xi}\mathbb{Z}) = 0$ we find

$$\mu = \sum_{j\in\mathbb{Z}} p_j \delta_{\frac{2\pi}{\xi}\mathbb{Z}}$$

with $p_j := \mu(\frac{2\pi}{\xi}j)$. From the definition of the Fourier transform we get

$$\begin{aligned}\hat{\mu}(\eta) &= \frac{1}{2\pi}\int e^{-ix\eta}\mu(dx)\\ &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} p_j \exp\left[-i\left(\frac{2\pi}{\xi}j\right)\eta\right]\end{aligned}$$

for all $\eta \in \mathbb{R}$. Setting $\eta = \xi$, we see

$$\begin{aligned}\hat{\mu}(\xi) &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} p_j \underbrace{\exp(-i2\pi j)}_{1}\\ &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} p_j \exp(-i\,0) = \hat{\mu}(0).\end{aligned}$$

$\Rightarrow$: From $\hat{\mu}(\xi) = \hat{\mu}(0)$ we conclude

$$2\pi(\hat{\mu}(0) - \hat{\mu}(\xi)) = \int (1 - e^{-ix\xi})\,\mu(dx) = 0.$$

In particular, $\int (1 - e^{-ix\xi})\,\mu(dx) \in \mathbb{R}$, i.e.

$$\int (1 - e^{-ix\xi})\,\mu(dx) = \operatorname{Re}\int (1 - e^{-ix\xi})\,\mu(dx) = \int (1 - \cos(x\xi))\,\mu(dx) = 0.$$

Since $1 - \cos(x\xi) \geqslant 0$, this implies

$$\mu\{x \in \mathbb{R}; 1 - \cos(x\xi) > 0\} = 0.$$

Consequently,

$$0 = \mu\{x \in \mathbb{R}; \cos(x\xi) \neq 1\} = \mu\left(\mathbb{R} \setminus \frac{2\pi}{\xi}\mathbb{Z}\right).$$

(ii) Because of $|\widehat{\mu}(\xi_1)| = \widehat{\mu}(0)$ there is some $z_1 \in \mathbb{R}$ such that

$$\widehat{\mu}(\xi_1) = \widehat{\mu}(0)e^{i\, z_1\xi_1}.$$

Therefore,

$$\frac{1}{2\pi}\int e^{-i\,\xi_1(x+z_1)}\,\mu(dx) = \hat{\mu}(0).$$

Observe that the left-hand side is just the Fourier transform of the measure $\nu(B) := \mu(B-z_1)$, $B \in \mathscr{B}(\mathbb{R})$, and so

$$\hat{\nu}(\xi_1) = \hat{\mu}(0) = \hat{\nu}(0).$$

From part (i) we get that $\nu(\mathbb{R} \setminus \frac{2\pi}{\xi_1}\mathbb{Z}) = 0$. This is the same as

$$\mu\left\{\mathbb{R} \setminus \left(z_1 + \frac{2\pi}{\xi_1}\mathbb{Z}\right)\right\} = 0.$$

Using the same argument we find some $z_2 \in \mathbb{R}$, such that

$$\mu\left\{\mathbb{R} \setminus \left(z_2 + \frac{2\pi}{\xi_2}\mathbb{Z}\right)\right\} = 0.$$

Setting

$$A := \left(z_1 + \frac{2\pi}{\xi_1}\mathbb{Z}\right) \cap \left(z_2 + \frac{2\pi}{\xi_2}\mathbb{Z}\right)$$

we see that $\mu(\mathbb{R} \setminus A) = 0$. Let us show that A contains at most one element: Assume, on the contrary, that there are two distinct points in A, then there are $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{Z}$ such that

$$\begin{aligned} z_1 + \frac{2\pi}{\xi_1}n &= z_2 + \frac{2\pi}{\xi_2}n', \\ z_1 + \frac{2\pi}{\xi_1}m &= z_2 + \frac{2\pi}{\xi_2}m'. \end{aligned}$$

Subtracting these identities, we get

$$\begin{aligned} \frac{2\pi}{\xi_1}(n-m) &= \frac{2\pi}{\xi_2}(n'-m') \\ \Rightarrow \frac{\xi_2}{\xi_1} &= \frac{n'-m'}{n-m} \in \mathbb{Q}. \end{aligned}$$

This is clearly contradicting the assumption $\frac{\xi_1}{\xi_2} \notin \mathbb{Q}$.

■■

20 The Radon–Nikodým theorem. Solutions to Problems 20.1–20.9

Problem 20.1 Solution: The assumption $\nu \leqslant \mu$ immediately implies $\nu \ll \mu$. Indeed,

$$\mu(N) = 0 \;\Rightarrow\; 0 \leqslant \nu(N) \leqslant \mu(N) = 0 \;\Rightarrow\; \nu(N) = 0.$$

Using the Radon–Nikodým theorem we conclude that there exists a measurable function $f \in \mathcal{M}^+(\mathscr{A})$ such that $\nu = f \cdot \mu$. Assume that $f > 1$ on a set of positive μ-measure. Without loss of generality we may assume that the set has finite measure, otherwise we would consider the intersection $A_k \cap \{f > 1\}$ with some exhausting sequence $A_k \uparrow X$ and $\mu(A_k) < \infty$.

Then, for sufficiently small $\epsilon > 0$ we know that $\mu(\{f \geqslant 1 + \epsilon\}) > 0$ and so

$$\begin{aligned}
\nu(\{f \geqslant 1+\epsilon\}) &= \int_{\{f\geqslant 1+\epsilon\}} f\,d\mu \\
&\geqslant (1+\epsilon)\int_{\{f\geqslant 1+\epsilon\}} d\mu \\
&\geqslant (1+\epsilon)\mu(\{f \geqslant 1+\epsilon\}) \\
&\geqslant \mu(\{f \geqslant 1+\epsilon\})
\end{aligned}$$

which is impossible.

Problem 20.2 Solution: Because of our assumption both $\mu \ll \nu$ and $\nu \ll \mu$ which means that we know

$$\nu = f\mu \quad \text{and} \quad \mu = g\nu$$

for positive measurable functions f, g which are a.e. unique. Moreover,

$$\nu = f\mu = f \cdot g\nu$$

so that $f \cdot g$ is almost everywhere equal to 1 and the claim follows.

Because of Problem 20.4 (which is just Corollary 25.6) it is clear that $f, g < \infty$ a.e. and, by the same argument, $f, g > 0$ a.e.

Note that we do not have to specify *w.r.t. which measure* we understand the 'a.e.' since their null sets coincide anyway.

Problem 20.3 Solution: Take Lebesgue measure $\lambda := \lambda^1$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and the function $f(x) := x + \infty \cdot \mathbb{1}_{[0,1]^c}(x)$. Then $f \cdot \lambda$ is certainly not σ-finite.

Problem 20.4 Solution: See the proof of Corollary 25.6.

Problem 20.5 Solution: See the proof of Theorem 25.9.

Problem 20.6 Solution: (i) If F is AC, continuity is trivial, just take $N = 2$ in the very definition of AC functions.

To see that F is also BV, we take $\epsilon = 1$ and choose $\delta > 0$ such that for any subcollection $a \leqslant x_1 < y_1 < \cdots < x_N < y_N \leqslant b$ with $\sum_n (y_n - x_n) < \delta$ we have $\sum_n |F(y_n) - F(x_n)| < 1$. Let $M = [(b-a)/\delta]+1$ and $a_i = a+i(b-a)/M$ for $i = 0, 1, \dots, M$. Clearly, $a_i - a_{i-1} = (b-a)/M < \delta$ and, in particular, $V(f, [a_{i-1}, a_i]) < 1$ for all $i = 0, 1, \dots M$. Thus,

$$V(f; [a, b]) \leqslant \sum_{i=1}^{M} V(f, [a_{i-1}, a_i]) < M.$$

(ii) Following the hint, we see that f is increasing. Define $g := F - f$. We have to show that g is increasing. Let $x < y$. Obviously,

$$V(f; [a, x]) + F(y) - F(x) \leqslant V(f; [a, x]) + |F(y) - F(x)| \leqslant V(f; [a, y])$$

(since the points $x < y$ can be added to extend any partition of $[a, x]$ to give a partition of $[a, y]$). This gives $g(x) \leqslant g(y)$.

(iii) Fix $\epsilon > 0$ and pick $R = R(\epsilon)$ in such a way that

$$\int_{\{|f|>R\}} |f| \, d\lambda < \frac{\epsilon}{2}.$$

This is possible since f is integrable: use, e.g. monotone convergence. Now pick $x_1 < y_1 < x_2 < y_2 < \cdots < x_N < y_N$ with $\sum_{n=1}^N |y_n - x_n| < \delta$ where $\delta = \delta(\epsilon) := \epsilon/(2R)$ with the R we've just chosen. Then

$$\begin{aligned}|F(y_n) - F(x_n)| &\leqslant \int_{[x_n, y_n)} |f(t)| \, \lambda(dt) \\ &= \int_{[x_n, y_n) \cap \{|f| \leqslant R\}} |f(t)| \, \lambda(dt) + \int_{[x_n, y_n) \cap \{|f| > R\}} |f(t)| \, \lambda(dt).\end{aligned}$$

Summing over $n = 1, \dots, N$ gives

$$\sum_{n=1}^N |F(y_n) - F(x_n)| \leqslant R \sum_{n=1}^N |y_n - x_n| + \sum_{n=1}^N \int_{[x_n, y_n) \cap \{|f|>R\}} |f| \, d\lambda \leqslant R\delta + \int_{\{|f|>R\}} |f| \, d\lambda \leqslant \epsilon.$$

(iv) Write $F = f_1 - f_2$ with f_i increasing (see part (ii)). From (ii) we know that we can pick $f_1(x) = V(F,[a,x])$. Since F is absolutely continuous, so is f_1, hence f_2. This follows from the observation that

$$V(F,[a,y]) - V(F,[a,x]) = V(F,[x,y]) \quad \forall\, x < y.$$

Since the f_i are continuous, the set-functions $\mu_i[a,x) := f_i(x) - f_i(a)$ are pre-measures and extend to measures on the Borel σ-algebra – see also Problem 6.1.

Now let N be a Lebesgue null-set. For every $\delta > 0$ we can cover N by countably many intervals $[x_k, y_k)$ such that $\sum_{k\in\mathbb{N}}(y_k - x_k) < \delta$. This follows from the Carathéodory extension of Lebesgue measure defined on the half-open intervals (Theorem 6.1 and Proposition 6.3). Set

$$R_m := \bigcup_{k=1}^{m}[x_k, y_k) \uparrow R = \bigcup_{k\in\mathbb{N}}[x_k, y_k) \supset N.$$

Fix m. Without loss of generality we can assume that the intervals in R_m are non-overlapping and their length is still $< \delta$. (Otherwise, we could merge the overlapping intervals into one, reducing the number of intervals, their total lenght is still $< \delta$).

Since the f_i are AC, we find for every ϵ some δ such that

$$\sum_{k=1}^{m} |f_i(y_k) - f_i(x_k)| < \epsilon.$$

In particular, using the continuity of measures,

$$\mu_i(N) \leqslant \mu(R) \leqslant \lim_{m\to\infty} \mu(R_m) \leqslant \lim_{m\to\infty} \sum_{k=1}^{m} \mu_i([x_k, y_k)) = \lim_{m\to\infty} \sum_{k=1}^{m} |f_i(y_k) - f_i(x_k)| < \epsilon,$$

which shows that the Lebesgue null-set is also a μ_i-null set, i.e. $\mu_i \ll \lambda$ and therefore the claim follows from the Radon–Nikodým theorem.

Problem 20.7 Solution: This problem is somewhat ill-posed. We should first embed it into a suitable context, say, on the measurable space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Denote by $\lambda = \lambda^1$ one-dimensional Lebesgue measure. Then

$$\mu = \mathbb{1}_{[0,2]}\lambda \quad\text{and}\quad \nu = \mathbb{1}_{[1,3]}\lambda$$

and from this it is clear that

$$\nu = \mathbb{1}_{[1,2]}\nu + \mathbb{1}_{(2,3]}\nu = \mathbb{1}_{[1,2]}\lambda + \mathbb{1}_{(2,3]}\lambda$$

and from this we read off that

$$\mathbb{1}_{[1,2]}\nu \ll \mu$$

while

$$\mathbb{1}_{(2,3]}\nu \perp \mu.$$

It is interesting to note how 'big' the null-set of ambiguity for the Lebesgue decomposition is—it is actually $\mathbb{R} \setminus [0,3]$ a, from a Lebesgue (i.e. λ) point of view, huge and infinite set, but from a μ-ν-perspective a negligible, namely null, set.

Problem 20.8 Solution: Since we deal with a bounded measure we can use $F(x) := \mu(-\infty, x)$ rather than the more cumbersome definition for F employed in Problem 6.1 (which is good for locally finite measures!).

With respect to one-dimensional Lebesgue measure λ we can decompose μ according to Theorem 20.4 into

$$\mu = \mu^\circ + \mu^\perp \quad \text{where} \quad \mu^\circ \ll \lambda, \quad \mu^\perp \perp \lambda.$$

Now *define* $\mu_2 := \mu^\circ$ and $F_2 := \mu^\circ(-\infty, x)$. We have to prove property (2). For this we observe that μ° is a finite measure (since $\mu^\circ \leqslant \mu$ and that, therefore, $\mu^\circ = f \cdot \lambda$ with a function $f \in L^1(\lambda)$. Thus, for every $R > 0$

$$\begin{aligned}
F(y_j) - F(x_j) &= \mu^\circ(x_j, y_j) \\
&= \int_{(x_j,y_j)} f(t)\,\lambda(dt) \\
&= \int_{\{f<R\}\cap(x_j,y_j)} f(t)\,\lambda(dt) + \int_{\{f\geqslant R\}\cap(x_j,y_j)} f(t)\,\lambda(dt) \\
&\leqslant R \int_{(x_j,y_j)} \lambda(dt) + \int_{\{f\geqslant R\}\cap(x_j,y_j)} f(t)\,\lambda(dt).
\end{aligned}$$

Summing over $j = 1, 2, \dots, N$ gives

$$\sum_{j=1}^{N} |F_2(y_j) - F_2(x_j)| \leqslant R \cdot \delta + \int_{\{f\geqslant R\}} f(t)\,\lambda(dt)$$

since $\dot\bigcup_j (x_j, y_j) \subset \mathbb{R}$. Now we choose for given $\epsilon > 0$

- First $R = R(\epsilon)$ such that $\int_{\{f\geqslant R\}} f(t)\,\lambda(dt) \leqslant \epsilon/2$
- and then $\delta := \epsilon/(2R)$

to confirm that

$$\sum_{j=1}^{N} |F_2(y_j) - F_2(x_j)| \leqslant \epsilon$$

this settles b).

Now consider the measure $\mu^{\perp}$. Its distribution function $F^{\perp}(x) := \mu^{\perp}(-\infty, x)$ is increasing, left-continuous but not necessarily continuous. Such a function has, by Lemma 14.14 at most countably many discontinuities (jumps), which we denote by J. Thus, we can write

$$\mu^{\perp} = \mu_1 + \mu_3$$

with the jump (or saltus) $\Delta F(y) := F(y+) - F(y-)$ if $y \in J$.

$$\mu_1 := \sum_{y \in J} \Delta F(y) \cdot \delta_y, \quad \text{and} \quad \mu_3 := \mu^{\perp} - \mu_1;$$

μ_1 is clearly a measure (the sum being countable) with $\mu_1 \leqslant \mu^{\perp}$ and so is, therefore, μ_2 (since the defining difference is always positive). The corresponding distribution functions are

$$F_1(x) := \sum_{y \in J, y < x} \Delta F(y)$$

(called the jump or saltus function) and

$$F_2(x) := F^{\perp}(x) - F_1(x).$$

It is clear that F_2 is increasing and, more importantly, continuous so that the problem is solved.

It is interesting to note that our problem shows that we can decompose every left- or right-continuous monotone function into an absolutely continuous and singular part and the singular part again into a continuous and discontinuous part:

$$g = g_{\text{ac}} + g_{\text{sc}} + g_{\text{sd}}$$

where

- g —is a monotone left- or right-continuous function;
- g_{ac} —is a monotone absolutely continuous (and in particular continuous) function;
- g_{sc} —is a monotone continuous but singular function;
- g_{sd} —is a monotone discontinuous (even: pure jump), but nevertheless left- or right-continuous, and singular function.

Problem 20.9 Solution:

(i) In the following picture F_1 is represented by a black line, F_2 by a grey line and F_3 is a dotted black line.

(ii),(iii) The construction of the F_n's also shows that

$$|F_n(x) - F_{n+1}(x)| \leqslant \frac{1}{2^{n+1}}$$

since we modify F_n only on a set I^{ℓ}_{n+1} by replacing a diagonal line by a combination of diagonal-flat-diagonal and all this happens only within a range of 2^{-n} units. Since the flat bit is in the middle, we get that the maximal deviation between F_n and F_{n+1} is at most $\frac{1}{2} \cdot 2^{-n}$. Just look at the pictures!

Thus the convergence of $F_n \to F$ is uniform, i.e. it preserves continuity and F is continuous as all the F_n's are. That F is increasing is already inherited from the pointwise limit of the F_n's:

$$\begin{aligned} x < y \;\Rightarrow\; \forall\, n \,:\, F_n(x) \leqslant F_n(y) \\ \Rightarrow\; F(x) = \lim_n F_n(x) \leqslant \lim_n F_n(y) = F(y). \end{aligned}$$

(iv) Let C denote the Cantor set. Then for $x \in [0,1] \setminus C$ we find n and ℓ such that $x \in I_n^\ell$ (which is an open set!) and, since on those pieces F_n and F do not differ any more

$$F_n(x) = F(x) \;\Rightarrow\; F'(x) = F_n'(x) = 0$$

where we use that $F_n|I_n^\ell$ is constant. Since $\lambda(C) = 0$ (see Problem 7.12) we have $\lambda([0,1] \setminus C) = 1$ so that F' exists a.e. and satisfies $F' = 0$ a.e.

(v) We have $I_n^\ell = (a_\ell, b_\ell)$ (we suppress the dependence of a_ℓ, b_ℓ on n with, because of our ordering of the middle-thirds sets (see the problem):

$$a_1 < b_1 < a_2 < \cdots < a_{2^n-1} < b_{2^n-1}$$

and

$$\sum_{\ell=1}^{2^n-1} \big[F(b_\ell) - F(a_\ell)\big] = F(b_{2^n-1}) - F(a_1) \xrightarrow[n\to\infty]{} F(1) - F(0) = 1$$

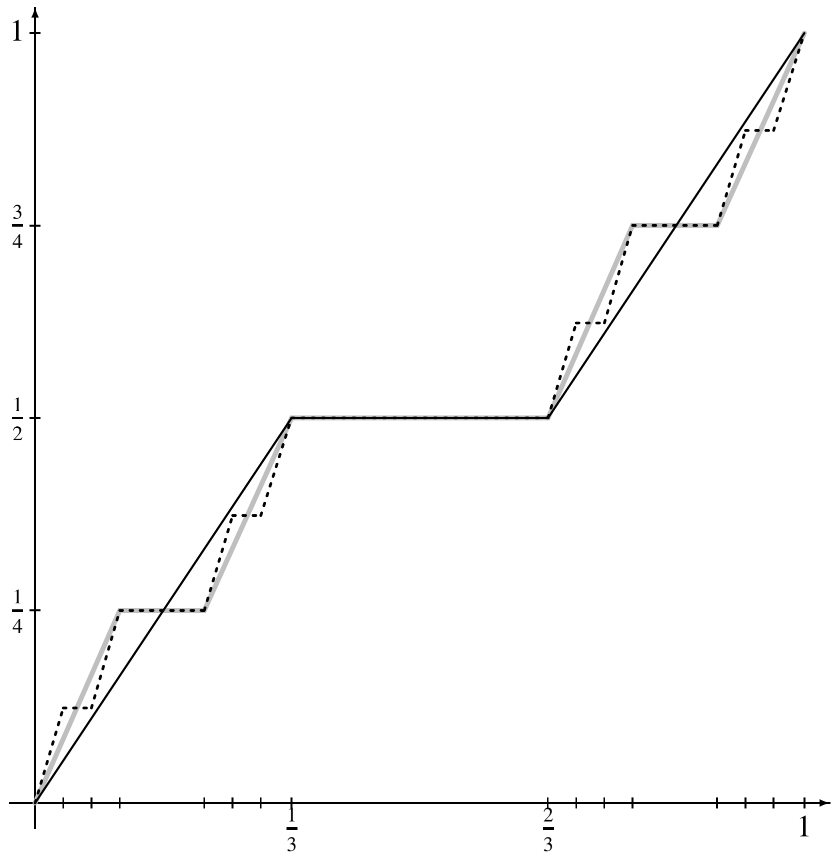
while (with the convention that $a_0 := 0$)

$$\sum_{\ell=1}^{2^n-1} (a_\ell - b_{\ell-1}) \xrightarrow[n\to\infty]{} 0.$$

This leads to a contradiction since, because of the first equality, the sum

$$\sum_{\ell=1}^{2^n-1} \big[F(a_\ell) - F(b_{\ell-1})\big]$$

will never become small.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21 Riesz representation theorems. Solutions to Problems 21.1–21.7

Problem 21.1 Solution:

(i) Let $f \in L^p(\mu)$ and $g \in L^q(\mu)$ such that $\|g\|_q \leqslant 1$. Hölder's inequality (13.5) gives

$$\|f \cdot g\|_1 \leqslant \|f\|_p \|g\|_q \leqslant \|f\|_p.$$

Therefore

$$\|f\|_p \geqslant \sup\left\{ \int f g \, d\mu \; : \; g \in L^q(\mu), \; \|g\|_q \leqslant 1 \right\}.$$

For the converse inequality '$\leqslant$' we use $g := \operatorname{sgn}(f) \cdot |f|^{p-1}$. Since $q = \frac{p}{p-1}$, we have

$$|g|^q = |f|^{(p-1)q} = |f|^p \in L^1(\mu),$$

and so $g \in L^q(\mu)$ and $\|g\|_q = \|f\|_p^{p/q}$. Setting $\widetilde{g} := g/\|g\|_q \in L^q(\mu)$ we find $\|\widetilde{g}\|_q \leqslant 1$ as well as

$$\int f \widetilde{g} \, d\mu = \frac{1}{\|g\|_q} \int |f|^p \, d\mu = \frac{1}{\|f\|_p^{p/q}} \|f\|_p^p = \|f\|_p^{(p(1-1/q)} = \|f\|_p.$$

In the last stepe we use $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let $\mathcal{D} \subset L^q(\mu)$ be a dense subset. Since $\mathcal{D} \subset L^q(\mu)$ we obviously have

$$\|f\|_p \geqslant \sup\left\{ \int f g \, d\mu \; : \; g \in \mathcal{D}, \; \|g\|_q \leqslant 1 \right\}.$$

Converesly, let $\epsilon > 0$. Because of (i) there is some $g \in L^q(\mu)$, $\|g\|_q \leqslant 1$ such that

$$\int f g \, d\mu \geqslant \|f\|_p - \epsilon.$$

Since $\mathcal{D}$ is dense, there is some $h \in \mathcal{D}$ with $\|g - h\|_q \leqslant \epsilon$. The Hölder inequality now shows

$$\begin{aligned}
\int f h \, d\mu &= \int f(h-g) \, d\mu + \int f g \, d\mu \\
&\geqslant -\|f\|_p \|h-g\|_q + \int f g \, d\mu \\
&\geqslant -\|f\|_p \epsilon + \int f g \, d\mu \\
&\geqslant -\|f\|_p \epsilon + \|f\|_p - \epsilon \\
&= \|f\|_p (1-\epsilon) - \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ proves the claim.

(iii) If $fg \in L^1(\mu)$ for all $g \in L^q(\mu)$, then $I_f(g) := \int |f| g \, d\mu$ is a positive linear functional on $L^q(\mu)$. From Theorem 21.5 we know that there exists a unique $\widetilde{f} \in L^q(\mu)$ such that

$$I_f(g) = \int \widetilde{f} g \, d\mu \quad \forall g \in L^q(\mu).$$

Therefore, $f = \widetilde{f} \in L^q(\mu)$.

Problem 21.2 Solution:

(i) We use a classical diagonal argument (as in the proof of Theorem 21.18). Let $(g_n)_{n\in\mathbb{N}}$ denote an enumeration of $\mathcal{D}_q$. Hölder's inequality (13.5) tells us

$$\left|\int u_n g_i \, d\mu\right| \leqslant \|u_n\|_p \|g_i\|_q \leqslant \left(\sup_{n\in\mathbb{N}} \|u_n\|_p\right) \|g_i\|_q$$

for all $i, n \in \mathbb{N}$. If $i = 1$, the sequence $(\int u_n g_1 \, d\mu)_{n\in\mathbb{N}}$ is bounded. Therefore, the Bolzano–Weierstraß theorem shows the existence of a subsequence $(u_n^1)_{n\in\mathbb{N}}$ such that the limit

$$\lim_{n\to\infty} \int u_n^1 g_1 \, d\mu$$

exists. We pick recursively subsequences $(u_n^{i+1})_{n\in\mathbb{N}} \subset (u_n^i)_{n\in\mathbb{N}}$ such that the limits

$$\lim_{n\to\infty} \int u_n^{i+1} g_{i+1} \, d\mu$$

exist. Because of the recursive thinning, we see that

$$\lim_{n\to\infty} \int u_n^i g_k \, d\mu$$

exists for all $k = 1, 2, \dots, i$. Thus, for the diagonal sequence $v_n := u_n^n$ the limits $\lim_{n\to\infty} \int v_n g_i \, d\mu$ exist for each $i \in \mathbb{N}$.

(ii) Let $g \in L^q(\mu)$ and $(u_{n(i)})_{i\in\mathbb{N}}$ be the diagonal sequence constructed in (i). Since $\mathbb{R}$ is complete, it is enough to show that $\left(\int u_{n(i)} g \, d\mu\right)_{i\in\mathbb{N}}$ is a Cauchy sequence. Fix $\epsilon > 0$. By assumption, $\mathcal{D}_q$ is dense in $L^q(\mu)$, i.e. there exists some $h \in \mathcal{D}_q$ such that $\|g - h\|_q \leqslant \epsilon$. Part (i) shows that we can take $N \in \mathbb{N}$ with

$$\left|\int u_{n(i)} h \, d\mu - \int u_{n(k)} h \, d\mu\right| \leqslant \epsilon \quad \forall i, k \geqslant N. \tag{$\star$}$$

Hölder's inequality and the triangle inequality show

$$\begin{aligned}
&\left|\int u_{n(i)} g \, d\mu - \int u_{n(k)} g \, d\mu\right| \\
&= \left|\int (u_{n(i)} - u_{n(k)})(g - h) \, d\mu + \int (u_{n(i)} - u_{n(k)}) h \, d\mu\right| \\
&\leqslant \left|\int (u_{n(i)} - u_{n(k)})(g - h) \, d\mu\right| + \underbrace{\left|\int (u_{n(i)} - u_{n(k)}) h \, d\mu\right|}_{\leqslant \epsilon \text{ b/o } (\star)}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \|u_{n(i)} - u_{n(k)}\|_p \|g-h\|_q + \epsilon \\
&\leqslant (\|u_{n(i)}\|_p + \|u_{n(k)}\|_p)\|g-h\|_q + \epsilon \\
&\leqslant 2\sup_{n\in\mathbb{N}} \|u_n\|_p \|g-h\|_q + \epsilon \\
&\leqslant \left(2\sup_{n\in\mathbb{N}} \|u_n\|_p + 1\right)\epsilon
\end{aligned}$$

for any $i, k \geqslant N$. This proves that $\left(\int u_{n(i)} g\,d\mu\right)_{i\in\mathbb{N}}$ is Cauchy.

(iii) Without loss of generality we may assume that the limits

$$I(g) := \lim_{i\to\infty} \int u^+_{n(i)} g\,d\mu, \quad \text{and} \quad J(g) := \lim_{i\to\infty} \int u^-_{n(i)} g\,d\mu$$

exist for all $g \in L^q(\mu)$. *Indeed:* From (i),(ii) we see that there is a subsequence such that $I(g)$ exists for all $g \in L^q(\mu)$. Thinning out this subsequence once again, we see that $J(g)$ exists for all $g \in L^q(\mu)$. Since I and J are positive linear functionals on $L^q(\mu)$, Theorem 21.5 proves that there are unique functions $v, w \in L^q(\mu)$, $v, w \geqslant 0$ representing these functionals:

$$I(g) = \int vg\,d\mu \quad \text{and} \quad J(g) = \int wg\,d\mu.$$

Therefore,

$$\begin{aligned}
\lim_{i\to\infty} \int u_{n(i)} g\,d\mu &= \lim_{i\to\infty} \int u^+_{n(i)} g\,d\mu - \lim_{i\to\infty} \int u^-_{n(i)} g\,d\mu \\
&= \int (v-w) g\,d\mu.
\end{aligned}$$

The claim follows if we use $u := v - w \in L^q(\mu)$.

Problem 21.3 Solution:

(i) By Problem 19.7(i) or 21.4(a), $\widehat{\mu}_k$ is positive semidefinite, i.e. for any choice of $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ and $\xi_1, \dots, \xi_m \in \mathbb{R}^n$ we have

$$\sum_{i,k=1}^m \widehat{\mu}_k(\xi_i - \xi_k)\lambda_i\bar{\lambda}_k \geqslant 0.$$

Since $\lim_{i\to\infty} \widehat{\mu}_i(\xi) = \phi(\xi)$, we see

$$\sum_{i,k=1}^m \phi(\xi_i - \xi_k)\lambda_i\bar{\lambda}_k \geqslant 0.$$

Since $\widehat{\mu}_i(-\xi) = \overline{\widehat{\mu}_i(\xi)}$, this also holds for the limit

$$\phi(-\xi) = \lim_{i\to\infty} \widehat{\mu}_i(-\xi) = \lim_{i\to\infty} \overline{\widehat{\mu}_i(\xi)} = \overline{\phi(\xi)} \quad \forall \xi \in \mathbb{R}^n.$$

This shows that ϕ is positive semidefinite. If $m = 1$ resp. $m = 2$, we see that the matrices

$$\begin{pmatrix} \phi(0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi(0) & \phi(-\xi) \\ \phi(\xi) & \phi(0) \end{pmatrix}$$

are positve hermitian for all $\xi \in \mathbb{R}^n$. Since determinants of positive hermitian matrices are positive, we find $\phi(0) \geqslant 0$ and

$$0 \leqslant \phi(0)^2 - \phi(\xi)\phi(-\xi) = \phi(0)^2 - \phi(\xi)\overline{\phi(\xi)} = \phi(0)^2 - |\phi(\xi)|^2.$$

(ii) First of all we show that the limit exists. Pick $u \in C_c^\infty(\mathbb{R}^n)$. Because of Theorem 19.23, $\mathcal{F}^{-1}u \in \mathcal{S}(\mathbb{R}^n)$ and we can use Plancherel's theorem (Theorem 19.12), to get

$$\int u\,d\mu_i = \int \mathcal{F}(\mathcal{F}^{-1}u)\,d\mu_i = \int \mathcal{F}^{-1}u(\xi)\widehat{\mu}_i(\xi)\,d\xi.$$

Since $|\widehat{\mu}_i(\xi)| \leqslant \widehat{\mu}_i(0) \to \phi(0)$ is uniformly bounded, we can use dominated convergence and find that

$$\Lambda(u) := \lim_{i\to\infty} \int u\,d\mu_i = \int \mathcal{F}^{-1}u(\xi)\phi(\xi)\,d\xi$$

is well-defined. The linearity of Λ follows from the linearity of the integral Moreover, if $u \geqslant 0$, then

$$\Lambda u = \lim_{i\to\infty} \int u\,d\mu_i \geqslant 0.$$

(iii) The continuity of Λ follows from

$$|\Lambda u| \leqslant \limsup_{i\to\infty} \int |u|\,d\mu_i \leqslant \|u\|_\infty \limsup_{i\to\infty} \underbrace{\mu_i(\mathbb{R}^n)}_{(2\pi)^n\widehat{\mu}_i(0)} = (2\pi)^n\phi(0)\|u\|_\infty.$$

Since $C_c^\infty(\mathbb{R}^n)$ is uniformly dense in $C_c(\mathbb{R}^n)$, (see Problem 15.13, the proof resembles the argument of Theorem 15.11), we can extend Λ to a positive linear functional on $C_c(\mathbb{R}^n)$: For $u \in C_c(\mathbb{R}^n)$ we take $(u_i)_{i\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$, such that $\|u_i - u\|_\infty \to 0$. Since

$$|\Lambda(u_i) - \Lambda(u_k)| = |\Lambda(u_i - u_k)| \leqslant (2\pi)^n\phi(0)\|u_i - u_k\|_\infty,$$

we conclude that $(\Lambda u_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, the limit $\Lambda u := \lim_{i\to\infty}\Lambda u_i$ exists and defines a positive linear functional on $C_c(\mathbb{R}^n)$. By Riesz's representation theorem, Theorem 21.8, there exists a unique regular measure representing the functional Λ

$$\Lambda u = \int u\,d\mu \quad \forall u \in C_c(\mathbb{R}^n).$$

(iv) Let $\epsilon > 0$. Since ϕ is continuous at $\xi = 0$, there is some $\delta > 0$ such that

$$|\phi(\xi) - \phi(0)| < \epsilon \quad \forall|\xi| \leqslant \delta.$$

Because of Lévy's truncation inequality, Problem 19.6,

$$\mu_i(\mathbb{R}^n \setminus [-R,R]^n) \leqslant 2(R\pi)^n \int_{[-1/R,1/R]^n} (\widehat{\mu}_i(0) - \operatorname{Re}\widehat{\mu}_i(\xi))\,d\xi$$

(note that $\check{\mu}_i(\xi) = (2\pi)^n\widehat{\mu}_i(-\xi)$). With the dominated convergence theorem we get

$$\limsup_{i\to\infty} \mu_i(\mathbb{R}^n \setminus [-R,R]^n) \leqslant 2(R\pi)^n \int_{[-1/R,1/R]^n} (\phi(0) - \operatorname{Re}\phi(\xi))\,d\xi$$

$$\leqslant 2(2\pi)^n \epsilon$$

for $R \geqslant \frac{1}{\delta}$. In particular we find for $i \geqslant n_0(\epsilon)$, $\mu_i(\mathbb{R}^n \setminus [-R, R]^n) \leqslant 3(2\pi)^n\epsilon$. In order to get $\mu_i(\mathbb{R}^n \setminus [-R, R]^n) \leqslant 3(2\pi)^n\epsilon$ for $i = 1, \dots, n_0(\epsilon)$, we can increase R, if needed.

(v) Let $(\chi_k)_{k\in\mathbb{N}} \subset C_c(\mathbb{R}^n)$ be a sequence of functions such that $0 \leqslant \chi_k \leqslant 1$ and $\chi_k \uparrow \mathbb{1}_{\mathbb{R}^n}$ (use, e.g. Urysohn functions, cf. page 239, or construct the χ_k directly). Because of (iii) we have

$$\int \chi_k \, d\mu = \Lambda(\chi_k) \leqslant (2\pi)^n \phi(0).$$

The monotone convergence theorem shows that μ is a finite measure:

$$\mu(\mathbb{R}^n) = \sup_{k\in\mathbb{N}} \int \chi_k \, d\mu \leqslant (2\pi)^n \phi(0).$$

Moreover, $M := \sup_{i\in\mathbb{N}} \mu_i(\mathbb{R}^n) < \infty$ since $\mu_i(\mathbb{R}^n) = (2\pi)^n \widehat{\mu}_i(0) \to \phi(0)$. It remains to show that μ_i converges weakly to μ. First of all,

$$\int u \, d\mu_i \xrightarrow[i\to\infty]{} \int u \, d\mu \quad \forall u \in C_c(\mathbb{R}^n). \tag{$\star$}$$

Let $u \in C_c(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$, there is a sequence $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\|f_k - u\|_\infty \to 0$. Thus,

$$\begin{aligned}
&\left|\int u \, d\mu_i - \int u \, d\mu\right| \\
&\leqslant \left|\int (u - f_k) \, d\mu_i\right| + \left|\int f_k \, d\mu_i - \int f_k \, d\mu\right| + \left|\int (f_k - u) \, d\mu\right| \\
&\leqslant \|u - f_k\|_\infty \mu_i(\mathbb{R}^n) + \left|\int f_k \, d\mu_i - \int f_k \, d\mu\right| + \|f_k - u\|_\infty \mu(\mathbb{R}^n) \\
&\leqslant \|u - f_k\|_\infty (M + \mu(\mathbb{R}^n)) + \left|\int f_k \, d\mu_i - \int f_k \, d\mu\right| \\
&\xrightarrow[i\to\infty]{\text{(ii)}} \|u - f_k\|_\infty (M + \mu(\mathbb{R}^n)) \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

Assume that $f \in C_b(\mathbb{R}^n)$. For $\epsilon > 0$, Party (iv) shows that there is some $R > 0$ such that with $K := [-R, R]^n$

$$\mu_i(K_n^c) = \mu_i(\mathbb{R}^n \setminus K) \leqslant \epsilon.$$

Without loss of generality we may assume that $\mu(\mathbb{R}^n \setminus K) \leqslant \epsilon$. Pick $\chi \in C_c(\mathbb{R}^n)$, $0 \leqslant \chi \leqslant 1$ and $\chi|_K = 1$. Then

$$\begin{aligned}
&\left|\int f \, d\mu_i - \int f \, d\mu\right| \\
&\leqslant \left|\int f\chi \, d\mu_i - \int f\chi \, d\mu\right| + \left|\int (1-\chi) f \, d\mu_i + \int (1-\chi) f \, d\mu\right| \\
&\leqslant \left|\int f\chi \, d\mu_i - \int f\chi \, d\mu\right| + \|f\|_\infty \left(\int \mathbb{1}_{K^c} \, d\mu_i + \int \mathbb{1}_{K^c} \, d\mu\right) \\
&\leqslant \left|\int f\chi \, d\mu_i - \int f\chi \, d\mu\right| + 2\|f\|_\infty \epsilon.
\end{aligned}$$

Since $f \cdot \chi \in C_c(\mathbb{R}^n)$, the first term on the right vanishes as $i \to \infty$, cf. (iii). So,

$$\limsup_{i\to\infty} \left|\int f \, d\mu_i - \int f \, d\mu\right| \leqslant 2\|f\|_\infty \epsilon \xrightarrow[\epsilon\to 0]{} 0.$$

(vi) Let $(\mu_k)_{k\in\mathbb{N}}$ be a weakly convergent sequence of finite measures. Define $f(x) := e^{-ix\cdot\xi}$, $\xi \in \mathbb{R}^n$, we get

$$\widehat{\mu}_k(\xi) = \frac{1}{(2\pi)^n}\int e^{-ix\cdot\xi}\,d\mu_k(x) \xrightarrow[k\to\infty]{} \frac{1}{(2\pi)^n}\int e^{-ix\cdot\xi}\,\mu(dx) = \widehat{\mu}(\xi),$$

i.e. the Fourier transforms converge pointwise. From part (iv) we know that the sequence $(\mu_k)_{k\in\mathbb{N}}$ is tight. For $\epsilon > 0$ there is some $R > 0$ such that $\mu_k(\mathbb{R}^n \setminus K) \leqslant \epsilon$ for $K := [-R, R]^n$. Without loss of generality we can enlarge R to make sure that $\mu(\mathbb{R}^n \setminus K) \leqslant \epsilon$, too. Because of the (uniform) continuity of the function $\mathbb{R} \ni r \mapsto e^{ir}$ on compact sets, there is some $\delta > 0$ such that

$$|e^{i(\xi-\eta)\cdot x} - 1| \leqslant \epsilon \quad \forall |\xi - \eta| < \delta,\ x \in K.$$

If $k \in \mathbb{N}$, $\xi, \eta \in \mathbb{R}^n$ with $|\xi - \eta| < \delta$, then we see

$$\begin{aligned}
|\widehat{\mu}_k(\xi) - \widehat{\mu}_k(\eta)| &\leqslant \frac{1}{(2\pi)^n}\int |e^{i\xi\cdot x} - e^{i\eta\cdot x}|\,\mu_k(dx) = \frac{1}{(2\pi)^n}\int |e^{i(\xi-\eta)\cdot x} - 1|\,\mu_k(dx)\\
&= \frac{1}{(2\pi)^n}\int_K \underbrace{|e^{i(\xi-\eta)\cdot x} - 1|}_{\leqslant\epsilon}\,\mu_k(dx) + \frac{1}{(2\pi)^n}\int_{K^c} \underbrace{|e^{i(\xi-\eta)\cdot x} - 1|}_{\leqslant 2}\,\mu_k(dx)\\
&\leqslant \frac{\mu_k(\mathbb{R}^n)}{(2\pi)^n}\epsilon + \frac{2}{(2\pi)^n}\mu_i(K^c)\\
&\leqslant \frac{1}{(2\pi)^n}(M+2)\epsilon
\end{aligned}$$

where $M := \sup_{k\in\mathbb{N}} \mu_k(\mathbb{R}^n) < \infty$. This proves the equicontinuity of the sequence $(\widehat{\mu}_k)_{k\in\mathbb{N}}$.

(vii) Let $\xi \in \mathbb{R}^n$ and $\epsilon > 0$. Use equicontinuity of the sequence $(\widehat{\mu}_k)_{k\in\mathbb{N}}$ to pick some $\delta > 0$. Since $\widehat{\mu}$ is continuous, we can ensure that δ is such that

$$|\widehat{\mu}(\xi) - \widehat{\mu}(\eta)| \leqslant \epsilon \quad \forall |\xi - \eta| \leqslant \delta.$$

This entails for all $\eta \in \mathbb{R}^n$ satisfying $|\eta - \xi| \leqslant \delta$:

$$\begin{aligned}
|\widehat{\mu}_k(\eta) - \widehat{\mu}(\eta)| &\leqslant \underbrace{|\widehat{\mu}_k(\eta) - \widehat{\mu}_k(\xi)|}_{\leqslant\epsilon} + |\widehat{\mu}_k(\xi) - \widehat{\mu}(\xi)| + \underbrace{|\widehat{\mu}(\xi) - \widehat{\mu}(\eta)|}_{\leqslant\epsilon}\\
\Rightarrow \sup_{\eta\in B_\delta(\xi)} |\widehat{\mu}_k(\eta) - \widehat{\mu}(\eta)| &\leqslant 2\epsilon + |\widehat{\mu}_k(\xi) - \widehat{\mu}(\xi)| \xrightarrow[k\to\infty]{} 2\epsilon \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}$$

Here we use that $\widehat{\mu}_k$ converges pointwise to $\widehat{\mu}$, cf. (vi). The calculation shows that $\widehat{\mu}_k$ converges locally uniformly to $\widehat{\mu}$. Since locally uniform convergence is the same as uniform convergence on compact sets, we are done.

Problem 21.4 Solution:

(i) Since μ is a finite measure, the continuity of $\hat{\mu}$ follows directly from the continuity lemma, Theorem 12.4 (cf. also 19.3). In order to show positive definiteness, pick $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_m \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. We get

$$\begin{aligned}\sum_{j,k=1}^m \phi(\xi_j - \xi_k)\lambda_j\bar{\lambda}_k &= \frac{1}{(2\pi)^n}\sum_{j,k=1}^m \lambda_j\bar{\lambda}_k \int e^{-ix\cdot(\xi_j-\xi_k)}\,\mu(dx)\\ &= \frac{1}{(2\pi)^n}\sum_{j,k=1}^m \lambda_j\bar{\lambda}_k \int e^{-ix\cdot\xi_j}\overline{e^{-ix\cdot\xi_k}}\,\mu(dx)\\ &= \frac{1}{(2\pi)^n}\int\left(\sum_{j=1}^m \lambda_j e^{-ix\cdot\xi_j}\right)\overline{\left(\sum_{k=1}^m \lambda_k e^{-ix\cdot\xi_k}\right)}\mu(dx)\\ &= \frac{1}{(2\pi)^n}\int\left|\sum_{j=1}^m \lambda_j e^{-ix\cdot\xi_j}\right|^2 \mu(dx) \geqslant 0.\end{aligned}$$

(ii) For $m = 1$ and $\xi = 0$ the definition of positive definiteness implies that the matrix $(\phi(0))$ is positive definite; in particular, $\phi(0) \geqslant 0$.

If we have for a matrix (a_{ik}) that $\sum_{ik} a_{ik}\lambda_i\bar{\lambda}_j \geqslant 0$, then

$$0 \leqslant \sum_{ik} a_{ik}\lambda_i\bar{\lambda}_k = \overline{\sum_{ik} a_{ik}\lambda_i\bar{\lambda}_k} = \sum_{ik}\overline{a_{ik}}\bar{\lambda}_i\lambda_k = \sum_{ki}\overline{a_{ki}}\bar{\lambda}_k\lambda_i$$

which means that $a_{ik} = \overline{a_{ki}}$. Apply this to the matrix $a_{ik} = \phi(\xi_i - \xi_k)$ with $m = 2$ and $\xi_1 = \xi$ and $\xi_2 = 0$ to infer that $\phi(\xi) = \overline{\phi(-\xi)}$. Moreover, the matrix

$$\begin{pmatrix}\phi(0) & \phi(-\xi)\\ \phi(\xi) & \phi(0)\end{pmatrix}$$

is positive semidefinite; in particular its determinant is positive:

$$0 \leqslant \phi(0)^2 - \phi(-\xi)\phi(\xi).$$

Since $\phi(-\xi) = \overline{\phi(\xi)}$, we get the inequality as claimed.

(iii) Because of $|\phi(\xi)| \leqslant \phi(0)$ we see that

$$\begin{aligned}&\left|\iint \phi(\xi-\eta)\left(e^{ix\cdot\xi}e^{-2\epsilon|\xi|^2}\right)\overline{\left(e^{ix\cdot\eta}e^{-2\epsilon|\eta|^2}\right)}\,d\xi\,d\eta\right|\\ &\qquad\leqslant |\phi(0)|\iint\left(e^{-2\epsilon|\xi|^2}e^{-2\epsilon|\eta|^2}\right)d\xi\,d\eta < \infty,\end{aligned}$$

i.e. ν_ϵ is well-defined. Let us show that $\nu_\epsilon \geqslant 0$. For this we cover $\mathbb{R}^n$ with countably many disjoint cubes $(I_i^k)_{i\in\mathbb{N}}$ with side-length $1/k$ and we pick any $\xi_i^k \in I_i^k$. Using the dominated convergence theorem and the positive definiteness of the function ϕ we get

$$\begin{aligned}\nu_\epsilon(x) &= \lim_{k\to\infty}\sum_{m,j\in\mathbb{N}}\int_{I_m^k}\int_{I_j^k}\phi(\xi_m^k - \xi_j^k)\left(e^{ix\cdot\xi_j^k}e^{-2\epsilon|\xi_j^k|^2}\right)\overline{\left(e^{ix\cdot\xi_m^k}e^{-2\epsilon|\xi_m^k|^2}\right)}\,d\xi\,d\eta\\ &= \lim_{k\to\infty}\sum_{m,j\in\mathbb{N}}\phi(\xi_j^k - \xi_m^k)\left(k^{-n}e^{ix\cdot\xi_j^k}e^{-2\epsilon|\xi_j^k|^2}\right)\overline{\left(k^{-n}e^{ix\cdot\xi_j^k}e^{-2\epsilon|\xi_j^k|^2}\right)}\end{aligned}$$

$$\geqslant 0.$$

Because of the parallelogram identity

$$2|\xi|^2 + 2|\eta|^2 = |\xi - \eta|^2 + |\xi + \eta|^2$$

we obtain

$$\begin{aligned} \nu_\epsilon(x) &= \iint \left(e^{ix\cdot\xi} \overline{\left(e^{ix\cdot\eta} e^{-2\epsilon|\eta|^2 - 2\epsilon|\xi|^2} \right)} \right) d\xi\, d\eta \\ &= \iint \left(e^{ix\cdot(\xi-\eta)} e^{-|\xi-\eta|^2 - |\xi+\eta|^2} \right) d\xi\, d\eta. \end{aligned}$$

Changing variables according to

$$\begin{pmatrix} t \\ s \end{pmatrix} := \begin{pmatrix} \xi - \eta \\ \xi + \eta \end{pmatrix} = \begin{pmatrix} \mathrm{id}_n & -\mathrm{id}_n \\ \mathrm{id}_n & \mathrm{id}_n \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} =: A \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

leads to

$$\begin{aligned} \nu_\epsilon(x) &= \frac{1}{|\det A|} \iint \phi(t) e^{ix\cdot t} e^{-\epsilon(|t|^2 + |s|^2)}\, dt\, ds \\ &= \frac{1}{c} \int \phi(t) e^{-\epsilon|t|^2} e^{ix\cdot t}\, dt \\ &= \frac{1}{c} \int \phi_\epsilon(t) e^{ix\cdot t}\, dt. \end{aligned} \tag{$\star$}$$

(iv) Define

$$g_t(x) := \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{2t} \right).$$

Applying Theorem 19.12 for the finite measure $\mu(dx) := e^{-t|x|^2}\, dx$ yields

$$\int \nu_\epsilon(x) e^{-\frac{t}{2}|x|^2}\, dx \overset{(\star)}{=} \frac{1}{c} \int \mathcal{F}^{-1}(\phi_\epsilon)(x) e^{-\frac{t}{2}|x|^2}\, dx = \frac{1}{c} \int \phi_\epsilon(\xi) \mathcal{F}^{-1}(e^{-\frac{t}{2}|\cdot|^2})(\xi)\, d\xi$$

for all $t > 0$ (observe: $\phi_\epsilon \in L^1(\mathbb{R}^n)$). Example 19.2(iii) shows $\mathcal{F}(g_t)(x) = (2\pi)^{-n} \exp(-t|x|^2/2)$. Therefore, $\mathcal{F}^{-1}(e^{-\frac{t}{2}|\cdot|^2})(\xi) = (2\pi)^n g_t(\xi)$. Since $|\phi(\xi)| \leqslant \phi(0)$ and $\int g_t(x)\, dx = 1$ we thus get

$$\int \nu_\epsilon(x) e^{-\frac{t}{2}|x|^2}\, dx = \frac{(2\pi)^n}{c} \int \phi_\epsilon(\xi) g_t(\xi)\, d\xi \leqslant \frac{(2\pi)^n}{c} \phi(0).$$

Fatou's lemma (Theorem 9.11) finally shows

$$\begin{aligned} \int \nu_\epsilon(x)\, dx &= \int \lim_{k\to\infty} \nu_\epsilon(x) e^{-\frac{1}{2k}|x|^2}\, dx \\ &\leqslant \liminf_{k\to\infty} \int \nu_\epsilon(x) e^{-\frac{1}{2k}|x|^2}\, dx \\ &\leqslant \frac{(2\pi)^n}{c} \phi(0). \end{aligned}$$

Since $\nu_\epsilon \geqslant 0$, see (iii), this means that $\nu_\epsilon \in L^1(\mathbb{R}^n)$.

(v) Parts (iii) and (iv) show that $\widehat{\mu}_\epsilon = \phi_\epsilon$ for the finite measure $\mu_\epsilon(dx) := c\nu_\epsilon(x)\,dx$. Since $\phi_\epsilon \to \phi$, Lévy's continuity theorem (Problem 21.3) shows that there exists a measure μ which is the weak limit of the family μ_ϵ as $\epsilon \to 0$ and $\widehat{\mu} = \phi$.

Problem 21.5 Solution:

(i) Since uniform convergence preserves continuity, we see that every $u \in \overline{C_c(X)}$ is continuous. By construction, the set $\{|u| \geqslant \epsilon\}$ is compact since there is some $u_\epsilon \in C_c(X)$ such that $\|u - u_\epsilon\|_\infty < \epsilon$. This means that u vanishes at infinity. In particular $\overline{C_c(X)} \subset C_\infty(X)$.

Conversely, if $u \in C_\infty(X)$ and $\epsilon > 0$, there is some compact set K_ϵ such that $|u| \leqslant \epsilon$ outside of K_ϵ. Now we use Urysohn's lemma and construct a function $\chi_\epsilon \in C_c(X)$ such that $\mathbb{1}_{K_\epsilon} \leqslant \chi_\epsilon \leqslant 1$. Then we get $u_\epsilon := \chi_\epsilon u \in C_c(X)$ as well as

$$|u - u_\epsilon| = (1 - \chi_\epsilon)|u| \leqslant \epsilon$$

uniformly for all x.

(ii) It is obvious that $C_\infty(X)$ is a vector space and that $\|\cdot\|_\infty$ is a norm in this space. The completeness follows from part (i) since $\overline{C_\infty(X)} = \overline{\overline{C_c(X)}} = \overline{C_c(X)}$.

(iii) Let $u \in C_\infty(X)$ and $\epsilon > 0$. Urysohn's lemma shows that there is a $\chi \in C_c(X)$, $0 \leqslant \chi \leqslant 1$, such that $|u| \leqslant \epsilon$ on the set $\{\chi < 1\} = \{\chi = 1\}^c$. Therefore,

$$\begin{aligned}
&\left|\int u\,d\mu_n - \int u\,d\mu\right| \\
&\leqslant \left|\int u\chi\,d\mu_n - \int u\chi\,d\mu\right| + \left|\int u(1-\chi)\,d\mu_n - \int u(1-\chi)\,d\mu\right| \\
&\leqslant \left|\int u\chi\,d\mu_n - \int u\chi\,d\mu\right| + \epsilon\left[\mu_n(X) + \mu(X)\right] \\
&\overset{21.16}{\leqslant} \left|\int u\chi\,d\mu_n - \int u\chi\,d\mu\right| + 2\epsilon \sup_{m\in\mathbb{N}} \mu_m(X).
\end{aligned}$$

Since $u\chi \in C_c(X)$, we find as $n \to \infty$

$$\limsup_{n\to\infty} \left|\int u\,d\mu_n - \int u\,d\mu\right| \leqslant 2\epsilon \sup_{m\in\mathbb{N}} \mu_m(X) \xrightarrow[\epsilon\to 0]{} 0.$$

Problem 21.6 Solution:

(i) First we consider $u \in C_c^\infty(\mathbb{R}^n)$. According to Theorem 19.23, $\mathcal{F}^{-1}u \in \mathcal{S}(\mathbb{R}^n)$, and Plancherel's theorem (Theorem 19.12) gives

$$\int u\,d\mu_i = \int \mathcal{F}(\mathcal{F}^{-1}u)\,d\mu_i = \int \mathcal{F}^{-1}u(\xi)\widehat{\mu}_i(\xi)\,d\xi.$$

Since $|\widehat{\mu}_i(\xi)| \leqslant \widehat{\mu}_i(0) \to \phi(0)$ is uniformly bounded, we can use the dominated convergence theorem to see

$$\Lambda(u) := \lim_{i\to\infty} \int u\,d\mu_i = \int \mathcal{F}^{-1}u(\xi)\phi(\xi)\,d\xi$$

i.e. $\Lambda(u)$ is well-defined. Moreover,

$$\mu_i(\mathbb{R}^n) = (2\pi)^n \widehat{\mu}_i(0) \xrightarrow[i\to\infty]{} (2\pi)^n \phi(0),$$

i.e. $M := \sup_i \mu_i(\mathbb{R}^n) < \infty$. Assume now that $u \in C_c(X)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ (with respect to uniform convergence, cf. Problem 15.13), there is a sequence $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_\infty \to 0$. Thus,

$$\begin{aligned}
&\left|\int u\,d\mu_i - \int u\,d\mu_j\right| \\
&\qquad \leqslant \left|\int (u-u_k)\,d\mu_i\right| + \left|\int (u-u_k)\,d\mu_j\right| + \left|\int u_k\,d\mu_i - \int u_k\,d\mu_j\right| \\
&\qquad \leqslant \|u-u_k\|_\infty \big(\mu_i(\mathbb{R}^n) + \mu_j(\mathbb{R}^n)\big) + \left|\int u_k\,d\mu_i - \int u_k\,d\mu_j\right| \\
&\qquad \leqslant 2\|u-u_k\|_\infty M + \left|\int u_k\,d\mu_i - \int u_k\,d\mu_j\right| \\
&\qquad \xrightarrow[i,j\to\infty]{} 2\|u-u_k\|_\infty M \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

This shows that $\big(\int u\,d\mu_i\big)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Thus, the limit $\Lambda(u) := \lim_{i\to\infty}\int u\,d\mu_i$ exists. Since convergent sequences are bounded, we see

$$\sup_{i\in\mathbb{N}} \left|\int u\,d\mu_i\right| < \infty.$$

Since $u \in C_c(\mathbb{R}^n) \implies |u| \in C_c(\mathbb{R}^n)$, we get

$$\sup_{n\in\mathbb{N}} \int |u|\,d\mu_i < \infty \quad \forall u \in C_c(\mathbb{R}^n),$$

i.e. the sequence $(\mu_i)_{i\in\mathbb{N}}$ is vaguely bounded. According to Theorem 21.18, $(\mu_i)_{i\in\mathbb{N}}$ has a vaguely convergent subsequence $\mu_{n(i)} \to \mu$.

(ii) We can use part (i) for any subsequence of $(\mu_i)_{i\in\mathbb{N}}$. We will show the the subsequential limits do not depend on the subsequence. Pick any two subsequences $(\mu_{n(i)})_{i\in\mathbb{N}}$ and $(\mu_{m(i)})_{i\in\mathbb{N}}$ of $(\mu_i)_{n\in\mathbb{N}}$ and assume that $\mu_{n(i)} \xrightarrow{\text{v}} \mu$, $\mu_{m(i)} \xrightarrow{\text{v}} \nu$. By definition, we find for all $u \in C_c(\mathbb{R}^n)$

$$\begin{aligned}
\lim_{i\to\infty}\int u\,d\mu_{n(i)} &= \int u\,d\mu, \\
\lim_{i\to\infty}\int u\,d\mu_{m(i)} &= \int u\,d\nu.
\end{aligned}$$

On the other hand, we have seen in (i) that $\Lambda(u) = \lim_{i\to\infty}\int u\,d\mu_i$. Thus,

$$\int u\,d\mu = \Lambda(u) = \int u\,d\nu.$$

Since this holds for all $u \in C_c(\mathbb{R}^n)$, we can use the regularity of the measures μ and ν to conclude that $\mu = \nu$. Since the limit does not depend on the subsequence, we already have vague convergence of the *full* sequence $(\mu_i)_{i\in\mathbb{N}}$. (Compare this with the following subsequence principle: A sequence $(a_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ converges if, and only if, every subsequence of $(a_i)_{i\in\mathbb{N}}$ has a convergent subsequence, and all subsequential limits coincide.)

(iii) In view of Theorem 21.17 it is enough to show that the sequence $(\mu_i)_{i\in\mathbb{N}}$ is tight

Fix $\epsilon > 0$. Since ϕ is continuous at $\xi = 0$, there is some $\delta > 0$ such that

$$|\phi(\xi) - \phi(0)| < \epsilon \quad \forall |\xi| \leqslant \delta.$$

From Lévy's truncation inequality, Problem 19.6, we get

$$\mu_i(\mathbb{R}^n \setminus [-R, R]^n) \leqslant 2(R\pi)^n \int_{[-1/R,1/R]^n} (\widehat{\mu}_i(0) - \operatorname{Re} \widehat{\mu}_i(\xi))\, d\xi$$

(observe, that $\check{\mu}_i(\xi) = (2\pi)^n \widehat{\mu}_i(-\xi)$). Now we can use dominated convergence to get

$$\begin{aligned}\limsup_{i\to\infty} \mu_i(\mathbb{R}^n \setminus [-R, R]^n) &\leqslant 2(R\pi)^n \int_{[-1/R,1/R]^n} (\phi(0) - \operatorname{Re} \phi(\xi))\, d\xi \\ &\leqslant 2(2\pi)^n \epsilon\end{aligned}$$

for all $R \geqslant \frac{1}{\delta}$. In particular, we find $\mu_i(\mathbb{R}^n \setminus [-R, R]^n) \leqslant 3(2\pi)^n\epsilon$ for $i \geqslant n_0(\epsilon)$. In order to ensure $\mu_i(\mathbb{R}^n \setminus [-R, R]^n) \leqslant 3(2\pi)^n\epsilon$ for $i = 1, \dots, n_0(\epsilon)$, we can enlarge R, if need be.

Problem 21.7 Solution: Since

$$\int_B u\, d\mu_n = \int_{B\cap \operatorname{supp} u} u\, d\mu_n$$

we can assume, without loss of generality, that B is contained in a compact set. Denote by $K := \overline{B}$ the closure of B and by $U := B^\circ$ the open interior of B. Moreover, we can assume that $u \geqslant 0$ – otherwise we consider $u^\pm$ separately.

According to Urysohn's lemma (Lemma B.2 or (21.6) & (21.7)), there are sequences $(w_k)_{k\in\mathbb{N}} \subset C_c(X)$, $(v_k)_{k\in\mathbb{N}} \subset C_c(X)$, $0 \leqslant v_k \leqslant 1$, $0 \leqslant w_k \leqslant 1$, with $w_k \uparrow \mathbb{1}_U$ and $v_k \downarrow \mathbb{1}_K$. By assumption $\mu_n \xrightarrow{\text{v}} \mu$ and so

$$\int_B u\, d\mu_n \leqslant \int_K u\, d\mu_n \leqslant \int u \cdot v_k\, d\mu_n \xrightarrow[n\to\infty]{} \int u \cdot v_k\, d\mu.$$

Beppo Levi's theorem implies

$$\limsup_{n\to\infty} \int_B u\, d\mu_n \leqslant \inf_{k\in\mathbb{N}} \int u \cdot v_k\, d\mu = \int_K u\, d\mu.$$

Similarly, we get from

$$\int_B u\,d\mu_n \geqslant \int_U u\,d\mu_n \geqslant \int u\cdot w_k\,d\mu_n \xrightarrow[n\to\infty]{} \int u\cdot w_k\,d\mu.$$

and the monotone convergence theorem

$$\liminf_{n\to\infty}\int_B u\,d\mu_n \geqslant \sup_{k\in\mathbb{N}}\int u\cdot w_k\,d\mu = \int_U u\,d\mu.$$

Finally, since $\mu(K\setminus U)=\mu(\partial B)=0$, we see that

$$\limsup_{n\to\infty}\int_B u\,d\mu_n \leqslant \int_K u\,d\mu = \int_U u\,d\mu \leqslant \liminf_{n\to\infty}\int_B u\,d\mu_n.$$

■■

22 Uniform integrability and Vitali's convergence theorem. Solutions to Problems 22.1–22.17

Problem 22.1 Solution: First, observe that

$$\lim_j u_j(x) = 0 \iff \lim_j |u_j(x)| = 0.$$

Thus,

$$\begin{aligned}
x \in \{\lim_j u_j = 0\} &\iff \forall\, \epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad \forall\, j \geqslant N_\epsilon \,:\, |u_j(x)| \leqslant \epsilon \\
&\iff \forall\, \epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad :\, \sup_{j \geqslant N_\epsilon} |u_j(x)| \leqslant \epsilon \\
&\iff \forall\, \epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad :\, x \in \{\sup_{j \geqslant N_\epsilon} |u_j| \leqslant \epsilon\} \\
&\iff \forall\, \epsilon > 0 \,:\, x \in \bigcup_{N \in \mathbb{N}} \{\sup_{j \geqslant N} |u_j| \leqslant \epsilon\} \\
&\iff \forall\, k \in \mathbb{N} \,:\, x \in \bigcup_{N \in \mathbb{N}} \{\sup_{j \geqslant N} |u_j| \leqslant 1/k\} \\
&\iff x \in \bigcap_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \{\sup_{j \geqslant N} |u_j| \leqslant 1/k\}.
\end{aligned}$$

Equivalently,

$$\{\lim_j u_j = 0\}^c = \bigcup_{k \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \{\sup_{j \geqslant N} |u_j| > 1/k\}.$$

By assumption and the continuity of measures,

$$\mu\Big(\bigcap_{N \in \mathbb{N}} \{\sup_{j \geqslant N} |u_j| > 1/k\} \Big) = \lim_N \mu\Big(\{\sup_{j \geqslant N} |u_j| > 1/k\} \Big) = 0$$

and, since countable unions of null sets are again null sets, we conclude that

$$\{\lim_j u_j = 0\} \quad \text{has full measure.}$$

■■

Problem 22.2 Solution: Note that

$$x \in \Big\{ \sup_{j \geqslant k} |u_j| > \epsilon \Big\} \iff \sup_{j \geqslant k} |u_j(x)| > \epsilon$$

$$\iff \exists\, j \geqslant k \,:\, |u_j(x)| > \epsilon$$
$$\iff x \in \bigcup_{j\geqslant k}\{|u_j| > \epsilon\}$$

and since

$$\bigcup_{j\geqslant k}\{|u_j| > \epsilon\} \downarrow \bigcap_{k\in\mathbb{N}}\bigcup_{j\geqslant k}\{|u_j| > \epsilon\} \stackrel{\text{def}}{=} \limsup_{j\to\infty}\{|u_j| > \epsilon\}$$

we can use the continuity of measures to get

$$\lim_k \mu\Big(\sup_{j\geqslant k}|u_j| > \epsilon\Big) = \lim_k \mu\Big(\bigcup_{j\geqslant k}\{|u_j| > \epsilon\}\Big) = \mu\Big(\bigcap_{k\in\mathbb{N}}\bigcup_{j\geqslant k}\{|u_j| > \epsilon\}\Big).$$

This, and the result of Problem 22.1 show that either of the following two equivalent conditions

$$\lim_{k\to\infty}\mu\Big(\sup_{j\geqslant k}|u_j| \geqslant \epsilon\Big) = 0 \qquad \forall\,\epsilon > 0;$$
$$\mu\Big(\limsup_{j\to\infty}\{|u_j| \geqslant \epsilon\}\Big) = 0 \qquad \forall\,\epsilon > 0;$$

ensure the almost everywhere convergence of $\lim_j u_j(x) = 0$.

Problem 22.3 Solution:

- Assume first that $u_j \to u$ in μ-measure, that is,

  $$\forall\,\epsilon > 0, \quad \forall\, A \in \mathscr{A},\ \mu(A) < \infty \,:\, \lim_j \mu\big(\{|u_j - u| > \epsilon\} \cap A\big) = 0.$$

  Since

  $$|u_j - u_k| \leqslant |u_j - u| + |u - u_k| \quad \forall j,k \in \mathbb{N}$$

  we see that

  $$\{|u_j - u_k| > 2\epsilon\} \subset \{|u_j - u| > \epsilon\} \cup \{|u - u_k| > \epsilon\}$$

  (since, otherwise $|u_j - u_k| \leqslant \epsilon + \epsilon = 2\epsilon$). Thus, we get for every measurable set A with finite μ-measure that

  $$\begin{aligned}&\mu\big(\{|u_j - u_k| > 2\epsilon\} \cap A\big)\\ &\quad\leqslant \mu\big[(\{|u_j - u| > \epsilon\} \cap A) \cup (\{|u_k - u| > \epsilon\} \cap A)\big]\\ &\quad\leqslant \mu\big[\{|u_j - u| > \epsilon\} \cap A\big] + \mu\big[\{|u_k - u| > \epsilon\} \cap A\big]\end{aligned}$$

  and each of these terms tend to infinity as $j,k \to \infty$.

- Assume now that $|u_j - u_k| \to 0$ in μ-measure as $j,k \to \infty$. Let $(A_\ell)_\ell$ be an exhausting sequence such that $A_\ell \uparrow X$ and $\mu(A_j) < \infty$.

  The problem is to identify the limiting function.

Fix ℓ. By assumption, we can choose $N_j \in \mathbb{N}$, $j \in \mathbb{N}$, such that

$$\forall\, m, n \geqslant N_j \;:\; \mu\big(\{|u_m - u_n| > 2^{-j}\} \cap A_\ell\big) < 2^{-j}.$$

(Note that N_j may depend on ℓ, but we suppress this dependency as ℓ is fixed.) By enlarging N_j, if needed, we can always assume that

$$N_1 < N_2 < \cdots < N_j < N_{j+1} \to \infty.$$

Consequently, there is an exceptional set $E_j \subset A_\ell$ with $\mu(E_j \cap A_\ell) < 2^{-j}$ such that

$$|u_{N_{j+1}}(x) - u_{N_j}(x)| \leqslant 2^{-j} \quad \forall\, x \in A_\ell \setminus E_j$$

and, if $E_i^* := \bigcup_{j\geqslant i} E_j$ we have $\mu(E_i \cap A_\ell) \leqslant 2 \cdot 2^{-i}$ as well as

$$|u_{N_{j+1}}(x) - u_{N_j}(x)| \leqslant 2^{-j} \quad \forall\, j \geqslant i, \quad \forall\, x \in A_\ell \setminus E_i^*.$$

This means that

$$\sum_j (u_{N_{j+1}} - u_{N_j}) \text{ converges uniformly for } x \in A_\ell \setminus E_i^*$$

so that $\lim_j u_{N_j}$ exists uniformly on $A_\ell \setminus E_i^*$ for all i. Since $\mu(E_i^* \cap A_\ell) < 2 \cdot 2^{-i}$ we conclude that

$$\lim_j u_{N_j} \mathbb{1}_{A_\ell} = u^{(\ell)} \mathbb{1}_{A_\ell} \text{ exists almost everywhere}$$

for some $u^{(\ell)}$. Since, however, a.e. limits are unique (up to a null set, that is) we know that $u^{(\ell)} = u^{(m)}$ a.e. on $A_\ell \cap A_m$ so that there is a (up to null sets) unique limit function u satisfying

$$\lim_j u_{N_j} = u \text{ exists a.e., hence in measure by Lemma 22.4.} \tag{*}$$

Thus, we have found a candidate for the limit of our Cauchy sequence. In fact, since

$$|u_k - u| \leqslant |u_k - u_{N_j}| + |u_{N_j} - u|$$

we have

$$\begin{aligned}
&\mu(\{|u_k - u| > \epsilon\} \cap A_\ell) \\
&\qquad \leqslant \mu(\{|u_k - u_{N_j}| > \epsilon\} \cap A_\ell) + \mu(\{|u_{N_j} - u| > \epsilon\} \cap A_\ell)
\end{aligned}$$

and the first expression on the right-hand side tends to zero (as $k, N(j) \to \infty$) because of the assumption, while the second term tends to zero (as $N(j) \to \infty$) because of (*))

Problem 22.4 Solution:

(i) This sequence converges in measure to $f \equiv 0$ since for $\epsilon \in (0,1)$

$$\lambda(|f_{n,j}| > \epsilon) = \lambda[(j-1)/n, j/n] = \tfrac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

This means, however, that potential a.e. and $\mathcal{L}^p$-limits must be $f \equiv 0$, too. Since for every x

$$\liminf f_{n,j}(x) = 0 < \infty = \limsup f_{n,j}$$

the sequence cannot converge at any point.

Also the $\mathcal{L}^p$-limit (if $p \geqslant 1$) does not exist, since

$$\int |f_{n,j}|^p \, d\lambda = n^p \lambda[(j-1)/n, j/n] = n^{p-1}.$$

(ii) As in (i) we see that $g_n \xrightarrow{\mu} g \equiv 0$. Similarly,

$$\int |g_n|^p \, d\mu = n^p \lambda(0, 1/n) = n^{p-1}$$

so that the $\mathcal{L}^p$-limit does not exist. The pointwise limit, however, exists since

$$\lim_{n\to\infty} n\mathbb{1}_{(0,n)}(x) = 0.$$

for every $x \in (0,1)$.

(iii) The shape of g_n is that of a triangle with base $[0, 1/n]$. Thus, for every $\epsilon > 0$,

$$\lambda(|h_n| > \epsilon) \leqslant \lambda[0, 1/n] = \tfrac{1}{n}$$

which shows that $h_n \xrightarrow{\mu} h \equiv 0$. This must be, if the respective limits exist, also the limiting function for a.e. and $\mathcal{L}^p$-convergence. Since

$$\int |h_n|^p \, d\lambda = a_n^p \tfrac{1}{2} \lambda[0, 1/n] = \frac{a_n^p}{2n}$$

we have $\mathcal{L}^p$-convergence if, and only if, the sequence a_n^p/n tends to zero as $n \to \infty$.

We have, however, always a.e. convergence since the support of the function h_n is $[0, 1/n]$ and this shrinks to $\{0\}$ which is a null set. Thus,

$$\lim_n a_n(1 - nx)^+ = 0$$

except, possibly, at $x = 0$.

Problem 22.5 Solution: We claim that

(i) $au_j + bw_j \to au + bw$;

(ii) $\max(u_j, w_j) \to \max(u, w)$;

(iii) $\min(u_j, w_j) \to \min(u, w)$;

(iv) $|u_j| \to |u|$.

Note that

$$|au_j + bw_j - au - bw| \leqslant |a||u_j - u| + |b||w_j - w|$$

so that

$$\{|au_j + bw_j - au - bw| > 2\epsilon\} \subset \{|u_j - u| > \epsilon/|a|\} \cup \{|w_j - w| > \epsilon/|b|\}.$$

This proves the first limit.

Since, by the lower triangle inequality,

$$||u_j| - |u|| \leqslant |u_j - u|$$

we get

$$\{||u_j| - |u|| > \epsilon\} \subset \{|u_j - u| > \epsilon\}$$

and $|u_j| \to |u|$ follows.

Finally, since

$$\max u_j, w_j = \frac{1}{2}\Big(u_j + w_j + |u_j - w_j|\Big)$$

we get $\max u_j, w_j \to \max u, w$ by using rules (i) and (iv) several times. The minimum is treated similarly.

Problem 22.6 Solution: The hint is somewhat misleading since this construction is not always possible (or sensible). Just imagine $\mathbb{R}$ with the counting measure. Then $X_{\sigma f}$ would be all of $\mathbb{R}$... What I had in mind when giving this hint was a construction along the following lines:

Consider Lebesgue measure λ in $\mathbb{R}$ and define $f := \mathbb{1}_F + \infty\mathbb{1}_{F^c}$ where $F = [-1, 1]$ (or any other set of finite Lebesgue measure). Then $\mu := f \cdot \lambda$ is a not σ-finite measure. Moreover, Take any sequence $u_n \xrightarrow{\lambda} u$ converging in λ-measure. Then

$$\mu(\{|u_n - u| > \epsilon\} \cap A) = \lambda(\{|u_n - u| > \epsilon\} \cap A)$$

since all sets A with $\mu(A) < \infty$ are contained in F and $\lambda(F) = \mu(F) < \infty$. Thus, $u_n \xrightarrow{\mu} u$.

However, changing u *arbitrarily* on $\mathbb{1}_{F^c}$ also yields a limit point in μ-measure since, as mentioned above, all sets of finite μ-measure are within F.

This pathology cannot happen in a σ-finite measure space, cf. Lemma 22.6.

Problem 22.7 Solution:

(i) Fix $\epsilon > 0$. Then

$$\begin{aligned}\int |u - u_j| \, d\mu &= \int_A |u - u_j| \, d\mu \\ &= \int_{A\cap\{|u-u_j|\leqslant\epsilon\}} |u - u_j| \, d\mu + \int_{A\cap\{|u-u_j|>\epsilon\}} |u - u_j| \, d\mu \\ &\leqslant \int_{A\cap\{|u-u_j|\leqslant\epsilon\}} \epsilon \, d\mu + \int_{A\cap\{|u-u_j|>\epsilon\}} (|u| + |u_j|) \, d\mu \\ &\leqslant \epsilon\mu(A) + 2C\,\mu\big(A \cap \{|u - u_j| > \epsilon\}\big) \\ &\xrightarrow[j\to\infty]{} \epsilon\mu(A) \\ &\xrightarrow[\epsilon\to 0]{} 0.\end{aligned}$$

(ii) Note that u_j converges almost everywhere and in λ-measure to $u \equiv 0$. However,

$$\int |u_j| \, d\lambda = \lambda[j, j+1] = 1 \neq 0$$

so that the limit—if it exists—cannot be $u \equiv 0$. Since this is, however, the canonical candidate, we conclude that there is no $\mathcal{L}^1$ convergence.

(iii) The limit depends on the set A which is fixed. This means that we are, essentially, dealing with a finite measure space.

Problem 22.8 Solution: A pseudo-metric is symmetric (d_2) and satisfies the triangle inequality (d_3).

(i) First we note that $\rho_\mu(\xi, \eta) \in [0, 1]$ is well-defined. That it is symmetric (d_2) is obvious. For the triangle inequality we observe that for three random variables ξ, η, ζ and numbers $\epsilon, \delta > 0$ we have

$$|\xi - \zeta| \leqslant |\xi - \eta| + |\eta - \zeta|$$

implying that

$$\{|\xi - \zeta| > \epsilon + \delta\} \subset \{|\xi - \eta| > \epsilon\} \cup \{|\eta - \zeta| > \delta\}$$

so that

$$\mathbb{P}(|\xi - \zeta| > \epsilon + \delta) \leqslant \mathbb{P}(|\xi - \eta| > \epsilon) + \mathbb{P}(|\eta - \zeta| > \delta).$$

If $\epsilon > \rho_{\mathbb{P}}(\xi, \eta)$ and $\delta > \rho_{\mathbb{P}}(\eta, \zeta)$ we find

$$\mathbb{P}(|\xi - \zeta| > \epsilon + \delta) \leqslant \mathbb{P}(|\xi - \eta| > \epsilon) + \mathbb{P}(|\eta - \zeta| > \delta) \leqslant \epsilon + \delta$$

which means that

$$\rho_{\mathbb{P}}(\xi, \zeta) \leqslant \epsilon + \delta.$$

Passing to the infimum of all possible δ- and ϵ-values we get

$$\rho_{\mathbb{P}}(\xi, \zeta) \leqslant \rho_{\mathbb{P}}(\xi, \eta) + \rho_{\mathbb{P}}(\eta, \zeta).$$

(ii) Assume first that $\rho_{\mathbb{P}}(\xi_j,\xi)\xrightarrow[j\to\infty]{}0$. Then

$$\begin{aligned}\rho_{\mathbb{P}}(\xi_j,\xi)\xrightarrow[j\to\infty]{}0 &\iff \exists\,(\epsilon_j)_j\subset\mathbb{R}_+ : \mathbb{P}(|\xi-\xi_j|>\epsilon_j)\leqslant\epsilon_j\\ &\Rightarrow\ \forall\,\epsilon>\epsilon_j : \mathbb{P}(|\xi-\xi_j|>\epsilon)\leqslant\epsilon_j.\end{aligned}$$

Thus, for given $\epsilon>0$ we pick $N=N(\epsilon)$ such that $\epsilon>\epsilon_j$ for all $j\geqslant N$ (possible as $\epsilon_j\to 0$). Then we find

$$\forall\,\epsilon>0\ \exists\,N(\epsilon)\in\mathbb{N}\ \forall\,j\geqslant N(\epsilon) : \mathbb{P}(|\xi-\xi_j|>\epsilon)\leqslant\epsilon_j;$$

this means, however, that $\mathbb{P}(|\xi-\xi_j|>\epsilon)\xrightarrow[j\to\infty]{}0$ for any choice of $\epsilon>0$.

Conversely, assume that $\xi_j\xrightarrow{\mathbb{P}}0$. Then

$$\begin{aligned}&\forall\,\epsilon>0 : \lim_j\mathbb{P}(|\xi-\xi_j|>\epsilon)=0\\ &\iff\ \forall\,\epsilon,\delta>0\ \exists\,N(\epsilon,\delta)\ \forall\,j\geqslant N(\epsilon,\delta) : \mathbb{P}(|\xi-\xi_j|>\epsilon)<\delta\\ &\Rightarrow\ \forall\,\epsilon>0\ \exists\,N(\epsilon)\ \forall\,j\geqslant N(\epsilon) : \mathbb{P}(|\xi-\xi_j|>\epsilon)<\epsilon\\ &\Rightarrow\ \forall\,\epsilon>0\ \exists\,N(\epsilon)\ \forall\,j\geqslant N(\epsilon) : \rho_{\mathbb{P}}(\xi,\xi_j)\leqslant\epsilon\\ &\Rightarrow\ \lim_j\rho_{\mathbb{P}}(\xi,\xi_j)=0.\end{aligned}$$

(iii) We have

$$\begin{aligned}\rho(\xi_j,\xi_k)\xrightarrow[j,k\to\infty]{}0 &\overset{\text{(ii)}}{\iff} \xi_j-\xi_k\xrightarrow[j,k\to\infty]{\mathbb{P}}0\\ &\overset{\text{P22.3}}{\iff} \exists\,\xi : \xi_k\xrightarrow[k\to\infty]{\mathbb{P}}\xi\\ &\overset{\text{(ii)}}{\iff} \exists\,\xi : \rho(\xi,\xi_k)\xrightarrow[k\to\infty]{}0\end{aligned}$$

(iv) Note that for $x,y>0$

$$\frac{x+y}{1+x+y}=\frac{x}{1+x+y}+\frac{y}{1+x+y}\leqslant\frac{x}{1+x}+\frac{y}{1+y}$$

and

$$(x+y)\wedge 1=\begin{cases}x+y=(x\wedge 1)+(y\wedge 1) & \text{if } x+y\leqslant 1;\\ 1\leqslant(x\wedge 1)+(y\wedge 1) & \text{if } x+y\geqslant 1.\end{cases}$$

This means that both $g_{\mathbb{P}}$ and $d_{\mathbb{P}}$ satisfy the triangle inequality, that is (d_3). Symmetry, i.e. (d_2), is obvious.

Moreover, since for all $x\geqslant 0$

$$\frac{x}{1+x}\leqslant x\wedge 1\leqslant 2\,\frac{x}{1+x}$$

(consider the cases $x \leqslant 1$ and $x \geqslant 1$ separately), we have

$$g_{\mathbb{P}}(\xi,\eta) \leqslant d_{\mathbb{P}}(\xi,\eta) \leqslant 2g_{\mathbb{P}}(\xi,\eta)$$

which shows that $g_{\mathbb{P}}$ and $d_{\mathbb{P}}$ have the same Cauchy sequences. Moreover, for all $\epsilon \leqslant 1$,

$$\begin{aligned}\mathbb{P}(|\xi-\eta|>\epsilon) &= \mathbb{P}(|\xi-\eta|\wedge 1>\epsilon)\\ &\leqslant \frac{1}{\epsilon}\int |\xi-\eta|\wedge 1\,d\mathbb{P}\\ &= \frac{1}{\epsilon}\,d_{\mathbb{P}}(\xi,\eta)\end{aligned}$$

so that (because of (iii)) any $d_{\mathbb{P}}$ Cauchy sequence is a $\rho_{\mathbb{P}}$ Cauchy sequence. And since for all $\epsilon \leqslant 1$ also

$$\begin{aligned}d_{\mathbb{P}}(\xi,\eta) &= \int_{|\xi-\eta|>\epsilon} |\xi-\eta|\wedge 1\,d\mathbb{P} + \int_{|\xi-\eta|\leqslant\epsilon} |\xi-\eta|\wedge 1\,d\mathbb{P}\\ &\leqslant \int_{|\xi-\eta|>\epsilon} 1\,d\mathbb{P} + \int_{|\xi-\eta|\leqslant\epsilon} \epsilon\,d\mathbb{P}\\ &\leqslant \mathbb{P}(|\xi-\eta|>\epsilon)+\epsilon,\end{aligned}$$

all $\rho_{\mathbb{P}}$ Cauchy sequences are $d_{\mathbb{P}}$ Cauchy sequences, too.

Problem 22.9 Solution:

(i) Fix $\epsilon > 0$. We have

$$\begin{aligned}\int |u_n-u|\wedge \mathbb{1}_A\,d\mu &= \int_{\{|u_n-u|\leqslant\epsilon\}} |u_n-u|\wedge \mathbb{1}_A\,d\mu + \int_{\{|u_n-u|>\epsilon\}} |u_n-u|\wedge \mathbb{1}_A\,d\mu\\ &\leqslant \epsilon\mu(A) + \mu(\{|u_n-u|>\epsilon\}\cap A).\end{aligned}$$

Letting first $n\to\infty$ and then $\epsilon\to 0$ yields

$$\limsup_n \int |u_n-u|\wedge \mathbb{1}_A\,d\mu \leqslant \epsilon\mu(A) \xrightarrow[\epsilon\to 0]{} 0.$$

(ii) WLOG we show that $(u_n)_n$ contains an a.e. convergent subsequence. Let $(A_k)_k$ be as in the hint and fix i. By (i) we know that $|u-u_n|\wedge \mathbb{1}_{A_i} \to 0$ in L^1. By Corollary 13.8 we see that there is a subsequence $u_n^{(i)}$ such that $|u-u_n^{(i)}|\wedge \mathbb{1}_{A_i} \to 0$ almost everywhere. Now take repeatedly subsequences as $i \rightsquigarrow i+1 \rightsquigarrow i+2 \rightsquigarrow \dots$ etc. and then take the diagonal sequence. This will furnish a subsequence $(u_n'')_n \subset (u_n)_n$ which converges a.e. to u on $\bigcup_i A_i = X$.

(iii) We are now in the setting of Corollary 13.8: $|u_n|, |u| \leqslant w$ for some $w \in \mathcal{L}^p(\mu)$ and $u_n \xrightarrow{\mu} u$. Thus, every subsequence $(u_n')_n \subset (u_n)_n$ converges in measure to the same limit u and by (ii) there is some $(u_n'')_n \subset (u_n')_n$ such that $u_n'' \xrightarrow{\text{a.e.}} u$. Now we can use the dominated convergence theorem (Theorem 12.2 or Theorem 13.9) to show that $\lim_n \|u_n''-u\|_p = 0$.

Assume now that u_n does not converge to u in L^p. This means that $\limsup_n \|u_n - u\|_p > 0$, i.e. there is some subsequence such that $\liminf_n \|u'_n - u\|_p > 0$. On the other hand, there is some $(u''_n)_n \subset (u'_n)_n$ such that

$$0 = \lim_n \|u''_n - u\|_p \geqslant \liminf_n \|u'_n - u\|_p > 0$$

and this is a contradiction.

Problem 22.10 Solution: Note that the sets A_j are of finite μ-measure. Observe that the functions $f_j := u\mathbb{1}_{A_j}$

- converge in μ-measure to $f \equiv 0$:

$$\mu(\{|f_j| > \epsilon\} \cap A_j) \leqslant \mu(A_j) \xrightarrow[j\to\infty]{} 0.$$

- are uniformly integrable:

$$\sup_j \int_{\{|f_j|>|u|\}} |f_j| \, d\mu = 0$$

since $|f_j| = |u\mathbb{1}_{A_j}| \leqslant |u|$ and $|u|$ is integrable.

Therefore, Vitali's Theorem shows that $f_j \to 0$ in $\mathcal{L}^1$ so that $\int f_j \, d\mu = \int_{A_j} u \, d\mu \to 0$.

Problem 22.11 Solution:

(i) Trivial. More interesting is the assertion that

A sequence $(x_n)_n \subset \mathbb{R}$ converges to 0 if, and only if, every subsequence $(x_{n_k})_k$ contains some sub-subsequence $(\widetilde{x}_{n_k})_k$ which converges to 0.

Necessity is again trivial. Sufficiency: assume that $(x_n)_n$ does not converge to 0. Then the sequence $(\min\{|x_n|, 1\})_n$ is bounded and still does not converge to 0. Since this sequence is bounded, it contains a convergent subsequence $(x_{n_k})_k$ with some limit $\alpha \neq 0$. But then $(x_{n_k})_k$ cannot contain a sub-subsequence $(\widetilde{x}_{n_k})_k$ which is a null sequence.

(ii) If $u_n \xrightarrow{\mu} u$, then every subsequence $u_{n_k} \xrightarrow{\mu} u$. Thus, using the argument from the proof of Problem 22.3 we can extract a sub-subsequence $(\widetilde{u}_{n_k})_k \subset (u_{n_k})_k$ such that

$$\lim_k \widetilde{u}_{n_k}(x)\mathbb{1}_A(x))u(x)\mathbb{1}_A(x) \text{ almost everywhere.} \tag{*}$$

Note that (unless we are in a σ-finite measure space) the exceptional set may depend on the testing set A.

Conversely, assume that every subsequence $(u_{n_k})_k \subset (u_n)_n$ has a sub-subsequence $(\widetilde{u}_{n_k})_k$ satisfying (*). Because of Lemma 22.4 we have

$$\lim_k \mu\big(\{|\widetilde{u}_{n_k} - u| > \epsilon\} \cap A\big) = 0.$$

Assume now that u_n does not converge in μ-measure on A to u. Then

$$x_n := \mu(\{|u_n - u| > \epsilon\} \cap A) \not\to 0.$$

Since the whole sequence $(x_n)_n$ is bounded (by $\mu(A)$) there exists some subsequence $(x_{n_k})_k$ given by $(u_{n_k})_k$ such that

$$x_{n_k} = \mu(\{|u_{n_k} - u| > \epsilon\} \cap A) \to \alpha \neq 0.$$

This contradicts, however, the fact that x_{n_k} has itself a subsequence converging to zero.

(iii) Fix some set A of finite μ-measure. All conclusions below take place relative to resp. on this set only.

If $u_n \xrightarrow{\mu} u$ we have for every subsequence $(u_{n_k})_k$ a sub-subsequence $(\widetilde{u}_{n_k})_k$ with $\widetilde{u}_{n_k} \to u$ a.e. Since Φ is continuous, we get $\Phi \circ \widetilde{u}_{n_k} \to \Phi \circ u$ a.e.

This means, however, that every subsequence $(\Phi \circ u_{n_k})_k$ of $(\Phi \circ u_n)_n$ has a sub-subsequence $(\Phi \circ \widetilde{u}_{n_k})_k$ which converges a.e. to $\Phi \circ u$. Thus, part (ii) says that $\Phi \circ u_n \xrightarrow{\mu} \Phi \circ u$.

Problem 22.12 Solution: Since $\mathcal{F}$ and $\mathcal{G}$ are uniformly integrable, we find for any given $\epsilon > 0$ functions $f_\epsilon, g_\epsilon \in \mathcal{L}^1_+$ such that

$$\sup_{f \in \mathcal{F}} \int_{\{|f| > f_\epsilon\}} |f| \, d\mu \leqslant \epsilon \quad \text{and} \quad \sup_{g \in \mathcal{G}} \int_{\{|g| > g_\epsilon\}} |g| \, d\mu \leqslant \epsilon.$$

We will use this notation throughout.

(i) Since $f := |f_1| + \cdots + |f_n| \in \mathcal{L}^1_+$ we find that

$$\int_{\{|f_j| > f\}} |f_j| \, d\mu = \int_{\emptyset} |f_j| \, d\mu = 0$$

uniformly for all $1 \leqslant j \leqslant n$. This proves uniform integrability.

(ii) Instead of $\{f_1, \dots, f_N\}$ (which is uniformly integrable because of (i)) we show that $\mathcal{F} \cup \mathcal{G}$ is uniformly integrable.

Set $h_\epsilon := f_\epsilon + g_\epsilon$. Then $h_\epsilon \in \mathcal{L}^1_+$ and

$$\{|w| \geqslant f_\epsilon + g_\epsilon\} \subset \{|w| \geqslant f_\epsilon\} \cap \{|w| \geqslant g_\epsilon\}$$

which means that we have

$$\int_{\{|w| > h_\epsilon\}} |w| \, d\mu \leqslant \begin{cases} \int_{\{|w| > f_\epsilon\}} |w| \, d\mu \leqslant \epsilon & \text{if } w \in \mathcal{F} \\ \int_{\{|w| > g_\epsilon\}} |w| \, d\mu \leqslant \epsilon & \text{if } w \in \mathcal{G}. \end{cases}$$

Since this is uniform for all $w \in \mathcal{F} \cup \mathcal{G}$, the claim follows.

(iii) Set $h_\epsilon := f_\epsilon + g_\epsilon \in \mathcal{L}^1_+$. Since $|f+g| \leqslant |f| + |g|$ we have

$$\begin{aligned}\{|f+g| > h_\epsilon\} \subset \{|f| > h_\epsilon\} &\cup \{|g| > h_\epsilon\} \\ &= \big[\{|f| > h_\epsilon\} \cap \{|g| > h_\epsilon\}\big] \\ &\quad \cup \big[\{|f| > h_\epsilon\} \cap \{|g| \leqslant h_\epsilon\}\big] \\ &\quad \cup \big[\{|f| \leqslant h_\epsilon\} \cap \{|g| > h_\epsilon\}\big]\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\{|f+g|>h_\epsilon\}} |f+g|\,d\mu \\ &\leqslant \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} \big(|f|+|g|\big)\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|\leqslant h_\epsilon\}}} |f|\vee|g|\,d\mu + \int_{\substack{\{|f|\leqslant h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |f|\vee|g|\,d\mu \\ &= \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |f|\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |g|\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|\leqslant h_\epsilon\}}} |f|\,d\mu + \int_{\substack{\{|f|\leqslant h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |g|\,d\mu \\ &\leqslant \int_{\{|f|>h_\epsilon\}} |f|\,d\mu + \int_{\{|g|>h_\epsilon\}} |g|\,d\mu + \int_{\{|f|>h_\epsilon\}} |f|\,d\mu + \int_{\{|g|>h_\epsilon\}} |g|\,d\mu \\ &\leqslant \int_{\{|f|>f_\epsilon\}} |f|\,d\mu + \int_{\{|g|>g_\epsilon\}} |g|\,d\mu + \int_{\{|f|>f_\epsilon\}} |f|\,d\mu + \int_{\{|g|>g_\epsilon\}} |g|\,d\mu \\ &\leqslant 4\epsilon\end{aligned}$$

uniformly for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

(iv) This follows from (iii) if we set

- $t\mathcal{F} \rightsquigarrow \mathcal{F}$,
- $(1-t)\mathcal{F} \rightsquigarrow \mathcal{G}$,
- $tf_\epsilon \rightsquigarrow f_\epsilon$,
- $(1-t)f_\epsilon \rightsquigarrow g_\epsilon$,

and observe that the calculation is uniform for all $t \in [0,1]$.

(v) Without loss of generality we can assume that $\mathcal{F}$ is convex, i.e. coincides with its convex hull.

Let u be an element of the $\mathcal{L}^1$-closure of (the convex hull of) $\mathcal{F}$. Then there is a sequence

$$(f_j)_j \subset \mathcal{F} \,:\, \lim_j \|u - f_j\|_1 = 0.$$

We have, because of $|u| \leqslant |u - f_j| + |f_j|$,

$$\begin{aligned}\{|u| > f_\epsilon\} \subset \{|u-f_j| > f_\epsilon\} &\cup \{|f_j| > f_\epsilon\} \\ &= \big[\{|u-f_j| > f_\epsilon\} \cap \{|f_j| > f_\epsilon\}\big] \\ &\quad \cup \big[\{|u-f_j| > f_\epsilon\} \cap \{|f_j| \leqslant f_\epsilon\}\big]\end{aligned}$$

$$\cup \left[\{|u-f_j| \leqslant f_\epsilon\} \cap \{|f_j| > f_\epsilon\}\right]$$

that

$$\begin{aligned}
&\int_{\{|u|>f_\epsilon\}} |u|\,d\mu \\
&\leqslant \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u|\,d\mu + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|\leqslant f_\epsilon\}}} |u|\,d\mu + \int_{\substack{\{|u-f_j|\leqslant f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u|\,d\mu \\
&\leqslant \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u-f_j|\,d\mu + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |f_j|\,d\mu \\
&\qquad + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|\leqslant f_\epsilon\}}} |u-f_j| \vee |f_j|\,d\mu + \int_{\substack{\{|u-f_j|\leqslant f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u-f_j| \vee |f_j|\,d\mu \\
&\leqslant \|u-f_j\|_1 + \int_{\{|f_j|>f_\epsilon\}} |f_j|\,d\mu + \|u-f_j\|_1 + \int_{\{|f_j|>f_\epsilon\}} |f_j|\,d\mu \\
&\leqslant 2\,\|u-f_j\|_1 + 2\epsilon \\
&\xrightarrow[j\to\infty]{} 2\epsilon.
\end{aligned}$$

Since this holds uniformly for all such u, we are done.

Problem 22.13 Solution: By assumption,

$$\forall\,\epsilon>0\ \exists\,w_\epsilon \in \mathcal{L}^1_+ \,:\, \sup_{f\in\mathcal{F}} \int_{\{|f|>w_\epsilon\}} |f|\,d\mu \leqslant \epsilon.$$

Now observe that

$$\begin{aligned}
&\int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}} \sup_{1\leqslant j\leqslant k} |f_j|\,d\mu \\
&\leqslant \sum_{\ell=1}^k \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}\cap\{|f_\ell|=\sup_{1\leqslant j\leqslant k}|f_j|\}} |f_\ell|\,d\mu \\
&\leqslant \sum_{\ell=1}^k \int_{\{|f_\ell|>w_\epsilon\}} |f_\ell|\,d\mu \\
&\leqslant \sum_{\ell=1}^k \epsilon \\
&= k\,\epsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \sup_{1\leqslant j\leqslant k} |f_j|\,d\mu \\
&\leqslant \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|\leqslant w_\epsilon\}} \sup_{1\leqslant j\leqslant k} |f_j|\,d\mu + \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}} \sup_{1\leqslant j\leqslant k} |f_j|\,d\mu
\end{aligned}$$

$$\leqslant \int w_\epsilon \, d\mu + k\epsilon$$

and we get

$$\lim_{k\to\infty} \frac{1}{k} \int \sup_{1\leqslant j\leqslant k} |f_j| \, d\mu \leqslant \lim_{k\to\infty} \frac{1}{k} \int w_\epsilon \, d\mu + \epsilon = \epsilon$$

which proves our claim as $\epsilon > 0$ was arbitrary.

Problem 22.14 Solution: Since the function $u \equiv R$, $R > 0$, is integrable w.r.t. the probability measure $\mathbb{P}$, we get

$$\begin{aligned}
\int_{\{|u_j|>R\}} |u_j| \, d\mathbb{P} &\leqslant \int_{\{|u_j|>R\}} |u_j| \frac{|u_j|^{p-1}}{R^{p-1}} \, d\mathbb{P} \\
&= \frac{1}{R^{p-1}} \int_{\{|u_j|>R\}} |u_j|^p \, d\mathbb{P} \\
&\leqslant \frac{1}{R^{p-1}} \int |u_j|^p \, d\mathbb{P} \\
&\leqslant \frac{1}{R^{p-1}} \sup_k \int |u_k|^p \, d\mathbb{P} \\
&= \frac{1}{R^{p-1}} \sup_k \|u_k\|_p^p
\end{aligned}$$

which converges to zero as $R \to \infty$. This proves uniform integrability.

Counterexample:

Vitali's theorem implies that a counterexample should satisfy

$$u_j \xrightarrow[j\to\infty]{\mathbb{P}} u, \qquad \|u_j\|_1 = 1, \qquad u_j \text{ does not converge in } \mathcal{L}^1.$$

Consider, for example, the probability space $((0,1), \mathscr{B}(0,1), dx)$ and the sequence

$$u_j := j \cdot \mathbb{1}_{(0,1/j)}.$$

Then $u_j \to 0$ pointwise (everywhere!), hence in measure. This is also the expected $\mathcal{L}^1$ limit, if it exists. Moreover,

$$\|u_j\|_1 = \int u_j \, dx = 1$$

which means that u_j cannot converge in $\mathcal{L}^1$ to the expected limit $u \equiv 0$, i.e. it does not converge in $\mathcal{L}^1$.

Vitali's theorem shows now that $(u_j)_j$ cannot be uniformly integrable.

We can verify this fact also directly: for $R > 0$ and all $j > R$ we get

$$\int_{\{|u_j|>R\}} |u_j| \, dx = \int u_j \, dx = 1$$

which proves

$$\sup_j \int_{\{|u_j|>R\}} |u_j|\,dx = 1 \qquad \forall\, R>0$$

and $(u_j)_j$ cannot be uniformly integrable (in view of the equivalent characterizations of uniform integrability on finite measure spaces, cf. Theorem 22.9)

Problem 22.15 Solution: We have

$$\begin{aligned}
\sum_{j=k}^{\infty} j\,\mu(j<|f|\leqslant j+1) &= \sum_{j=k}^{\infty}\int_{\{j<|f|\leqslant j+1\}} j\,d\mu \\
&\leqslant \sum_{j=k}^{\infty}\int_{\{j<|f|\leqslant j+1\}} |f|\,d\mu \\
&= \int_{\{|f|>k\}} |f|\,d\mu,
\end{aligned}$$

and, since $2j \geqslant j+1$ for all $j \in\in \mathbb{N}$, also

$$\begin{aligned}
2\sum_{j=k}^{\infty} j\,\mu(j<|f|\leqslant j+1) &= \sum_{j=k}^{\infty} 2j\,\mu(j<|f|\leqslant j+1) \\
&= \sum_{j=k}^{\infty}\int_{\{j<|f|\leqslant j+1\}} 2j\,d\mu \\
&\geqslant \sum_{j=k}^{\infty}\int_{\{j<|f|\leqslant j+1\}} |f|\,d\mu \\
&= \int_{\{|f|>k\}} |f|\,d\mu.
\end{aligned}$$

This shows that

$$\int_{\{|f|>k\}} |f|\,d\mu \leqslant 2\sum_{j=k}^{\infty} j\,\mu(j<|f|\leqslant j+1) \leqslant 2\int_{\{|f|>k\}} |f|\,d\mu$$

and this implies

$$\sup_{f\in\mathcal{F}} \int_{\{|f|>k\}} |f|\,d\mu \simeq \sup_{f\in\mathcal{F}} \sum_{j=k}^{\infty} j\,\mu(j<|f|\leqslant j+1).$$

This proves the claim (since we are in a finite measure space where $u\equiv k$ is an integrable function!)

Problem 22.16 Solution: Fix $\epsilon>0$. By assumption there is some $w=w_\epsilon\in\mathcal{L}^1_+$ such that

$$\sup_i \int_{\{|f_i|>w\}} |f_i|\,d\mu \leqslant \epsilon.$$

Since $|u_i|\leqslant|f_i|$ we infer that $\{|u_i|>w\}\subset\{|f_i|>w\}$, and so

$$\int_{\{|u_i|>w\}} |u_i|\,d\mu \leqslant \int_{\{|f_i|>w\}} |f_i|\,d\mu \leqslant \epsilon \;\text{ uniformly for all } i\in I.$$

Problem 22.17 Solution: Let $g \in \mathcal{L}^1_+(\mu)$. Then

$$0 \leqslant \int (|u| - g \wedge |u|)\, d\mu = \int_{\{|u|\geqslant g\}} (|u| - g)\, d\mu \leqslant \int_{\{|u|\geqslant g\}} |u|\, d\mu.$$

This implies that uniform integrability of the family $\mathcal{F}$ implies that the condition of Problem 22.17 holds. On the other hand,

$$\begin{aligned}
\int_{\{|u|\geqslant g\}} |u|\, d\mu &= \int_{\{|u|\geqslant g\}} (2|u| - |u|)\, d\mu \\
&\leqslant \int_{\{|u|\geqslant g\}} (2|u| - g)\, d\mu \\
&\leqslant \int_{\{2|u|\geqslant g\}} (2|u| - g)\, d\mu \\
&= 2 \int_{\{|u|\geqslant \frac{1}{2} g\}} \big(|u| - \tfrac{1}{2} g\big)\, d\mu \\
&= 2 \int_{\{|u|\geqslant \frac{1}{2} g\}} \big(|u| - \big[\tfrac{1}{2} g\big] \wedge |u|\big)\, d\mu
\end{aligned}$$

and since $g \in \mathcal{L}^1$ if, and only if, $\frac{1}{2} g \in \mathcal{L}^1$, we see that the condition given in Problem 22.17 entails uniform integrability.

In finite measure spaces this conditions is simpler: constants are integrable functions in finite measure spaces; thus we can replace the condition given in Problem 22.17 by

$$\lim_{R\to\infty} \sup_{u\in\mathcal{F}} \int (|u| - R \wedge |u|)\, d\mu = 0.$$

---

23 Martingales. Solutions to Problems 23.1–23.16

Problem 23.1 Solution: Since $\mathscr{A}_0 = \{\emptyset, X\}$ an $\mathscr{A}_0$-measurable function u must satisfy $\{u = s\} = \emptyset$ or $= X$, i.e. all $\mathscr{A}_0$-measurable functions are constants.

So if $(u_j)_{j\in\mathbb{N}_0}$ is a martingale, u_0 is a constant and we can calculate its value because of the martingale property:

$$\int_X u_0 \, d\mu = \int_X u_1 \, d\mu \implies u_0 = \mu(X)^{-1} \int_X u_1 \, d\mu. \tag{*}$$

Conversely, since $\mathscr{A}_0 = \{\emptyset, X\}$ and since

$$\int_\emptyset u_0 \, d\mu = \int_\emptyset u_1 \, d\mu$$

always holds, it is clear that the calculation and choice in (*) is necessary and sufficient for the claim.

Problem 23.2 Solution: We consider only the martingale case, the other two cases are similar.

(a) Since $\mathscr{B}_j \subset \mathscr{A}_j$ we get

$$\begin{aligned} \int_A u_j \, d\mu &= \int_A u_{j+1} \, d\mu \quad \forall A \in \mathscr{A}_j \\ \implies \int_B u_j \, d\mu &= \int_B u_{j+1} \, d\mu \quad \forall B \in \mathscr{B}_j \end{aligned}$$

showing that $(u_j, \mathscr{B}_j)_j$ is a martingale.

(b) It is clear that the above implication cannot hold if we enlarge $\mathscr{A}_j$ to become $\mathscr{C}_j$. Just consider the following 'extreme' case (to get a counterexample): $\mathscr{C}_j = \mathscr{A}$ for all j. Any martingale $(u_j, \mathscr{C})_j$ must satisfy,

$$\int_A u_j \, d\mu = \int_A u_{j+1} \, d\mu \quad \forall A \in \mathscr{A}.$$

Considering the sets $A := \{u_j < u_{j+1}\} \in \mathscr{A}$ and $A' := \{u_j > u_{j+1}\} \in \mathscr{A}$ we conclude that

$$0 = \int_{\{u_j > u_{j+1}\}} (u_j - u_{j+1}) \, d\mu \implies \mu(\{u_j > u_{j+1}\}) = 0$$

and, similarly $\mu(\{u_j < u_{j+1}\}) = 0$ so that $u_j = u_{j+1}$ almost everywhere and for all j. This means that, if we start with a non-constant martingale $(u_j, \mathscr{A}_j)_j$, then this can never be a martingale w.r.t. the filtration $(\mathscr{C}_j)_j$.

Problem 23.3 Solution: For the notation etc. we refer to Problem 4.15. Since the completion $\overline{\mathscr{A}}_j$ is given by

$$\overline{\mathscr{A}}_j = \sigma(\mathscr{A}_j, \mathscr{N}), \quad \mathscr{N} := \{M \subset X : \exists\, N \in \mathscr{A},\ N \supset M,\ \mu(N) = 0\}$$

we find that for all $A_j^* \in \mathscr{A}_j^*$ there exists some $A_j \in \mathscr{A}_j$ such that

$$A_j^* \setminus A_j \cup A_j \setminus A_j^* \in \mathscr{N}.$$

Writing $\bar{\mu}$ for the unique extension of μ onto $\overline{\mathscr{A}}$ (and thus onto $\overline{\mathscr{A}}_j$ for all j) we get for A_j^*, A_j as above

$$\begin{aligned}
\left|\int_{A_j^*} u_j\,d\bar{\mu} - \int_{A_j} u_j\,d\mu\right| &= \left|\int_{A_j^*} u_j\,d\bar{\mu} - \int_{A_j} u_j\,d\bar{\mu}\right| \\
&= \left|\int (\mathbb{1}_{A_j^*} - \mathbb{1}_{A_j}) u_j\,d\bar{\mu}\right| \\
&\leqslant \int \left|\mathbb{1}_{A_j^*} - \mathbb{1}_{A_j}\right| u_j\,d\bar{\mu} \\
&= \int \mathbb{1}_{A_j^* \setminus A_j \cup A_j \setminus A_j^*} u_j\,d\bar{\mu} \\
&\leqslant \int \mathbb{1}_N u_j\,d\mu = 0
\end{aligned}$$

for a suitable μ-null-set $N \supset A_j^* \setminus A_j \cup A_j \setminus A_j^*$. This proves that

$$\int_{A_j^*} u_j\,d\bar{\mu} = \int_{A_j} u_j\,d\mu$$

and we see easily from this that $(u_j, \mathscr{A}_j^*)_j$ is again a (sub-, super-)martingale if $(u_j, \mathscr{A}_j)_j$ is a (sub-, super-)martingale.

Problem 23.4 Solution: To see that the condition is sufficient, set $k = j + 1$. For the necessity, assume that $k = j + m$. Since $\mathscr{A}_j \subset \mathscr{A}_{j+1} \subset \cdots \subset \mathscr{A}_{j+m} = \mathscr{A}_k$ we get from the submartingale property

$$\int_A u_j\,d\mu \leqslant \int_A u_{j+1}\,d\mu \leqslant \int_A u_{j+2}\,d\mu \leqslant \cdots \leqslant \int_A u_{j+m}\,d\mu = \int_A u_k\,d\mu.$$

For supermartingales resp. martingales the conditions obviously read:

$$\int_A u_j\,d\mu \geqslant \int_A u_k\,d\mu \qquad \forall\, j < k,\ \forall\, A \in \mathscr{A}_j$$

resp.

$$\int_A u_j\,d\mu = \int_A u_k\,d\mu \qquad \forall\, j < k,\ \forall\, A \in \mathscr{A}_j.$$

Problem 23.5 Solution: We have $\mathcal{S}_j = \{A \in \mathcal{A}_j : \mu(A) < \infty\}$ and we have to check conditions (S_1)–(S_3) for a semiring, cf. page 39. Indeed

$$\emptyset \in \mathcal{A}_j, \mu(\emptyset) = 0 \implies \emptyset \in \mathcal{S}_j \implies (S_1);$$

and

$$\begin{aligned} A, B \in \mathcal{S}_j &\implies A \cap B \in \mathcal{A}_j,\ \mu(A \cap B) \leqslant \mu(A) < \infty \\ &\implies A \cap B \in \mathcal{S}_j \implies (S_2); \end{aligned}$$

and

$$\begin{aligned} A, B \in \mathcal{S}_j &\implies A \setminus B \in \mathcal{A}_j,\ \mu(A \setminus B) \leqslant \mu(A) < \infty \\ &\implies A \setminus B \in \mathcal{S}_j \implies (S_3). \end{aligned}$$

Since $\mathcal{S}_j \subset \mathcal{A}_j$ also $\sigma(\mathcal{S}_j) \subset \mathcal{A}_j$. On the other hand, if $A \in \mathcal{A}_j$ with $\mu(A) = \infty$ we can, because of σ-finiteness find a sequence $(A_k)_k \subset \mathcal{A}_0 \subset \mathcal{A}_j$ such that $\mu(A_k) < \infty$ and $A_k \uparrow X$. Thus, $A_k \cap A \in \mathcal{S}_j$ for all k and $A = \bigcup_k (A_k \cap A)$. This shows that $\mathcal{A}_j \subset \sigma(\mathcal{S}_j)$.

The rest of the problem is identical to remark 23.2(i) when combined with Lemma 16.6.

Problem 23.6 Solution: Using Lemma 17.2 we can approximate $u_j \in \mathcal{L}^2(\mathcal{A}_j)$ by simple functions in $\mathcal{E}(\mathcal{A}_j)$, i.e. with functions of the form $f_j^\ell = \sum_m c_j^{\ell,m} \mathbb{1}_{A_j^{\ell,m}}$ (the sum is a finite sum!) where $c_j^\ell \in \mathbb{R}$ and $A_j^\ell \in \mathcal{A}_j$. Using the Cauchy–Schwarz inequality we also see that

$$\int (f_j^\ell - u_j) u_j \, d\mu \leqslant \|f_j^\ell - u_j\|_{L^2} \cdot \|u_j\|_{L^2} \xrightarrow[j \text{ fixed}]{\ell \to \infty} 0.$$

Using the martingale property we find for $j \leqslant k$:

$$\int \mathbb{1}_{A_j^{\ell,m}} u_k \, d\mu = \int \mathbb{1}_{A_j^{\ell,m}} u_j \, d\mu \qquad \forall \ell,\ m$$

and therefore

$$\int f_j^\ell u_k \, d\mu = \int f_j^\ell u_j \, d\mu \qquad \forall \ell$$

and since the limit $\ell \to \infty$ exists

$$\int u_j u_k \, d\mu = \lim_\ell \int f_j^\ell u_k \, d\mu = \lim_\ell \int f_j^\ell u_j \, d\mu = \int u_j^2 \, d\mu.$$

Problem 23.7 Solution: Since the f_j's are bounded, it is clear that $(f \bullet u)_k$ is integrable. Now take $A \in \mathcal{A}_k$. Then

$$\int_A (f \bullet u)_{k+1} \, d\mu = \int_A \sum_{j=1}^{k+1} f_{j-1}(u_j - u_{j-1}) \, d\mu$$

$$
\begin{aligned}
&= \int_A (f \bullet u)_k + f_k(u_{k+1} - u_k)\, d\mu \\
&= \int_A (f \bullet u)_k \, d\mu + \int (\mathbb{1}_A \cdot f_k)(u_{k+1} - u_k)\, d\mu
\end{aligned}
$$

Using Remark 23.2(iii) we find

$$
\begin{aligned}
\int (\mathbb{1}_A \cdot f_k)(u_{k+1} - u_k)\, d\mu &= \int \mathbb{1}_A \cdot f_k u_{k+1}\, d\mu - \int \mathbb{1}_A \cdot f_k u_k \, d\mu \\
&= \int \mathbb{1}_A \cdot f_k u_k \, d\mu - \int \mathbb{1}_A \cdot f_k u_k \, d\mu \\
&= 0
\end{aligned}
$$

and we conclude that

$$
\int_A (f \bullet u)_{k+1}\, d\mu = \int_A (f \bullet u)_k \, d\mu \qquad \forall\, A \in \mathscr{A}_k.
$$

Problem 23.8 Solution:

(i) Note that

$$
S_{n+1}^2 - S_n^2 = (S_n + \xi_{n+1})^2 - S_n^2 = \xi_{n+1}^2 + \xi_{n+1} S_n.
$$

If $A \in \mathscr{A}_n$, then $\mathbb{1}_A S_n$ is independent of ξ_{n+1} and we find, therefore,

$$
\begin{aligned}
\int_A (S_{n+1}^2 - S_n^2)\, d\mathbb{P} &= \int_A \xi_{n+1}^2 \, d\mathbb{P} + \int_A \xi_{n+1} S_n \, d\mathbb{P} \\
&\geqslant \int_A \xi_{n+1} S_n \, d\mathbb{P} \\
&= \int \xi_{n+1} (\mathbb{1}_A S_n)\, d\mathbb{P} \\
&= \underbrace{\int \xi_{n+1}\, d\mathbb{P}}_{=0} \int \mathbb{1}_A S_n \, d\mathbb{P} \\
&= 0.
\end{aligned}
$$

(ii) Observe, first of all, that due to independence

$$
\begin{aligned}
\int S_n^2 \, d\mathbb{P} &= \sum_{j=1}^n \int \xi_j^2 \, d\mathbb{P} + \sum_{j \neq k} \int \xi_j \xi_k \, d\mathbb{P} \\
&= n \int \xi_1^2 \, d\mathbb{P} + \sum_{j \neq k} \underbrace{\int \xi_j \, d\mathbb{P}}_{=0} \int \xi_k \, d\mathbb{P} \\
&= n \int \xi_1^2 \, d\mathbb{P}
\end{aligned}
$$

so that $\kappa := \int \xi_1^2 \, d\mathbb{P}$ is a reasonable candidate for the assertion. Using the calculation of part (i) of this problem we see

$$[S_{n+1}^2 - \kappa(n+1)] - [S_n^2 - \kappa n] = \xi_{n+1}^2 + \xi_{n+1} S_n - \kappa$$

and integrating over $\int_A \dots d\mathbb{P}$ for any $A \in \mathscr{A}_n$ gives, just as in (i), because of independence of $\mathbb{1}_A$ and ξ_{n+1} resp. $\mathbb{1}_A S_n$ and ξ_{n+1}

$$\begin{aligned}
&\int_A \Big([S_{n+1}^2 - \kappa(n+1)] - [S_n^2 - \kappa n] \Big) \, d\mathbb{P} \\
&\quad = \int \mathbb{1}_A \cdot \xi_{n+1}^2 \, d\mathbb{P} + \int \xi_{n+1} \, d\mathbb{P} \int \mathbb{1}_A \cdot S_n \, d\mathbb{P} - \kappa \int_A d\mathbb{P} \\
&\quad = \mathbb{P}(A) \int \xi_{n+1}^2 \, d\mathbb{P} - \kappa \int_A d\mathbb{P} \\
&\quad = 0
\end{aligned}$$

since ξ_1 and ξ_{n+1} are identically distributed implying that $\kappa = \int \xi_{n+1}^2 \, d\mathbb{P} = \int \xi_1^2 \, d\mathbb{P}$.

Problem 23.9 Solution: As in Problem 23.8 we find

$$M_{n+1} - M_n = \xi_{n+1}^2 + S_n \xi_{n+1} - \sigma_{n+1}^2.$$

Integrating over $A \in \mathscr{A}_n$ yields

$$\begin{aligned}
&\int_A (M_{n+1} - M_n) \, d\mathbb{P} \\
&\quad = \int_A \xi_{n+1}^2 \, d\mathbb{P} + \int_A S_n \xi_{n+1} \, d\mathbb{P} - \sigma_{n+1}^2 \int_A d\mathbb{P} \\
&\quad = \mathbb{P}(A) \underbrace{\int_\Omega \xi_{n+1}^2 \, d\mathbb{P}}_{=\sigma_{n+1}^2} + \int_A S_n \, d\mathbb{P} \underbrace{\int_\Omega \xi_{n+1} \, d\mathbb{P}}_{=0} - \sigma_{n+1}^2 \mathbb{P}(A) \\
&\quad = 0,
\end{aligned}$$

where we use the independence of $\mathbb{1}_A$ and ξ_{n+1} resp. of $\mathbb{1}_A S_n$ and ξ_{n+1} and the hint given in the statement of the problem.

Problem 23.10 Solution: We find that for $A \in \mathscr{A}_n$

$$\int_A u_{n+1} \, d\mu = \int_A (u_n + d_{n+1}) \, d\mu = \int_A u_n \, d\mu + \int_A d_{n+1} \, d\mu = \int_A u_n \, d\mu$$

which shows that $(u_n, \mathscr{A}_n)_n$ is a martingale, hence $(u_n^2, \mathscr{A}_n)_n$ is a submartingale—cf. Example 23.3(vi).

Now

$$\int u_n^2 \, d\mu = \sum_j \int d_j^2 \, d\mu + 2 \sum_{j<k} \int d_j d_k \, d\mu$$

but, just as in Problem 23.6, we can approximate d_j by $\mathscr{A}_j$-measurable simple functions $(f_j^\ell)_{\ell\in\mathbb{N}}$ which shows, since $\int_A d_k\,d\mu = 0$ for any $A \in \mathscr{A}_j$ and $k > j$:

$$\int d_j d_k\,d\mu = \lim_\ell \int f_j^\ell d_k\,d\mu = 0.$$

Problem 23.11 Solution: For $A \in \mathscr{A}_n$ we find

$$\begin{aligned}
&\int_A \left[\left(\frac{1-p}{p}\right)^{S_{n+1}} - \left(\frac{1-p}{p}\right)^{S_n}\right] d\mathbb{P}\\
&\qquad = \int_A \left(\frac{1-p}{p}\right)^{S_n}\left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}} - 1\right] d\mathbb{P}\\
&\qquad = \int_A \left(\frac{1-p}{p}\right)^{S_n} d\mathbb{P}\cdot\int_\Omega \left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}} - 1\right] d\mathbb{P}
\end{aligned}$$

where we use that $\mathbb{1}_A\left(\frac{1-p}{p}\right)^{S_n}$ and $\left(\frac{1-p}{p}\right)^{\xi_{n+1}} - 1$ are independent, see formulae (23.6) and (23.7). But since ξ_{n+1} is a Bernoulli random variable we find

$$\begin{aligned}
&\int_\Omega \left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}} - 1\right] d\mathbb{P}\\
&\qquad = \left[\left(\frac{1-p}{p}\right)^{1} - 1\right]\cdot p + \left[\left(\frac{1-p}{p}\right)^{-1} - 1\right]\cdot(1-p)\\
&\qquad = [1-2p] + [2p-1]\\
&\qquad = 0.
\end{aligned}$$

The integrability conditions for martingales are obviously satisfied.

Problem 23.12 Solution: A solution in a more general context can be found in Example 25.4 on page 297 of the textbook.

Problem 23.13 Solution: By definition, a supermartingale satisfies

$$\int_A u_j\,d\mu \geqslant \int_A u_{j+1}\,d\mu \qquad \forall j \in \mathbb{N},\ A \in \mathscr{A}_j.$$

If we take $A = X$ and if $u_k = 0$, then this becomes

$$0 = \int_X u_k\,d\mu \geqslant \int_X u_{k+1}\,d\mu \geqslant 0$$

and since, by assumption, $u_{k+1} \geqslant 0$, we conclude that $u_{k+1} = 0$.

Problem 23.14 Solution: By definition,

$$A \in \mathscr{A}_\tau \iff A \in \mathscr{A} \quad \text{and} \quad \forall j : A \cap \{\tau \leqslant j\} \in \mathscr{A}_j.$$

Thus,

- $\emptyset \in \mathscr{A}_\tau$ is obvious;
- if $A \in \mathscr{A}_\tau$, then

$$A^c \cap \{\tau \leqslant j\} = \{\tau \leqslant j\} \setminus A = \underbrace{\{\tau \leqslant j\}}_{\in \mathscr{A}_j} \setminus \underbrace{(A \cap \{\tau \leqslant j\})}_{\in \mathscr{A}_j} \in \mathscr{A}_j$$

  thus $A^c \in \mathscr{A}_\tau$.
- if $A_\ell \in \mathscr{A}_\tau$, $\ell \in \mathbb{N}$, then

$$\left[\bigcup_\ell A_\ell\right] \cap \{\tau \leqslant j\} = \bigcup_\ell \underbrace{\left[A_\ell \cap \{\tau \leqslant j\}\right]}_{\in \mathscr{A}_j} \in \mathscr{A}_j$$

  thus $\bigcup A_\ell \in \mathscr{A}_\tau$.

■■

Problem 23.15 Solution: By definition, τ is a stopping time if

$$\forall n \in \mathbb{N}_0 : \{\tau \leqslant n\} \in \mathscr{A}_n.$$

Thus, if τ is a stopping time, we find for $n \geqslant 1$

$$\{\tau < n\} = \{\tau \leqslant n-1\} \in \mathscr{A}_{n-1} \subset \mathscr{A}_n$$

and, therefore, for all $n \in \mathbb{N}_0$

$$\{\tau = n\} = \{\tau \leqslant n\} \setminus \{\tau < n\} \in \mathscr{A}_n.$$

Conversely, if $\{\tau = n\} \in \mathscr{A}_n$ for all n, we get

$$\{\tau \leqslant k\} = \{\tau = 0\} \cup \{\tau = 1\} \cup \cdots \cup \{\tau = k\} \in \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_k \subset \mathscr{A}_k.$$

■■

Problem 23.16 Solution: Since $\sigma \wedge \tau \leqslant \sigma$ and $\sigma \wedge \tau \leqslant \tau$, we find from Lemma 23.6 that

$$\mathscr{F}_{\sigma\wedge\tau} \subset \mathscr{F}_\sigma \cap \mathscr{F}_\tau.$$

Conversely, if $A \in \mathscr{F}_\sigma \cap \mathscr{F}_\tau$ we know that

$$A \cap \{\sigma \leqslant j\} \in \mathscr{F}_j \ \text{ and } \ A \cap \{\tau \leqslant j\} \in \mathscr{F}_j \qquad \forall j \in \mathbb{N}_0.$$

Thus,

$$A \cap \{\sigma \wedge \tau \leqslant j\} = A \cap \big(\{\sigma \leqslant j\} \cup \{\tau \leqslant j\}\big) \in \mathscr{F}_j$$

and we get $A \in \mathscr{F}_{\sigma\wedge\tau}$.

■■

24 Martingale convergence theorems. Solutions to Problems 24.1–24.9

Problem 24.1 Solution: We have $\tau_0 = 0$ which is clearly a stopping time and since

$$\sigma_1 := \inf\{j > 0 : u_j \leqslant a\} \wedge N \qquad (\inf \emptyset = +\infty)$$

it is clear that

$$\{\sigma_1 > \ell\} = \{u_1 > a\} \cap \cdots \cap \{u_\ell > a\} \in \mathscr{A}_\ell.$$

The claim follows by induction once we have shown that σ_k and τ_k are stopping times for a generic value of k. Since the structure of their definitions are similar, we show this for σ_k only.

By induction assumption, let $\tau_0, \sigma_1, \tau_1, \ldots, \sigma_{k-1}, \tau_{k-1}$ be stopping times. By definition,

$$\sigma_k := \inf\{j > \tau_{k-1} : u_j \leqslant a\} \wedge N \qquad (\inf \emptyset = +\infty)$$

and we find for $\ell \in \mathbb{N}$ and $\ell < N$

$$\{\sigma_k > \ell\} = \{\tau_{k-1} > \ell\} \cup \Big(\{\tau_{k-1} \leqslant \ell\} \cap \{u_{\tau_{k-1}+1} > a\} \cap \cdots \cap \{u_{\ell-1} > a\} \cap \{u_\ell > a\}\Big) \in \mathscr{A}_\ell$$

while, by definition we get for $\ell = N$

$$\{\sigma_k > N\} = \emptyset \in \mathscr{A}_N.$$

Problem 24.2 Solution: Theorem 24.7 becomes for supermartingales: *Let $(u_\ell)_{\ell\in-\mathbb{N}}$ be a backwards supermartingale and assume that $\mu|_{\mathscr{A}_{-\infty}}$ is σ-finite. Then $\lim_{j\to\infty} u_{-j} = u_{-\infty} \in (-\infty, \infty]$ exists a.e. Moreover, L^1-$\lim_{j\to\infty} u_{-j} = u_{-\infty}$ if, and only if, $\sup_j \int u_{-j}\,d\mu < \infty$; in this case $(u_\ell, \mathscr{A}_\ell)_{\ell\in-\mathbb{N}}$ is a supermartingale and $u_{-\infty}$ is finitely-valued.*

Using this theorem the claim follows immediately from the supermartingale property:

$$-\infty < \int_A u_{-1}\,d\mu \leqslant \int_A u_{-j}\,d\mu \leqslant \int_A u_{-\infty}\,d\mu < \infty \quad \forall\, j \in \mathbb{N},\ A \in \mathscr{A}_{-\infty}$$

and, in particular, for $A = X \in \mathscr{A}_{-\infty}$.

Problem 24.3 Solution: Corollary 24.3 shows pointwise a.e. convergence. Using Fatou's lemma we get

$$\begin{aligned} 0 = \lim_{j\to\infty} \int u_j \, d\mu &= \liminf_{j\to\infty} \int u_j \, d\mu \\ &\geqslant \int \liminf_{j\to\infty} u_j \, d\mu \\ &= \int u_\infty \, d\mu \geqslant 0 \end{aligned}$$

so that $u_\infty = 0$ a.e.

Moreover, since $\int u_j \, d\mu \xrightarrow[j\to\infty]{} 0 = \int u_\infty \, d\mu$, Theorem 24.6 shows that $u_j \to u_\infty$ in L^1-sense.

Problem 24.4 Solution: From $L^1\text{-}\lim_{j\to\infty} u_j = f$ we conclude that $\sup_j \int |u_j| \, d\mu < \infty$ and we get that $\lim_{j\to\infty} u_j$ exists a.e. Since L^1-convergence also implies a.e. convergence of a subsequence, the limiting functions must be the same.

Problem 24.5 Solution: The quickest solution uses the famous Chung-Fuchs result that a simple random walk (this is just $S_j := \xi_1 + \cdots + \xi_j$ with ξ_k iid Bernoulli $p = q = \frac{1}{2}$) does not converge and that $-\infty = \liminf_j S_j < \limsup_j S_j = \infty$ a.e. Knowing this we are led to

$$P(u_j \text{ converges}) = P(\xi_0 + 1 = 0) = \frac{1}{2}.$$

It remains to show that u_j is a martingale. For $A \in \sigma(\xi_1, \dots, \xi_j)$ we get

$$\begin{aligned} \int_A u_{j+1} \, dP &= \int_A (\xi_0 + 1)(\xi_1 + \cdots + \xi_j + \xi_{j+1}) \, dP \\ &= \int_A (\xi_0 + 1)(\xi_1 + \cdots + \xi_j) \, dP + \int_A (\xi_0 + 1)\xi_{j+1} \, dP \\ &= \int_A u_j \, dP + \int_A (\xi_0 + 1) \, dP \int_\Omega \xi_{j+1} \, dP \\ &= \int_A u_j \, dP \end{aligned}$$

where the last step follows because of independence.

If you do not know the Chung-Fuchs result, you could argue as follows: assume that for some finite random variable S the limit $S_j(\omega) \to S(\omega)$ takes place on a set $A \subset \Omega$. Since the ξ_j's are iid, we have

$$\xi_2 + \xi_3 + \cdots \to S$$

and

$$\xi_1 + \xi_2 + \cdots \to S$$

which means that S and $S + \xi_1$ have the same probability distribution. But this entails that S is necessarily $\pm\infty$, i.e., S_j cannot have a finite limit.

Problem 24.6 Solution:

(i) Cf. the construction in Scholium 23.4.

(ii) Note that $n^2 - (n-1)^2 - 1 = 2n - 2$ is even.

The function $f : \mathbb{R}^{2n-2} \to \mathbb{R}$, $f(x_1, \dots, x_{2n-2}) = x_1 + \dots + x_{n^2-(n-1)^2}$ is clearly Borel measurable, i.e. the function

$$f(\xi_{(n-1)^2+2}, \dots, \xi_{n^2}) = \xi_{(n-1)^2+2} + \dots + \xi_{n^2}$$

is $\mathscr{A}_n$-measurable and so is the set A_n.

Moreover, $x \in A_n$ if, and only if, exactly half of $\xi_{(n-1)^2+2}, \dots, \xi_{n^2}$ are $+1$ and the other half is -1. Thus,

$$\lambda(A_n) = \binom{2n-2}{n-1}\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right)^{n-1} = \binom{2n-2}{n-1}\left(\frac{1}{2}\right)^{2n-2}$$

Using Stirling's formula, we get

$$\begin{aligned}\frac{1}{2^{2k}}\binom{2k}{k} &= \frac{(2k)!}{k!\,k!} \\ &\sim \frac{\sqrt{2\pi 2k}(2k)^{2k}e^k e^k}{2^{2k}\sqrt{2\pi k}\sqrt{2\pi k}k^k k^k e^{2k}} \\ &= \frac{1}{\sqrt{k\pi}} \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Setting $k = n - 1$ this shows both

$$\lim_n \lambda(A_n) = 0 \quad\text{and}\quad \sum_n \lambda(A_n) \sim \sum_n \frac{1}{\sqrt{n}} = \infty.$$

Finally, $\limsup_n \mathbb{1}_{A_n} = \mathbb{1}_{\limsup_n A_n} = 1$ a.e. while, by Fatou's lemma

$$0 \leqslant \int \liminf_n \mathbb{1}_{A_n}\,d\lambda \leqslant \liminf_n \int \mathbb{1}_{A_n}\,d\lambda = \liminf_n \lambda(A_n) = 0,$$

i.e., $\liminf_n \mathbb{1}_{A_n} = 0$ a.e. This means that $\mathbb{1}_{A_n}$ does not have a limit as $n \to \infty$.

(iii) For $A \in \mathscr{A}_n$ we have because of independence

$$\begin{aligned}&\int_A M_{n+1}\,d\lambda \\ &= \int_A M_n(1 + \xi_{n^2+1})\,d\lambda + \int_A \mathbb{1}_{A_n}\xi_{n^2+1}\,d\lambda \\ &= \int_A M_n\,d\lambda \int_{[0,1]} (1 + \xi_{n^2+1})\,d\lambda + \int_A \mathbb{1}_{A_n}\,d\lambda \int_{[0,1]} \xi_{n^2+1}\,d\lambda \\ &= \int_A M_n\,d\lambda.\end{aligned}$$

(iv) We have

$$\begin{aligned}&\{M_{n+1} \neq 0\} \\ &= \{M_{n+1} \neq 0, \xi_{n^2+1} = -1\} \cup \{M_{n+1} \neq 0, \xi_{n^2+1} = +1\} \\ &\subset A_n \cup \{M_n \neq 0, \xi_{n^2+1} = +1\}.\end{aligned}$$

(v) By definition,

$$M_{n+1} - M_n = M_n \xi_{n^2+1} + \mathbb{1}_{A_n} \xi_{n^2+1} = (M_n + \mathbb{1}_{A_n}) \xi_{n^2+1}$$

so that

$$|M_{n+1} - M_n| = |M_n + \mathbb{1}_{A_n}| \cdot |\xi_{n^2+1}| = |M_n + \mathbb{1}_{A_n}|.$$

This shows that for $x \in \{\lim_n M_n \text{ exists}\}$ the limit $\lim_n \mathbb{1}_{A_n}(x)$ exists. But, because of (ii), the latter is a null set, so that the pointwise limit of M_n cannot exist.

On the other hand, using the inequality (iv), shows

$$\lambda(M_{n+1} \neq 0) \leqslant \tfrac{1}{2} \lambda(M_n \neq 0) + \lambda(A_n)$$

and iterating this gives

$$\begin{aligned} \lambda(M_{n+k} \neq 0) &\leqslant \tfrac{1}{2^k} \lambda(M_n \neq 0) + \lambda(A_n) + \cdots \lambda(A_{n+k-1}) \\ &\leqslant \tfrac{1}{2^k} + \lambda(A_n) + \cdots \lambda(A_{n+k-1}). \end{aligned}$$

Letting first $n \to \infty$ and then $k \to \infty$ yields

$$\limsup_j \lambda(M_j \neq 0) = 0$$

so that $\lim_j \lambda(M_j = 0) = 0$.

Problem 24.7 Solution: Note that for $A \in \big\{\{1\}, \{2\}, \ldots, \{n\}, \{n+1, n+2, \ldots\}\big\}$ we have

$$\begin{aligned} \int_A \xi_{n+1} \, dP &= \int_A (n+2) \mathbb{1}_{[n+2,\infty)\cap\mathbb{N}} \, dP \\ &= \begin{cases} 0 & \text{if } A \text{ is a singleton} \\ \displaystyle\int_{[n+1,\infty)\cap\mathbb{N}} (n+2) \mathbb{1}_{[n+2,\infty)\cap\mathbb{N}} \, dP & \text{else} \end{cases} \end{aligned}$$

and in the second case we have

$$\begin{aligned} \int_{[n+1,\infty)\cap\mathbb{N}} (n+2) \mathbb{1}_{[n+2,\infty)\cap\mathbb{N}} \, dP &= \int_{[n+2,\infty)\cap\mathbb{N}} (n+2) \, dP \\ &= (n+2) \sum_{j=n+2}^{\infty} P(\{j\}) \\ &= (n+2) \sum_{j=n+2}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1} \right) \\ &= 1, \end{aligned}$$

The same calculation shows

$$\int_A \xi_n \, dP = \int_A (n+1) \mathbb{1}_{[n+1,\infty)\cap\mathbb{N}} \, dP$$

$$= \begin{cases} 0 & \text{if } A \text{ is a singleton} \\ \int_{[n+1,\infty)\cap\mathbb{N}} (n+1)\mathbb{1}_{[n+1,\infty)\cap\mathbb{N}}\,dP = 1 & \text{else} \end{cases}$$

so that

$$\int_A \xi_{n+1}\,dP = \int_A \xi_n\,dP$$

for all A from a generator of the σ-algebra which contains an exhausting sequence. This shows, by Remark 23.2(i) that $(\xi_n)_n$ is indeed a martingale.

The second calculation from above also shows that $\int \xi_n\,dP = 1$ while

$$\sup_n \xi_n = \infty \qquad \text{and} \qquad \lim_n \xi_n = 0$$

are obvious.

Problem 24.8 Solution:

(i) Using Problem 23.6 we get

$$\begin{aligned}\int (u_j - u_{j-1})^2\,d\mu &= \int u_j^2\,d\mu - 2\int u_j u_{j-1}\,d\mu + \int u_{j-1}^2\,d\mu \\ &= \int u_j^2\,d\mu - 2\int u_{j-1}^2\,d\mu + \int u_{j-1}^2\,d\mu \\ &= \int u_j^2\,d\mu - \int u_{j-1}^2\,d\mu\end{aligned}$$

which means that

$$\int u_N^2\,d\mu = \sum_{j=1}^N \int (u_j - u_{j-1})^2\,d\mu$$

and the claim follows.

(ii) Because of Example 23.3(vi), $p = 2$, we conclude that $(u_j^2)_j$ is a submartingale which, due to L^2-boundedness, satisfies the assumptions of Theorem 24.2 on submartingale convergence. This means that $\lim_j u_j^2 = u^2$ exists a.e. This is, alas, not good enough to get $u_j \to u$ a.e., it only shows that $|u_j| \to |u|$ a.e.

The following trick helps: let $(A_k)_k \subset \mathscr{A}_0$ be an exhausting sequence with $A_k \uparrow X$ and $\mu(A_k) < \infty$. Then $(\mathbb{1}_{A_k} u_j)_j$ is an L^1-bounded martingale: indeed, if $A \in \mathscr{A}_n$ then $A \cap A_k \in \mathscr{A}_n$ and it is clear that

$$\int_A \mathbb{1}_{A_k} u_n\,d\mu = \int_{A\cap A_k} u_n\,d\mu = \int_{A\cap A_k} u_{n+1}\,d\mu = \int_A \mathbb{1}_{A_k} u_{n+1}\,d\mu$$

while, by the Cauchy–Schwarz inequality,

$$\int |\mathbb{1}_{A_k} u_n|\,d\mu \leqslant \sqrt{\mu(A_k)} \cdot \sqrt{\sup_n \int u_n^2\,d\mu} \leqslant c_k.$$

Thus, we can use Theorem 24.2 and conclude that

$$\mathbb{1}_{A_k} u_n \xrightarrow[n\to\infty]{} \mathbb{1}_{A_k} u$$

almost everywhere with, because of almost-everywhere-uniqueness of the limits on each of the sets A_k, a single function u. This shows $u_n \to u$ a.e.

(iii) Following the hint and using the arguments of part (i) we find

$$\begin{aligned}\int (u_{j+k} - u_j)^2 \, d\mu &= \int (u_{j+k}^2 - u_j^2) \, d\mu \\ &= \sum_{\ell=j+1}^{j+k} \int (u_\ell^2 - u_{\ell-1}^2) \, d\mu \\ &= \sum_{\ell=j+1}^{j+k} \int (u_\ell - u_{\ell-1})^2 \, d\mu.\end{aligned}$$

Now we use Fatou's lemma and the result of part (ii) to get

$$\begin{aligned}\int \liminf_j (u - u_j)^2 \, d\mu &\leqslant \liminf_j \int (u - u_j)^2 \, d\mu \\ &\leqslant \limsup_j \int (u - u_j)^2 \, d\mu \\ &\leqslant \limsup_j \sum_{\ell=j+1}^{\infty} \int (u_\ell - u_{\ell-1})^2 \, d\mu \\ &= 0\end{aligned}$$

since, by L^2-boundedness, $\sum_{k=1}^\infty \int (u_k - u_{k-1})^2 \, d\mu < \infty$.

(iv) Since $\mu(\xi) < \infty$, constants are integrable and we find using the Cauchy–Schwarz and Markov inequalities

$$\begin{aligned}\int_{|u_k|>R} |u_k| \, d\mu &\leqslant \sqrt{\mu(|u_k| > R)} \cdot \sqrt{\int u_k^2 \, d\mu} \\ &\leqslant \frac{1}{R} \sqrt{\int u_k^2 \, d\mu} \cdot \sqrt{\int u_k^2 \, d\mu} \\ &\leqslant \frac{1}{R} \sup_k \int u_k^2 \, d\mu\end{aligned}$$

from which we get uniform integrability; the claim follows now from parts (i)–(iii) and Theorem 24.6.

∎∎

Problem 24.9 Solution:

(i) Note that $\int \epsilon_j \, dP = 0$ and $\int \epsilon_j^2 \, dP = 1$. Moreover, $\xi_n := \sum_{j=1}^n \epsilon_j y_j$ is a martingale w.r.t. the filtration $\mathscr{A}_n := \sigma(\epsilon_1, \dots, \epsilon_n)$ and

$$\int \xi_n^2 \, dP = \sum_{j=1}^n y_j^2.$$

Problem 24.8 now shows that $\sum_{j=1}^{\infty} y_j^2 < \infty$ means that the martingale $(\xi_n)_n$ is L^2-bounded, i.e. ξ_n converges a.e. The converse follows from part (iii).

(ii) This follows with the same arguments as in part (i) with $\mathscr{A}_n = \sigma(\xi_1, \dots, \xi_n)$.

(iii) We show that $S_n^2 - A_n$ is a martingale. Now for $A \in \mathscr{A}_n$

$$\begin{aligned}\int_A M_{n+1}\,dP &= \int_A (S_{n+1}^2 - A_{n+1})\,dP \\ &= \int_A (S_n^2 + 2\xi_{n+1}S_n + \xi_{n+1}^2 - A_n - \sigma_{n+1}^2)\,dP \\ &= \int_A (S_n^2 - A_n)\,dP + \int_A (2\xi_{n+1}S_n + \xi_{n+1}^2 - \sigma_{n+1}^2)\,dP \\ &= \int_A M_n\,dP + \int_A (2\xi_{n+1}S_n + \xi_{n+1}^2 - \sigma_{n+1}^2)\,dP\end{aligned}$$

But, because of independence,

$$\begin{aligned}&\int_A (2\xi_{n+1}S_n + \xi_{n+1}^2 - \sigma_{n+1}^2)\,dP \\ &= \int_A 2\xi_{n+1}\,dP \int_\Omega S_n\,dP + P(A)\int \xi_{n+1}^2\,dP - P(A)\sigma_{n+1}^2 \\ &= 0 + P(A)\sigma_{n+1}^2 - P(A)\sigma_{n+1}^2 \\ &= 0.\end{aligned}$$

and the claim is established.

Now define

$$\tau := \tau_\kappa := \inf\{j : |M_j| > \kappa\}.$$

By optional sampling, $(M_{n\wedge\tau_\kappa})_n$ is again a martingale and we have

$$\begin{aligned}|M_{n\wedge\tau}| &= M_n \mathbb{1}_{\{n<\tau\}} + |M_\tau|\mathbb{1}_{\{n\geqslant\tau\}} \\ &\leqslant \kappa\mathbb{1}_{\{n<\tau\}} + |M_\tau|\mathbb{1}_{\{n\geqslant\tau\}} \\ &\leqslant \kappa\mathbb{1}_{\{n<\tau\}} + |M_\tau - M_{\tau-1}|\mathbb{1}_{\{n\geqslant\tau\}} + |M_{\tau-1}|\mathbb{1}_{\{n\geqslant\tau\}} \\ &= \kappa\mathbb{1}_{\{n<\tau\}} + |\xi_\tau|\mathbb{1}_{\{n\geqslant\tau\}} + |M_{\tau-1}|\mathbb{1}_{\{n\geqslant\tau\}} \\ &\leqslant \kappa\mathbb{1}_{\{n<\tau\}} + |\xi_\tau|\mathbb{1}_{\{n\geqslant\tau\}} + \kappa\mathbb{1}_{\{n\geqslant\tau\}} \\ &\leqslant \kappa + C\end{aligned}$$

where we use, for the estimate of $M_{\tau-1}$, the definition of τ for the last estimate. Since $(M_{n\wedge\tau})_n$ is a martingale, this gives

$$\int (S_{n\wedge\tau}^2 - A_{n\wedge\tau})\,dP = \int (S_0^2 - A_0)\,dP = 0$$

so that

$$\int A_{n\wedge\tau}\,dP = \int S_{n\wedge\tau}^2\,dP \leqslant (\kappa + C)^2$$

uniformly in n.

Thus, by Beppo Levi's theorem,

$$\int A_\tau \, dP \leqslant (\kappa + C)^2 < \infty$$

which means that $A_\tau < \infty$ almost surely. But since $\sum_j \xi_j$ converges almost surely, $P(\tau = \infty) = 1$ for sufficiently large κ, and we are done.

■■

25 Martingales in action. Solutions to Problems 25.1–25.15

Problem 25.1 Solution: This problem is intimately linked with problem 25.7.

Without loss of generality we assume that μ and ν are finite measures, the case for σ-finite μ and arbitrary ν is exactly as in the proof of Theorem 25.2.

Let $(A_i)_i$ be as described in the problem and define the finite σ-algebras $\mathscr{A}_n := \sigma(A_1, \dots, A_n)$. Using the hint we can achieve that

$$\mathscr{A}_n = \sigma\big(C_1^n, \dots, C_{\ell(n)}^n\big)$$

with mutually disjoint C_i^k's and $\ell(n) \leqslant 2^n + 1$ and $\dot\bigcup_i C_i^n = X$. Then the construction of Example 25.4 yields a countably-indexed martingale since the σ-algebras $\mathscr{A}_i$ are increasing.

This means, that the countable version of the martingale convergence theorem is indeed enough for the proof.

Problem 25.2 Solution: "$\Rightarrow$": Assume first that (25.1) holds, i.e. that $\nu \ll \mu$. If $\mu(A \vartriangle B) = 0$ for some $A, B \in \mathscr{A}$ we get $\nu(A \vartriangle B) = 0$. By definition,

$$\nu(A \vartriangle B) = \nu(A \setminus B) + \nu(B \setminus A) = \nu(A \setminus (A \cap B)) + \nu(B \setminus (A \cap B)) = 0$$

so that

$$\nu(A \setminus (A \cap B)) = \nu(B \setminus (A \cap B)) = 0.$$

Assume that $\nu(A) < \infty$. Then $\nu(A \cap B) \leqslant \nu(A) < \infty$ and we see that

$$\nu(A) = \nu(A \cap B) \;\text{ and }\; \nu(B) = \nu(A \cap B)$$

which means that $\nu(A) = \nu(B)$.

If $\nu(A) = \infty$ the condition $\nu(A \setminus (A \cap B)) = 0$ shows that $\nu(A \cap B) = \infty$, otherwise $0 = \nu(A \setminus (A \cap B)) = \nu(A) - \nu(A \cap B) = \infty$ which is impossible. Again we have $\nu(A) = \infty = \nu(B)$.

"$\Leftarrow$": Assume now that the condition stated in the problem is satisfied. If $N \in \mathscr{A}$ is any μ-null set, we choose $A := N$ and $B := \emptyset$ and observe that $A \vartriangle B = N$. Thus,

$$\mu(N) = \mu(A \vartriangle B) = 0 \quad \Rightarrow \quad \nu(A) = \nu(B)$$

but this is just $\nu(N) = \nu(A) = \nu(\emptyset) = 0$. Condition (25.1) follows.

Problem 25.3 Solution: Using simply the Radon–Nikodým theorem, Theorem 25.2, gives

$$\forall\, t \quad \exists\, p_t(x) \text{ such that } \nu_t(dx) = p_t(x) \cdot \mu_t(dx)$$

with a measurable function $x \mapsto p_t(x)$; it is, however, far from being clear that $(t,x) \mapsto p_t(x)$ is jointly measurable.

A slight variation of the proof of Theorem 25.2 allows us to incorporate parameters provided the families of measures are measurable w.r.t. these parameters. Following the hint we set (notation as in the proof of 25.2)

$$p_\alpha(t,x) := \sum_{A\in\alpha} \frac{\nu_t(A)}{\mu_t(A)}\, I_A(x)$$

with the agreement that $\frac{0}{0} := 0$ (note that $\frac{a}{0}$ with $a \neq 0$ will not turn up because of the absolute continuity of the measures!). Since $t \mapsto \mu_t(A)$ and $t \mapsto \mu_t(A)$ are measurable, the above sum is measurable so that

$$(t,x) \mapsto p(t,x)$$

is a jointly measurable function. If we can show that

$$\lim_\alpha p_\alpha(t,x) = p(t,x)$$

exists (say, in L^1, t being fixed) then the limiting function is again jointly measurable.

Using exactly the arguments of the proof of Theorem 25.2 with t fixed we can confirm that this limit exists and defines a jointly measurable function with the property that

$$\nu_t(dx) = p(t,x) \cdot \nu_t(dx).$$

Because of the a.e. uniqueness of the Radon–Nikodým density the functions $p(t,x)$ and $p_t(x)$ coincide, for every t a.e. as functions of x; without additional assumptions on the nature of the dependence on the parameter, the exceptional set may, though, depend on t!

Problem 25.4 Solution: $\boxed{\nu \ll \lambda}$. We show that $\lambda(N) = 0 \implies \nu(N) = 0$. Let $N \in \mathscr{B}(\mathbb{R}^n)$ be a Lebesgue null set. Using the invariance of Lebesgue measure under shifts we get

$$\begin{aligned}
0 = \int \underbrace{\lambda(N)}_{=0}\, \nu(dy) &= \int \lambda(N-y)\, \nu(dy) \\
&= \iint \mathbb{1}_N(x+y)\, \lambda(dx)\nu(dy) \\
&\overset{\text{Tonelli}}{=} \iint \mathbb{1}_N(x+y)\, \nu(dy)\lambda(dx)
\end{aligned}$$

$$= \int \nu(N-y)\,\lambda(dy).$$

Therefore, $\nu(N-y)=0$ for Lebesgue almost every y, i.e. there is some x_0 such that $\nu(N+x_0)=0$. Now we use the quasi-invariance to get $\nu(N)=\nu((N+x_0)-x_0)=0$.

$\boxed{\lambda \ll \nu}$. We show that $\nu(N)=0 \implies \lambda(N)=0$. Let $N \in \mathscr{B}(\mathbb{R}^n)$ be a null set for the measure ν. Similar to the first part of the proof we get

$$0 = \int \underbrace{\nu(N-x)}_{=0}\,\lambda(dx) = \int \lambda(N-y)\,\nu(dy)$$
$$= \int \lambda(N)\,\nu(dy) = \lambda(N)\nu(\mathbb{R}^n).$$

This shows that $\lambda(N)=0$ (unless ν is trivial....).

Problem 25.5 Solution: Have a look at the respective solutions for Chapter 20.

Problem 25.6 Solution: We write $u^{\pm}$ for the positive resp. negative parts of $u \in \mathcal{L}^1(\mathscr{A})$, i.e. $u = u^+ - u^-$ and $u^{\pm} \geqslant 0$. Fix such a function u and define

$$\nu^{\pm}(F) := \int_F u^{\pm}(x)\,\mu(dx), \quad \forall\, F \in \mathscr{F}.$$

Clearly, $\nu^{\pm}$ are measures on the σ-algebra $\mathscr{F}$. Moreover

$$\forall\, N \in \mathscr{F},\ \mu(N)=0 \implies \nu^{\pm}(N) = \int_N u^{\pm}\,d\mu = 0$$

which means that $\nu^{\pm} \ll \mu$. By the Radon–Nikodým theorem we find (up to null-sets unique) positive functions $f^{\pm} \in \mathcal{L}^1(\mathscr{F})$ such that

$$\nu^{\pm}(F) = \int_F f^{\pm}\,d\mu \quad \forall\, F \in \mathscr{F}.$$

Thus, $u^{\mathscr{F}} := f^+ - f^- \in \mathcal{L}^1(\mathscr{F})$ clearly satisfies

$$\int_F u^{\mathscr{F}}\,d\mu = \int_F u\,d\mu \quad \forall\, F \in \mathscr{F}.$$

To see uniqueness, we assume that $w \in \mathcal{L}^1(\mathscr{F})$ also satisfies

$$\int_F w\,d\mu = \int_F u\,d\mu \quad \forall\, F \in \mathscr{F}.$$

Since then

$$\int_F u^{\mathscr{F}}\,d\mu = \int_F w\,d\mu \quad \forall\, F \in \mathscr{F}.$$

we can choose $f := \{w > u^{\mathscr{F}}\}$ and find

$$0 = \int_{\{w > u^{\mathscr{F}}\}} (w - u^{\mathscr{F}})\,d\mu$$

which is only possible if $\mu(\{w > u^{\mathscr{F}}\}) = 0$. Similarly we conclude that $\mu(\{w < u^{\mathscr{F}}\}) = 0$ from which we get $w = u^{\mathscr{F}}$ almost everywhere.

Reformulation of the submartingale property.

Recall that $(u_j, \mathscr{A}_j)_j$ is a submartingale if, for every j, $u_j \in \mathcal{L}^1(\mathscr{A}_j)$ and if

$$\int_A u_j \, d\mu \leqslant \int_A u_{j+1} \, d\mu \quad \forall A \in \mathscr{A}_j, \ \forall j.$$

We claim that this is equivalent to saying

$$u_j \leqslant u_{j+1}^{\mathscr{A}_j} \quad \text{almost everywhere}, \ \forall j.$$

The direction '⇒' is clear. To see '⇐' we fix j and observe that, since

$$\int_A u_j \, d\mu \leqslant \int_A u_{j+1} \, d\mu = \int_A u_{j+1}^{\mathscr{A}_j} \, d\mu \quad \forall A \in \mathscr{A}_j,$$

we get, in particular, for $A := \{u_{j+1}^{\mathscr{A}_j} < u_j\} \in \mathscr{A}_j$,

$$0 \leqslant \int_{\{u_{j+1}^{\mathscr{A}_j} < u_j\}} (u_{j+1}^{\mathscr{A}_j} - u_j) \, d\mu$$

which is only possible if $\mu(\{u_{j+1}^{\mathscr{A}_j} < u_j\}) = 0$.

Problem 25.7 Solution: Since both μ and ν are σ-finite, we can restrict ourselves, using the technique of the Proof of Theorem 25.2 to the case where μ and ν are finite. All we have to do is to pick an exhaustion $(K_\ell)_\ell$, $K_\ell \uparrow X$ such that $\mu(K_\ell), \mu(K_\ell) < \infty$ and to consider the measures $\mathbb{1}_{K_\ell}\mu$ and $\mathbb{1}_{K_\ell}\nu$ which clearly inherit the absolute continuity from μ and ν.

Using the Radon–Nikodým theorem (Theorem 25.2) we get that

$$\mu_j \ll \nu_j \implies \mu_j = u_j \cdot \nu_j$$

with an $\mathscr{A}_j$-measurable positive density u_j. Moreover, since μ is a finite measure,

$$\int_X u_j \, d\nu = \int_X u_j \, d\nu_j = \int_X d\mu_j = \mu_j(X) < \infty$$

so that all the $(u_j)_j$ are ν-integrable. Using exactly the same argument as at the beginning of the proof of Theorem 25.2 (ii)⇒(i), we get that $(u_j)_j$ is even uniformly ν-integrable. Finally, $(u_j)_j$ is a martingale (given the measure ν), since for $j, j+1$ and $A \in \mathscr{A}_j$ we have

$$\begin{aligned}
\int_A u_{j+1} \, d\nu &= \int_A u_{j+1} \, d\nu_{j+1} \\
&= \int_A d\mu_{j+1} && (u_{j+1} \cdot \nu_{j+1} = \mu_{j+1}) \\
&= \int_A d\mu_j && (A \in \mathscr{A}_j)
\end{aligned}$$

$$
\begin{aligned}
&= \int_A u_j \, d\nu_j && (\mu_j = u_j \cdot \nu_j) \\
&= \int_A u_j \, d\nu
\end{aligned}
$$

and we conclude that $u_j \to u_\infty$ a.e. and in $L^1(\nu)$ for some limiting function u_∞ which is still $L^1(\nu)$ and also $\mathscr{A}_\infty := \sigma(\bigcup_{j\in\mathbb{N}} \mathscr{A}_j)$-measurable. Since, by assumption, $\mathscr{A}_\infty = \mathscr{A}$, this argument shows also that

$$\mu = u_\infty \cdot \nu$$

and it reveals that

$$u_\infty = \frac{d\mu}{d\nu} = \lim_j \frac{d\mu_j}{d\nu_j}.$$

Problem 25.8 Solution: We can assume that $\mathbb{V}\xi_j < \infty$, otherwise the inequality would be trivial.

Note that the random variables $\xi_j - \mathbb{E}\xi_j$, $j = 1, 2, \dots, n$ are still independent and, of course, centered (= mean-zero). Thus, by Example 23.3(x) we get that

$$M_k := \sum_{j=1}^{k} (\xi_j - \mathbb{E}\xi_j) \text{ is a martingale}$$

and, because of Example 23.3(v), $(|M_k|)_k$ is a submartingale. Applying (25.10) in this situation proves the claimed inequality since

$$
\begin{aligned}
\mathbb{V}M_n &= \mathbb{E}(M_n^2) && (\text{since } \mathbb{E}M_n = 0) \\
&= \sum_{j=1}^{n} \mathbb{E}(\xi_j^2)
\end{aligned}
$$

where we use, for the last equality, what probabilists call *Theorem of Bienaymé* for the independent random variables ξ_j:

$$
\begin{aligned}
\mathbb{E}(M_n^2) &= \sum_{j,k=1}^{n} \mathbb{E}\big[(\xi_j - \mathbb{E}\xi_j)(\xi_k - \mathbb{E}\xi_k)\big] \\
&= \sum_{j=k=1}^{n} \mathbb{E}\big[(\xi_j - \mathbb{E}\xi_j)^2\big] + \sum_{j\neq k} \mathbb{E}\big[(\xi_j - \mathbb{E}\xi_j)\big]\mathbb{E}\big[(\xi_k - \mathbb{E}\xi_k)\big] && (\text{by independence}) \\
&= \sum_{j=k=1}^{n} \mathbb{E}\big[(\xi_j - \mathbb{E}\xi_j)^2\big] \\
&= \sum_{j=1}^{n} \mathbb{E}\big[M_j^2\big] \\
&= \sum_{j=1}^{n} \mathbb{V}M_j.
\end{aligned}
$$

Problem 25.9 Solution:

(i) As in the proof of Theorem 25.12 we find

$$\begin{aligned}
\int u^p\,d\mu &\overset{(14.9)}{=} p\int_0^\infty s^{p-1}\,\mu(\{u\geqslant s\})\,ds\\
&\leqslant p\int_0^\infty s^{p-2}\left(\int \mathbb{1}_{\{u\geqslant s\}}(x)w(x)\mu(dx)\right)ds\\
&= p\int\left(\int_0^\infty \mathbb{1}_{[0,u(x)]}(s)s^{p-2}\,ds\right)w(x)\,\mu(dx)\\
&= p\int \frac{u(x)^{p-1}}{p-1}\,w(x)\,\mu(dx)\\
&= \frac{p}{p-1}\int u^{p-1}w\,d\mu
\end{aligned}$$

Note that this inequality is meant in $[0,+\infty]$, i.e. we allow the cases $a\leqslant+\infty$ and $+\infty\leqslant+\infty$.

(ii) Pick conjugate numbers $p,q\in(1,\infty)$, i.e. $q=\frac{p}{p-1}$. Then we can rewrite the result of (i) and then apply Hölder's inequality to get

$$\begin{aligned}
\|u\|_p^p &\leqslant \frac{p}{p-1}\int u^{p-1}w\,d\mu\\
&\leqslant \frac{p}{p-1}\left(\int u^{(p-1)q}\,d\mu\right)^{1/q}\left(\int w^p\,d\mu\right)^{1/p}\\
&= \frac{p}{p-1}\left(\int u^p\,d\mu\right)^{1-1/p}\|w\|_p\\
&= \frac{p}{p-1}\|u\|_p^{p-1}\cdot\|w\|_p
\end{aligned}$$

and the claim follows upon dividing both sides by $\|u\|_p^{p-1}$. (Here we use the finiteness of this expression, i.e. the assumption $u\in\mathcal{L}^p$).

Problem 25.10 Solution: Only the first inequality needs proof. Note that

$$\max_{1\leqslant j\leqslant N}\int |u_j|^p\,d\mu\leqslant\int\max_{1\leqslant j\leqslant N}|u_j|^p\,d\mu=\int u_N^*\,d\mu$$

from which the claim easily follows.

Problem 25.11 Solution: Let $(A_k)_k\subset\mathscr{A}_0$ be an exhausting sequence, i.e. $A_k\uparrow X$ and $\mu(A_k)<\infty$. Since $(u_j)_j$ is L^1-bounded, we know that

$$\sup_j\|u_j\|_p\leqslant c<\infty$$

and we find, using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$

$$\int|\mathbb{1}_{A_k}u_j|\,d\mu\leqslant\left(\mu(A_k)\right)^{1/q}\cdot\|u_j\|_p\leqslant c\left(\mu(A_k)\right)^{1/q}$$

uniformly for all $j \in \mathbb{N}$. This means that the martingale $(\mathbb{1}_{A_k} u_j)_j$ (see the solution to Problem 24.8) is L^1-bounded and we get, as in Problem 24.8 that for some unique function u

$$\lim_j \mathbb{1}_{A_k} u_j = \mathbb{1}_{A_k} u \quad \forall k$$

a.e., hence $u_j \xrightarrow[j\to\infty]{} u$ a.e. Using Fatou's Lemma we get

$$\begin{aligned}
\int |u|^p \, d\mu &= \int \liminf_j |u_j|^p \, d\mu \\
&\leqslant \liminf \int |u_j|^p \, d\mu \\
&\leqslant \sup_j \int |u_j|^p \, d\mu \; < \; \infty
\end{aligned}$$

which means that $u \in L^p$.

For each $k \in \mathbb{N}$ the martingale $(\mathbb{1}_{A_k} u_j)_j$ is also uniformly integrable: using Hölder's and Markov's inequalities we arrive at

$$\begin{aligned}
\int_{\{\mathbb{1}_{A_k}|u_j| > \mathbb{1}_{A_k} R\}} \mathbb{1}_{A_k} |u_j| \, d\mu &\leqslant \int_{\{|u_j| > R\}} \mathbb{1}_{A_k} |u_j| \, d\mu \\
&\leqslant \big(\mu\{|u_j| > R\}\big)^{1/q} \|u_j\|_p \\
&\leqslant \left(\frac{1}{R^p} \|u_j\|_p^p\right)^{1/q} \|u_j\|_p \\
&\leqslant \frac{c^{p/q+1}}{R^{p/q}}
\end{aligned}$$

and the latter tends, uniformly for all j, to zero as $R \to \infty$. Since $\mathbb{1}_{A_k} \cdot R$ is integrable, the claim follows.

Thus, Theorem 24.6 applies and shows that for $u_\infty := u$ and every k the family $(u_j \mathbb{1}_{A_k})_{j \in \mathbb{N} \cup \{\infty\}}$ is a martingale. Because of Example 23.3(vi) $(|u_j|^p \mathbb{1}_{A_k})_{j \in \mathbb{N} \cup \{\infty\}}$ is a submartingale and, therefore, for all $k \in \mathbb{N}$

$$\int |\mathbb{1}_{A_k} u_j|^p \, d\mu \leqslant \int |\mathbb{1}_{A_k} u_{j+1}|^p \, d\mu \leqslant \int |\mathbb{1}_{A_k} u_\infty|^p \, d\mu = \int |\mathbb{1}_{A_k} u|^p \, d\mu,$$

Since, by Fatou's lemma

$$\int |\mathbb{1}_{A_k} u|^p \, d\mu = \int \liminf_j |\mathbb{1}_{A_k} u_j|^p \, d\mu \leqslant \liminf_j \int |\mathbb{1}_{A_k} u_j|^p \, d\mu$$

we see that

$$\int |\mathbb{1}_{A_k} u|^p \, d\mu = \lim_j \int |\mathbb{1}_{A_k} u_j|^p \, d\mu = \sup_j \int |\mathbb{1}_{A_k} u_j|^p \, d\mu.$$

Since suprema interchange, we get

$$\begin{aligned}
\int |u|^p \, d\mu &= \sup_k \int |\mathbb{1}_{A_k} u|^p \, d\mu \\
&= \sup_k \sup_j \int |\mathbb{1}_{A_k} u_j|^p \, d\mu
\end{aligned}$$

$$= \sup_j \sup_k \int |\mathbb{1}_{A_k} u_j|^p \, d\mu$$
$$= \sup_j \int |u_j|^p \, d\mu$$

and Riesz's convergence theorem, Theorem 13.10, finally proves that $u_j \to u$ in L^p.

Problem 25.12 Solution: Since f_k is a martingale and since

$$\begin{aligned}\int |f_k| \, d\lambda^n &\leqslant \sum_{z \in 2^{-k}\mathbb{Z}^n} \frac{1}{\lambda^n(Q_k(z))} \int_{Q_k(z)} |f| \, d\lambda^n \int \mathbb{1}_{Q_k(z)} \, d\lambda^n \\ &= \sum_{z \in 2^{-k}\mathbb{Z}^n} \int_{Q_k(z)} |f| \, d\lambda^n \\ &= \int |f| \, d\lambda^n < \infty\end{aligned}$$

we get from the martingale convergence theorem 24.2 that

$$f_\infty := \lim_k f_k$$

exists almost everywhere and that $f_\infty \in \mathcal{L}^1(\mathscr{B})$. The above calculation shows, on top of that, that for any set $Q \in \mathscr{A}_k^{[0]}$

$$\int_Q f_k \, d\lambda^n = \int_Q f \, d\lambda^n$$

and

$$\int_Q |f_k| \, d\lambda^n \leqslant \int_Q |f| \, d\lambda^n.$$

This allows us to show that $(f_k)_{k \in \mathbb{N}}$ is uniformly integrable. Indeed, fix R, take some $w \in L^1(\mathscr{A}_1^{[0]})$ with $w > 0$ (you can construct this easily using a convergent series with steps of type $z + [0, 2^{-1})^n$, $z \in \mathbb{Z}^n$ and suitable weights), and take $Q = \{|f_k| > Rw\}$ to get

$$\begin{aligned}\int_{\{|f_k|>Rw\}} |f_k| \, d\lambda^n &\leqslant \int_{\{|f_k|>Rw\}} |f| \, d\lambda^n \\ &= \int_{\{|f_k|>Rw\} \cap \{|f|>Rw/2\}} |f| \, d\lambda^n + \int_{\{|f_k|>Rw\} \cap \{|f| \leqslant Rw/2\}} |f| \, d\lambda^n \\ &\leqslant \int_{\{|f|>Rw/2\}} |f| \, d\lambda^n + \int_{\{|f_k|>Rw \geqslant 2|f|\}} |f| \, d\lambda^n \\ &\leqslant \int_{\{|f|>Rw/2\}} |f| \, d\lambda^n + \frac{1}{2} \int_{\{|f_k|>Rw \geqslant 2|f|\}} |f_k| \, d\lambda^n \\ &\leqslant \int_{\{|f|>Rw/2\}} |f| \, d\lambda^n + \frac{1}{2} \int_{\{|f_k|>Rw\}} |f_k| \, d\lambda^n\end{aligned}$$

and this we can re-arrange to become

$$\int_{\{|f_k|>Rw\}} |f_k| \, d\lambda^n \leqslant 2 \int_{\{|f|>Rw/2\}} |f| \, d\lambda^n.$$

The rhS is uniform for all k, so we can use dominated convergence and let $R \to \infty$ to get

$$\lim_{R\to\infty} \sup_k \int_{\{|f_k|>Rw\}} |f_k| \, d\lambda^n \leqslant 2 \lim_{R\to\infty} \int_{\{|f|>Rw/2\}} |f| \, d\lambda^n = 0.$$

This shows (Theorem 24.6) that $f_k \to f_\infty$ in L^1 and a.e. In particular, we get for any $k \in \mathbb{N}$ and $Q \in \mathscr{A}_k^{[0]}$

$$\int_Q f_\infty \, d\lambda^n \overset{L^1\text{-limit}}{=} \lim_{m\geqslant k, m\to\infty} \int_Q f_m \, d\lambda^n \overset{\forall m \geqslant k, \text{ def. of } f_k}{=} \int_Q f \, d\lambda^n$$

Thus, $\int_Q f_\infty \, d\lambda^n = \int_Q f \, d\lambda^n$ for all $Q \in \bigcup_k \mathscr{A}_k^{[0]}$. This is a $\cap$-stable system, so the equality also holds for $Q \in \sigma\left(\bigcup_k \mathscr{A}_k^{[0]}\right) = \mathscr{B}$.

Taking $Q = \{f > f_\infty\}$ and $Q = \{f < f_\infty\}$ – both are measurable sets since f, f_∞ are measurable – shows $f = f_\infty$ a.e. Thus, $(f_k)_{k\in\mathbb{N}\cup\{\infty\}}$ is UI.

■■

Problem 25.13 Solution: As one would expect, the derivative at x turns out to be $u(x)$. This is seen as follows (without loss of generality we can assume that $y > x$):

$$\begin{aligned}
&\left|\frac{1}{x-y}\left(\int_{[a,x]} u(t)\,dt - \int_{[a,x]} u(t)\,dt\right) - u(x)\right| \\
&\quad= \left|\frac{1}{x-y}\int_{[x,y]} \big(u(t)-u(x)\big)\,dt\right| \\
&\quad\leqslant \frac{1}{|x-y|}\int_{[x,y]} \Big|u(t)-u(x)\Big|\,dt \\
&\quad\leqslant \frac{1}{|x-y|}\,|x-y| \sup_{t\in[x,y]} \Big|u(t)-u(x)\Big| \\
&\quad= \sup_{t\in[x,y]} \Big|u(t)-u(x)\Big|
\end{aligned}$$

and the last expression tends to 0 as $|x-y| \to 0$ since u is uniformly continuous on compact sets.

If u is not continuous but merely of class L^1, we have to refer to Lebesgue's differentiation theorem, Theorem 25.20, in particular formula (25.19) which reads in our case

$$u(x) = \lim_{r\to 0} \frac{1}{2r} \int_{(x-r,x+r)} u(t)\,dt$$

for Lebesgue almost every $x \in (a,b)$.

■■

Problem 25.14 Solution: We follow the hint: first we remark that by Lemma 14.14 we know that f has at most countably many discontinuities. Since it is monotone, we also know that $F(t) := f(t+) = \lim_{s>t,s\to t} f(s)$ exists and is finite for every t and that $\{f \neq F\}$ is at most countable (since it is contained in the set of discontinuities of f), hence a Lebesgue null set.

If f is right-continuous, $\mu(a,b] := f(b) - f(a)$ extends uniquely to a measure on the Borel-sets and this measure is locally finite and σ-finite. If we apply Theorem 25.9 to μ and $\lambda = \lambda^1$ we can write $\mu = \mu^\circ + \mu^\perp$ with $\mu^\circ \ll \lambda$ and $\mu^\perp \perp \lambda$. By Corollary 25.22 $D\mu^\perp = 0$ a.e. and $D\mu^\circ$ exists a.e. and we get a.e.

$$D\mu(x) = \lim_{r\to 0} \frac{\mu(x-r,x+r)}{2r} = \lim_{r\to 0} \frac{\mu^\circ(x-r,x+r)}{2r} + 0$$

and we can set $f'(x) = D\mu(x)$ which is a.e. defined. Where it is not defined, we put it equal to 0.

Now we get

$$\begin{aligned} f(b) - f(a) &= \mu(a,b] \\ &\geqslant \mu(a,b) \\ &= \int_{(a,b)} d\mu \\ &\geqslant \int_{(a,b)} d\mu^\circ \\ &= \int_{(a,b)} D\mu(x)\,\lambda(dx) \\ &= \int_{(a,b)} f'(x)\,\lambda(dx). \end{aligned}$$

The above estimates show that we get equality if f is continuous and also absolutely continuous w.r.t. Lebesgue measure.

■■

Problem 25.15 Solution: Without loss of generality we may assume that $f_j(a) = 0$, otherwise we would consider the (still increasing) functions $x \mapsto f_j(x) - f_j(a)$ resp. their sum $x \mapsto s(x) - s(a)$. The derivatives are not influenced by this operation. As indicated in the hint call $s_n(x) := f_1(x) + \cdots + f_n(x)$ the nth partial sum. Clearly, s, s_n are increasing

$$\frac{s_n(x+h) - s_n(x)}{h} \leqslant \frac{s_{n+1}(x+h) - s_{n+1}(x)}{h} \leqslant \frac{s(x+h) - s(x)}{h}.$$

and possess, because of Problem 25.14, almost everywhere positive derivatives:

$$s_n'(x) \leqslant s_{n+1}'(x) \leqslant \cdots s'(x), \qquad \forall\, x \notin E$$

Note that the exceptional null-sets depend originally on the function s_n etc. but we can consider their (countable!!) union and get thus a universal exceptional null set E. This shows that the formally differentiated series

$$\sum_{j=1}^\infty f_j'(x) \quad \text{converges for all } x \notin E.$$

Since the sequence of partial sums is increasing, it will be enough to check that

$$s'(x) - s_{n_k}'(x) \xrightarrow[k\to\infty]{} 0 \quad \forall x \notin E.$$

Since, by assumption the sequence $s_k(x) \to s(x)$ we can choose a subsequence n_k in such a way that

$$s(b) - s_{n_k}(b) < 2^{-k} \quad \forall\, k \in \mathbb{N}.$$

Since

$$0 \leqslant s(x) - s_{n_k}(x) \leqslant s(b) - s_{n_k}(b)$$

the series

$$\sum_{k=1}^{\infty}(s(x) - s_{n_k}(x)) \leqslant \sum_{k=1}^{\infty} 2^{-k} < \infty \quad \forall\, x \in [a, b].$$

By the first part of the present proof, we can differentiate this series term-by-term and get that

$$\sum_{k=1}^{\infty}(s'(x) - s'_{n_k}(x)) \text{ converges} \quad \forall\, x \in (a, b) \setminus E$$

and, in particular, $s'(x) - s'_{n_k}(x) \xrightarrow[k\to\infty]{} 0$ for all $x \in (a, b) \setminus E$ which was to be proved.

26 Abstract Hilbert space. Solutions to Problems 26.1–26.19

Problem 26.1 Solution: If we set $\mu = \delta_1 + \cdots + \delta_n$, $X = \{1, 2, \dots, n\}$, $\mathscr{A} = \mathscr{P}(X)$ or $\mu = \sum_{j\in\mathbb{N}} \delta_j$, $X = \mathbb{N}$, $\mathscr{A} = \mathscr{P}(X)$, respectively, we can deduce 26.5(i) and (ii) from 26.5(iii).

Let us, therefore, only verify (iii). Without loss of generality (see the complexification of a real inner product space in Problem 26.3) we can consider the real case where $L^2 = L^2_{\mathbb{R}}$.

- L^2 is a vector space — this was done in Remark 13.5.
- $\langle u, v\rangle$ is finite on $L^2 \times L^2$ — this is the Cauchy–Schwarz inequality 13.3.
- $\langle u, v\rangle$ is bilinear — this is due to the linearity of the integral.
- $\langle u, v\rangle$ is symmetric — this is obvious.
- $\langle v, v\rangle$ is definite, and $\|u\|_2$ is a Norm — cf. Remark 13.5.

■■

Problem 26.2 Solution:

(i) We prove it for the complex case—the real case is simpler. Observe that

$$\begin{aligned}\langle u \pm w, u \pm w\rangle &= \langle u, u\rangle \pm \langle u, w\rangle \pm \langle w, u\rangle + \langle w, w\rangle\\ &= \langle u, u\rangle \pm \langle u, w\rangle \pm \overline{\langle u, w\rangle} + \langle w, w\rangle\\ &= \langle u, u\rangle \pm 2\,\mathrm{Re}\langle u, w\rangle + \langle w, w\rangle.\end{aligned}$$

Thus,

$$\langle u + w, u + w\rangle + \langle u - w, u - w\rangle = 2\langle u, u\rangle + 2\langle w, w\rangle.$$

Since $\|v\|^2 = \langle v, v\rangle$ we are done.

(ii) (SP_1): Obviously,

$$0 < (u, u) = \tfrac{1}{4}\,\|2v\|^2 = \|v\|^2 \;\Rightarrow\; v \neq 0.$$

(SP_1): is clear.

(iii) Using at the point (*) below the parallelogram identity, we have

$$\begin{aligned}4(u + v, w) &= 2(u + v, 2w)\\ &= \tfrac{1}{2}\left(\|u + v + 2w\|^2 - \|u + v - 2w\|^2\right)\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\left(\|(u+w)+(v+w)\|^2 - \|(u-w)+(v-w)\|^2\right)\\
&\overset{*}{=} \tfrac{1}{2}\Big[2\big(\|u+w\|^2 + \|v+w\|^2 - \|u-w\|^2 - \|v-w\|^2\big)\Big]\\
&= 4(u,w) + 4(v,w)
\end{aligned}
$$

and the claim follows.

(iv) We show $(qv,w) = q(v,w)$ for all $q \in \mathbb{Q}$. If $q = n \in \mathbb{N}_0$, we iterate (iii) n times and have

$$(nv,w) = n(v,w) \qquad \forall\, n \in \mathbb{N}_0 \tag{*}$$

(the case $n=0$ is obvious). By the same argument, we get for $m \in \mathbb{N}$

$$(v,w) = \left(m\,\tfrac{1}{m}\,v,w\right) = m\left(\tfrac{1}{m}\,v,w\right)$$

which means that

$$\left(\tfrac{1}{m}\,v,w\right) = \tfrac{1}{m}\,(v,w) \qquad \forall\, m \in \mathbb{N}. \tag{**}$$

Combining (*) and (**) then yields $(\frac{n}{m}\,v,w) = \frac{n}{m}\,(v,w)$. Thus,

$$(pu+qv,w) = p(u,w) + q(v,w) \qquad \forall\, p,q \in \mathbb{Q}.$$

(v) By the lower triangle inequality for norms we get for any $s,t \in \mathbb{R}$

$$
\begin{aligned}
\Big|\|tv \pm w\| - \|sv \pm w\|\Big| &\leqslant \|(tv \pm w) - (sv \pm w)\|\\
&= \|(t-s)v\|\\
&= |t-s| \cdot \|v\|.
\end{aligned}
$$

This means that the maps $t \mapsto tv \pm w$ are continuous and so is $t \mapsto (tv,w)$ as the sum of two continuous maps. If $t \in \mathbb{R}$ is arbitrary, we pick a sequence $(q_j)_{j\in\mathbb{N}} \subset \mathbb{Q}$ such that $\lim_j q_j = t$. Then

$$(tv,w) = \lim_j (q_j v,w) = \lim_j q_j(qv,w) = t(v,w)$$

so that

$$(su+tv,w) = (su,w) + (tv,w) = s(u,w) + t(v,w).$$

■■

Problem 26.3 Solution: This is actually a problem on complexification of inner product spaces... .

Since v and iw are vectors in $V \oplus iV$ and since $\|v\| = \|\pm iv\|$, we get

$$
\begin{aligned}
(v,iw)_{\mathbb{R}} &= \tfrac{1}{4}\left(\|v+iw\|^2 - \|v-iw\|^2\right)\\
&= \tfrac{1}{4}\left(\|i(w-iv)\|^2 - \|(-i)(w+iv)\|^2\right)\\
&= \tfrac{1}{4}\left(\|w-iv\|^2 - \|w+iv\|^2\right) \qquad (*)\\
&= (w,-iv)_{\mathbb{R}}\\
&= -(w,iv)_{\mathbb{R}}.
\end{aligned}
$$

In particular,

$$(v, iv) = -(v, iv) \implies (v, iv) = 0 \quad \forall v,$$

and we get

$$(v, v)_{\mathbb{C}} = (v, v)_{\mathbb{R}} > 0 \implies v = 0.$$

Moreover, using (*) we see that

$$\begin{aligned}(v, w)_{\mathbb{C}} &= (v, w)_{\mathbb{R}} + i(v, iw)_{\mathbb{R}} \\ &\overset{*}{=} (w, v)_{\mathbb{R}} - i(w, iv)_{\mathbb{R}} \\ &= (w, v)_{\mathbb{R}} + \bar{i} \cdot (w, iv)_{\mathbb{R}} \\ &= \overline{(w, v)_{\mathbb{R}} + i(w, iv)_{\mathbb{R}}} \\ &= \overline{(w, v)_{\mathbb{C}}}.\end{aligned}$$

Finally, for real $\alpha, \beta \in \mathbb{R}$ the linearity property of the real scalar product shows that

$$\begin{aligned}(\alpha u + \beta v, w)_{\mathbb{C}} &= \alpha(u, w)_{\mathbb{R}} + \beta(v, w)_{\mathbb{R}} + i\alpha(u, iw)_{\mathbb{R}} + i\beta(v, iw)_{\mathbb{R}} \\ &= \alpha(u, w)_{\mathbb{C}} + \beta(v, w)_{\mathbb{C}}.\end{aligned}$$

Therefore to get the general case where $\alpha, \beta \in \mathbb{C}$ we only have to consider the purely imaginary case:

$$\begin{aligned}(iv, w)_{\mathbb{C}} = (iv, w)_{\mathbb{R}} + i(iv, iw)_{\mathbb{R}} &\overset{*}{=} -(v, iw)_{\mathbb{R}} - i(v, -w)_{\mathbb{R}} \\ &= -(v, iw)_{\mathbb{R}} + i(v, w)_{\mathbb{R}} \\ &= i\big(i(v, iw)_{\mathbb{R}} + (v, w)_{\mathbb{R}}\big) \\ &= i(v, w)_{\mathbb{C}},\end{aligned}$$

where we use twice the identity (*). This shows complex linearity in the first coordinate, while skew-linearity follows from the conjugation rule $(v, w)_{\mathbb{C}} = \overline{(w, v)_{\mathbb{C}}}$.

Problem 26.4 Solution: The parallelogram law (stated for L^1) *would* say:

$$\left(\int_0^1 |u + w|\,dx\right)^2 + \left(\int_0^1 |u - w|\,dx\right)^2 = 2\left(\int_0^1 |u|\,dx\right)^2 + 2\left(\int_0^1 |w|\,dx\right)^2.$$

If $u \pm w, u, w$ have always only ONE sign (i.e. +ve or −ve), we could leave the modulus signs $|\cdot|$ away, and the equality would be correct! To show that there is no equality, we should therefore choose functions where we have some sign change. We try:

$$u(x) = 1/2, \qquad w(x) = x$$

(note: $u - w$ does change its sign!) and get

$$\begin{aligned}
\int_0^1 |u+w|\,dx &= \int_0^1 (\tfrac12 + x)\,dx = [\tfrac12(x+x^2)]_0^1 = 1\\
\int_0^1 |u-w|\,dx &= \int_0^{1/2} (\tfrac12 - x)\,dx + \int_{1/2}^1 (x-\tfrac12)\,dx\\
&= [\tfrac12(x-x^2)]_0^{1/2} + [\tfrac12(x^2-x)]_{1/2}^1\\
&= \tfrac14 - \tfrac18 - \tfrac18 + \tfrac14 = \tfrac14\\
\int_0^1 |u|\,dx &= \int_0^1 \tfrac12\,dx = \tfrac12\\
\int_0^1 |w|\,dx &= \int_0^1 x\,dx = [\tfrac12 x^2]_0^1 = \tfrac12
\end{aligned}$$

This shows that

$$1^2 + (\tfrac14)^2 = \tfrac{17}{16} \neq 1 = 2(\tfrac12)^2 + 2(\tfrac12)^2.$$

We conclude, in particular, that L^1 cannot be a Hilbert space (since in any Hilbert space the Parallelogram law is true....).

Problem 26.5 Solution:

(i) If $k = 0$ we have $\theta = 1$ and everything is obvious. If $k \neq 0$, we use the summation formula for the geometric progression to get

$$S := \frac1n \sum_{j=1}^n \theta^{jk} = \frac1n \sum_{j=1}^n \left(\theta^k\right)^j = \frac{\theta}{n}\,\frac{1-(\theta^k)^n}{1-\theta^k}$$

but $(\theta^k)^n = \exp(2\pi\frac{i}{n}\cdot k\cdot n) = \exp(2\pi i k) = 1$. Thus $S = 0$ and the claim follows.

(ii) Note that $\overline{\theta^j} = \theta^{-j}$ so that

$$\begin{aligned}
\|v+\theta^j w\|^2 &= \langle v+\theta^j w, v+\theta^j w\rangle\\
&= \langle v,v\rangle + \langle v,\theta^j w\rangle + \langle \theta^j w, v\rangle + \langle \theta^j w, \theta^j w\rangle\\
&= \langle v,v\rangle + \theta^{-j}\langle v,w\rangle + \theta^j\langle w,v\rangle + \theta^j\theta^{-j}\langle w,w\rangle\\
&= \langle v,v\rangle + \theta^{-j}\langle v,w\rangle + \theta^j\langle w,v\rangle + \langle w,w\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac1n\sum_{j=1}^n \theta^j \|v+\theta^j w\|^2\\
&= \frac1n\sum_{j=1}^n \theta^j\langle v,v\rangle + \frac1n\sum_{j=1}^n \langle v,w\rangle + \frac1n\sum_{j=1}^n \theta^{2j}\langle w,v\rangle + \frac1n\sum_{j=1}^n \theta^j\langle w,w\rangle\\
&= 0 + \langle v,w\rangle + 0 + 0
\end{aligned}$$

where we use the result from part (i) of the exercise.

(iii) Since the function $\phi \mapsto e^{i\phi}\|v + e^{i\phi}w\|^2$ is bounded and continuous, the integral exists as a (proper) Riemann integral, and we can use *any* Riemann sum to approximate the integral, see 12.6–12.12 in Chapter 12 or Corollary I.6 and Theorem I.8 of Appendix I. Before we do that, we change variables according to $\psi = (\phi + \pi)/2\pi$ so that $d\psi = d\phi/2\pi$ and

$$\frac{1}{2\pi}\int_{(-\pi,\pi]} e^{i\phi}\left\|v + e^{i\phi}w\right\|^2 d\phi = -\int_{(0,1]} e^{2\pi i\psi}\left\|v - e^{2\pi i\psi}w\right\|^2 d\psi.$$

Now using equidistant Riemann sums with step $1/n$ and nodes $\theta_n^j = e^{2\pi i\cdot\frac{1}{n}\cdot j}$, $j = 1, 2, \dots, n$ yields, because of part (ii) of the problem,

$$\begin{aligned}
-\int_{(0,1]} e^{2\pi i\psi}\left\|v - e^{2\pi i\psi}w\right\|^2 d\psi &= -\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\theta_n^j\|v - \theta_n^j w\|^2 \\
&= -\lim_{n\to\infty}\langle v, -w\rangle \\
&= \langle v, w\rangle.
\end{aligned}$$

■■

Problem 26.6 Solution: We assume that V is a $\mathbb{C}$-inner product space. Then,

$$\begin{aligned}
\|v + w\|^2 &= \langle v + w, v + w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2 \\
&= \|v\|^2 + 2\,\mathrm{Re}\langle v, w\rangle + \|w\|^2.
\end{aligned}$$

Thus

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2 \iff \mathrm{Re}\langle v, w\rangle = 0 \iff v\perp w.$$

■■

Problem 26.7 Solution: Let $(h_k)_k \subset \mathcal{H}$ such that $\lim_k \|h_k - h\| = 0$. By the triangle inequality

$$\|h_k - h_\ell\| \leqslant \underbrace{\|h_k - h\|}_{\to 0} + \underbrace{\|h - h_\ell\|}_{\to 0} \xrightarrow[k,\ell\to\infty]{} 0.$$

■■

Problem 26.8 Solution: Let $g, \widetilde{g} \in \mathcal{H}$. By the Cauchy–Schwarz inequality 26.3

$$\left|\langle g, h\rangle - \langle\widetilde{g}, h\rangle\right| \leqslant \left|\langle g - \widetilde{g}, h\rangle\right| \leqslant \|h\|\cdot\|\widetilde{g} - g\|$$

which proves continuity. Incidentally, this calculation shows also that, since $g \mapsto \langle g, h\rangle$ is linear, it would have been enough to check continuity at the point $g = 0$ (think about it!).

■■

Problem 26.9 Solution: Definiteness (N_1) and positive homogeneity (N_2) are obvious. The triangle inequality reads in this context ($g, g', h, h' \in \mathcal{H}$):

$$|||(g,h)+(g',h')||| \leqslant |||(g,h)||| + |||(g',h')||| \iff$$
$$\big(\|g+g'\|^p + \|h+h'\|^p\big)^{1/p} \leqslant \big(\|g\|^p + \|h\|^p\big)^{1/p} + \big(\|g'\|^p + \|h'\|^p\big)^{1/p}.$$

Since

$$\big(\|g+g'\|^p + \|h+h'\|^p\big)^{1/p} \leqslant \big(\big[\|g\|\|g'\|\big]^p + \big[\|h\| + \|h'\|\big]^p\big)^{1/p}$$

we can use the Minkowski inequality for sequences resp. in $\mathbb{R}^2$—which reads for numbers $a, A, b, B \geqslant 0$

$$\big((a+b)^p + (A+B)^p\big)^{1/p} \leqslant \big(a^p + A^p\big)^{1/p} + \big(b^p + B^p\big)^{1/p}$$

—and the claim follows.

Since $\mathbb{R}^2$ is only with the Euclidean norm a Hilbert space—the parallelogram identity fails for the norms $(|x|^p + |y|^p)^{1/p}$—this shows that also in the case at hand only $p = 2$ will be a Hilbert space norm.

■■

Problem 26.10 Solution: For the scalar product we have for all $g, g', h, h' \in \mathcal{H}$ such that $\|g-g'\|^2 + \|h-h'\|^2 < 1$

$$\Big|\langle g-g', h-h'\rangle\Big| \leqslant \|g-g'\| \cdot \|h-h'\| \leqslant \big[\|g-g'\|^2 + \|h-h'\|^2\big]^{1/2}$$

where we use the elementary inequality

$$ab \leqslant \frac{1}{2}(a^2+b^2) \leqslant a^2+b^2 \underbrace{\leqslant \sqrt{a^2+b^2}}_{\text{if } a^2+b^2\leqslant 1}.$$

Since $(g,h) \mapsto \big[\|g\|^2 + \|h\|^2\big]^{1/2}$ is a norm on $\mathcal{H}\times\mathcal{H}$ we are done.

Essentially the same calculation applies to $(t,h) \mapsto t\cdot h$.

■■

Problem 26.11 Solution: Assume that $\mathcal{H}$ has a countable maximal ONS, say $(e_j)_j$. Then, by definition, every vector $h \in \mathcal{H}$ can be approximated by a sequence made up of finite linear combinations of the $(e_j)_j$:

$$h_k := \sum_{j=1}^{n(k)} \alpha_j \cdot e_j$$

(note that $\alpha_j = 0$ is perfectly possible!). In view of problem 26.10 we can even assume that the α_j are rational numbers. This shows that the set

$$\mathcal{D} := \Big\{ \sum_{j=1}^{n} \alpha_j \cdot e_j \,:\, \alpha_j \in \mathbb{Q}, n \in \mathbb{N} \Big\}$$

is a countable dense subset of $\mathcal{H}$.

Conversely, if $\mathcal{D} \subset \mathcal{H}$ is a countable dense subset, we can use the Gram-Schmidt procedure and obtain from $\mathcal{D}$ an ONS. Then Theorem 26.24 proves the claim.

Problem 26.12 Solution: Let us, first of all, show that for a closed subspace $C \subset \mathcal{H}$ we have $C = (C^\perp)^\perp$.

Because of Lemma 26.12 we know that $C \subset (C^\perp)^\perp$ and that $C^\perp$ is itself a closed linear subspace of $\mathcal{H}$. Thus,

$$C \oplus C^\perp = \mathcal{H} = C^\perp \oplus (C^\perp)^\perp.$$

Thus C cannot be a proper subspace of $(C^\perp)^\perp$ and therefore $C = (C^\perp)^\perp$.

Applying this to the obviously closed subspace $C := \mathbb{K} \cdot w = \operatorname{span}(w)$ we conclude that $\operatorname{span}(w) = \operatorname{span}(w)^{\perp\perp}$.

By assumption, $M_w = \{w\}^\perp$ and $M_w^\perp = \{w\}^{\perp\perp}$ and we have $w \in \{w\}^{\perp\perp}$. The last expression is a (closed) subspace, so

$$w \in \{w\}^{\perp\perp} \implies \operatorname{span}(w) \subset \{w\}^{\perp\perp}$$

also. Further

$$\begin{aligned}\{w\} \subset \operatorname{span}(w) &\implies \{w\}^\perp \supset \operatorname{span}(w)^\perp \\ &\implies \{w\}^{\perp\perp} \subset \operatorname{span}(w)^{\perp\perp} = \operatorname{span}(w)\end{aligned}$$

and we conclude that

$$\{w\}^{\perp\perp} = \operatorname{span}(w)$$

which is either $\{0\}$ or a one-dimensional subspace.

Problem 26.13 Solution:

(i) By Pythagoras' Theorem 26.19

$$\|e_j - e_k\|^2 = \|e_j\|^2 + \|e_k\|^2 = 2 \qquad \forall\, j \neq k.$$

This shows that no subsequence $(e_j)_{j\in\not}$ can ever be a Cauchy sequence, i.e. it cannot converge.

If $h \in \mathcal{H}$ we get from Bessel's inequality 26.19 that the series

$$\sum_j |\langle e_j, h\rangle|^2 \leqslant \|h\|^2$$

is finite, i.e. converges. Thus the sequence with elements $\langle e_j, h\rangle$ must converge to 0 as $j \to \infty$.

(ii) Parseval's equality 26.19 shows that

$$\|h\|^2 = \sum_{j=1}^\infty |\langle e_j, h\rangle|^2 = \sum_{j=1}^\infty |c_j|^2 \leqslant \sum_{j=1}^\infty \frac{1}{j^2} < \infty$$

uniformly for all $h \in Q$, i.e. Q is a bounded set.

Let $(h_\ell)_\ell \subset Q$ be a sequence with $\lim_\ell h_\ell = h$ and write $c_j := \langle e_j, h\rangle$ and $c_j^\ell := \langle e_j, h_\ell\rangle$. Because of the continuity of the scalar product

$$|c_j| = |\langle e_j, h\rangle| = \lim_\ell |\langle e_j, h_\ell\rangle| = \lim_\ell |c_j^\ell| \leqslant \frac{1}{j}$$

which means that $h \in Q$ and that Q is closed.

Let $(h_\ell)_\ell \subset Q$ be a sequence and set $c_j(\ell) := \langle e_j, h_\ell\rangle$. Using the Bolzano-Weierstraß theorem for bounded sequences we get

$$|c_1(\ell)| \leqslant 1 \implies \exists \big(c_1(\ell_j^1)\big)_j \subset \big(c_1(\ell)\big)_\ell : \lim_j c_1(\ell_j^1) = \gamma_1$$

and

$$|c_2(\ell_j^1)| \leqslant \frac{1}{2} \implies \exists \big(c_2(\ell_j^2)\big)_j \subset \big(c_2(\ell_j^1)\big)_j : \lim_j c_2(\ell_j^2) = \gamma_2$$

and, recursively,

$$|c_k(\ell_j^{k-1})| \leqslant \frac{1}{k} \implies \exists \big(c_k(\ell_j^k)\big)_j \subset \big(c_k(\ell_j^{k-1})\big)_j : \lim_j c_k(\ell_j^k) = \gamma_k$$

and since we have considered sub-sub-etc.-sequences we get

$$c_k(\ell_m^m) \xrightarrow[m\to\infty]{} \gamma_k \quad \forall\, k \in \mathbb{N}.$$

Thus, we have constructed a subsequence $(h_{\ell_m^m})_m \subset (h_\ell)_\ell$ with

$$\langle e_k, h_{\ell_m^m}\rangle \xrightarrow[m\to\infty]{} \gamma_k \quad \forall\, k \in \mathbb{N} \tag{*}$$

so that $\gamma_j \leqslant 1/j$. Setting $h = \sum_j \gamma_j e_j$ we see (by Parseval's relation) that $h \in Q$. Further,

$$\begin{aligned}\|h - h_{\ell_m^m}\|^2 &= \sum_{j=1}^\infty |\gamma_j - c_j(\ell_m^m)|^2 \\ &\leqslant \sum_{j=1}^N |\gamma_j - c_j(\ell_m^m)|^2 + \sum_{j=N+1}^\infty \frac{4}{j^2}.\end{aligned}$$

Letting first $m \to \infty$ we get, because of (*)

$$\sum_{j=1}^N |\gamma_j - c_j(\ell_m^m)|^2 \xrightarrow[m\to\infty]{} 0,$$

and letting $N \to \infty$ gives

$$\limsup_m \|h - h_{\ell_m^m}\|^2 \leqslant \sum_{j=N+1}^\infty \frac{4}{j^2} \xrightarrow[N\to\infty]{} 0$$

so that $\lim_m \|h - h_{\ell_m^m}\|^2 = 0$.

(iii) R cannot be compact since $(e_j)_j \subset R$ does not have any convergent subsequence, see part (i).

R is bounded since $r \in R$ if, and only if, there is some $j \in \mathbb{N}$ such that

$$\|r - e_j\| \leqslant \frac{1}{j} \leqslant 1.$$

Thus, every $r \in R$ is bounded by

$$\|r\| \leqslant \|r - e_j\| + \|e_j\| \leqslant 2.$$

R is closed. Indeed, if $x_j \in B_{1/j}(e_j)$ we see that for $j \neq k$

$$\begin{aligned} \|x_j - x_k\| &= \|(x_j - e_j) + (e_j - e_k) + (e_k - x_k)\| \\ &\geqslant \|e_j - e_k\| - \|x_j - e_j\| - \|x_k - e_k\| \\ &\overset{\text{(i)}}{\geqslant} \sqrt{2} - \frac{1}{j} - \frac{1}{k}. \end{aligned}$$

This means that any sequence $(r_j)_r \subset R$ with $\lim_j r_j = r$ is in at most finitely many of the sets $\overline{B_{1/j}(e_j)}$. But a finite union of closed sets is closed so that $r \in R$.

(iv) Assume that $\sum_j \delta_j^2 < \infty$. Then closedness, boundedness and compactness follows exactly as in part (ii) of the problem with δ_j replacing $1/j$.

Conversely, assume that S is compact. Then the sequence

$$h_\ell = \sum_{j=1}^{\ell} \delta_j e_j \in S$$

and, by compactness, there is a convergent subsequence

$$h_{\ell_k} = \sum_{j=1}^{\ell_k} \delta_j e_j \xrightarrow[k\to\infty]{} h.$$

By Parseval's identity we get:

$$\|h_{\ell_k}\|^2 = \sum_{j=1}^{\ell_k} \delta_j^2 \xrightarrow[k\to\infty]{} \sum_{j=1}^{\infty} \delta_j^2 = \|h\|^2 < \infty.$$

■■

Problem 26.14 Solution:

(i) Note that for all $g \neq 0$

$$|\langle g, h\rangle| \leqslant \|g\| \cdot \|h\| \implies \frac{|\langle g, h\rangle|}{\|g\|} \leqslant \|h\|$$

so that

$$\sup_{g\neq 0} \frac{|\langle g, h\rangle|}{\|g\|} \leqslant \|h\|.$$

Since for $g = h$ the supremum is attained, we get equality.

Further, since $\left\| \frac{g}{\|g\|} \right\| = 1$, we have

$$\sup_{g \neq 0} \frac{|\langle g, h\rangle|}{\|g\|} = \sup_{g \neq 0} \left| \left\langle \frac{g}{\|g\|}, h \right\rangle \right| = \sup_{\gamma,\ \|\gamma\|=1} |\langle \gamma, h\rangle|.$$

Finally,

$$\|h\| = \sup_{g,\ \|g\|=1} |\langle g, h\rangle| \leqslant \sup_{g,\ \|g\|\leqslant 1} |\langle g, h\rangle| \leqslant \sup_{g,\ \|g\|\leqslant 1} \|g\| \cdot \|h\| \leqslant \|h\|.$$

(ii) Yes, since we can, by a suitable rotation $e^{i\theta}$ achieve that

$$\langle e^{i\theta} g, h\rangle = |\langle g, h\rangle|$$

while $\|g\| = \|e^{i\theta} g\|$.

(iii) Yes. If $D \subset \mathcal{H}$ is dense and $h \in \mathcal{H}$ we find a sequence $(d_j)_j \subset D$ with $\lim_j d_j = h$. Since the scalar product and the norm are continuous, we get

$$\lim_j \frac{\langle d_j, h\rangle}{\|d_j\|} = \frac{\langle h, h\rangle}{\|h\|} = \|h\|$$

and we conclude that

$$\|h\| \leqslant \sup_j |\langle d_j / \|d_j\|, h\rangle| \leqslant \sup_{d \in D,\ \|d\|=1} |\langle d, h\rangle|.$$

The reverse inequality is trivial.

Problem 26.15 Solution: Let $x, y \in \operatorname{span}\{e_j,\ j \in \mathbb{N}\}$. By definition, there exist numbers $m, n \in \mathbb{N}$ and 'coordinates' $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathbb{K}$ such that

$$x = \sum_{j=1}^{m} \xi_j e_j \quad \text{and} \quad y = \sum_{k=1}^{n} \eta_k e_k.$$

Without loss of generality we can assume that $m \leqslant n$. By defining

$$\xi_{m+1} := 0, \dots, \xi_n := 0$$

we can write for all $\alpha, \beta \in \mathbb{K}$

$$x = \sum_{j=1}^{n} \xi_j e_j \quad \text{and} \quad y = \sum_{k=1}^{n} \eta_k e_k \quad \text{and} \quad \alpha x + \beta y = \sum_{\ell=1}^{n} (\alpha \xi_\ell + \beta \eta_\ell) e_k.$$

This shows that $\operatorname{span}\{e_j,\ j \in \mathbb{N}\} \subset \mathcal{H}$ is a linear subspace.

Problem 26.16 Solution:

(i) Since $\sum_{j=1}^{\infty} a_j^2 = \infty$ there is some number $j_1 \in \mathbb{N}$ such that

$$\sum_{j=1}^{j_1} a_j^2 > 1.$$

Since the remaining tail of the series $\sum_{j>j_1} a_j^2 = \infty$ we can construct recursively a strictly increasing sequence $(j_k)_{k\in\mathbb{N}_0} \subset \mathbb{N}$, $j_0 := 1$, such that

$$\sum_{j\in J_k} a_j^2 > 1 \quad \text{where} \quad J_k := (j_k, j_{k+1}] \cap \mathbb{N}.$$

(ii) Define the numbers γ_k as, say,

$$\gamma_k := \frac{1}{k\sqrt{\sum_{j\in J_k} a_j^2}}.$$

Then

$$\begin{aligned}
\sum_j b_j^2 &= \sum_k \sum_{j\in J_k} \gamma_k^2 a_j^2 \\
&= \sum_k \gamma_k^2 \sum_{j\in J_k} a_j^2 \\
&= \sum_k \frac{\sum_{j\in J_k} a_j^2}{k^2 \sum_{j\in J_k} a_j^2} \\
&= \sum_k \frac{1}{k^2} < \infty.
\end{aligned}$$

Moreover, since

$$\frac{\sum_{j\in J_k} a_j^2}{\sqrt{\sum_{j\in J_k} a_j^2}} \geqslant 1,$$

we get

$$\begin{aligned}
\sum_j a_j b_j &= \sum_k \sum_{j\in J_k} \gamma_k a_j^2 \\
&= \sum_k \gamma_k \sum_{j\in J_k} a_j^2 \\
&= \sum_k \frac{1}{k} \frac{\sum_{j\in J_k} a_j^2}{\sqrt{\sum_{j\in J_k} a_j^2}} \\
&\geqslant \sum_k \frac{1}{k} = \infty.
\end{aligned}$$

(iii) We want to show (note that we renamed $\beta := a$ and $\alpha := b$ for notational reasons) that for any sequence $\alpha = (\alpha_j)_j$ we have:

$$\forall\, \beta \in \ell^2 : \langle \alpha, \beta \rangle < \infty \implies \alpha \in \ell^2.$$

Assume, to the contrary, that $\alpha \notin \ell^2$. Then $\sum_j \alpha_j^2 = \infty$ and, by part (i), we can find a sequence of j_k with the properties described in (i). Because of part (ii) there is a sequence $\beta = (\beta_j)_j \in \ell^2$ such that the scalar product $\langle \alpha, \beta \rangle = \infty$. This contradicts our assumption, i.e. α should have been in ℓ^2 in the first place.

(iv) Since, by Theorem 26.24 every separable Hilbert space has a basis $(e_j)_{j \in \mathbb{N}} \subset \mathcal{H}$, we can identify $h \in \mathcal{H}$ with the sequence of 'coordinates' $(\langle h, e_j \rangle)_{j \in \mathbb{N}}$ and it is clear that (iii) implies (iv).

Problem 26.17 Solution:

(i) Since $P^2 = P$ is obvious by the uniqueness of the minimizing element, this part follows already from Remark 26.15.

(ii) Note that for $u, v \in \mathcal{H}$ we have

$$\forall\, h \in \mathcal{H} \,:\, \langle u, h \rangle = \langle v, h \rangle \;\Rightarrow\; u = v.$$

Indeed, consider $h := u - v$. Then

$$\langle u, h \rangle = \langle v, h \rangle \;\Rightarrow\; 0 = \langle u - v, h \rangle = \langle u - v, u - v \rangle = |u - v|^2$$

so that $u = v$.

Linearity of P: Let $\alpha, \beta \in \mathbb{K}$ and $f, g, h \in \mathcal{H}$. Then

$$\begin{aligned}
\langle P(\alpha f + \beta g), h \rangle &= \langle \alpha f + \beta g, Ph \rangle \\
&= \alpha \langle f, Ph \rangle + \beta \langle g, Ph \rangle \\
&= \alpha \langle Pf, h \rangle + \beta \langle Pg, h \rangle \\
&= \langle \alpha Pf + \beta Pg, h \rangle
\end{aligned}$$

and we conclude that $P(\alpha f + \beta g) = \alpha Pf + \beta Pg$.

Continuity of P: We have for all $h \in \mathcal{H}$

$$\|Ph\|^2 = \langle Ph, Ph \rangle = \langle P^2 h, h \rangle = \langle Ph, h \rangle \leqslant \|Ph\| \cdot \|h\|$$

and dividing by $\|Ph\|$ shows that P is continuous, even a contraction.

Closedness of $P(\mathcal{H})$: Note that $f \in P(\mathcal{H})$ if, and only if, $f = Ph$ for some $h \in \mathcal{H}$. Since $P^2 = P$ we get

$$\begin{aligned}
f = Ph &\iff f - Ph = 0 \\
&\iff f - P^2 h = 0 \\
&\iff f - Pf = 0 \\
&\iff f \in (\mathrm{id} - P)^{-1}(\{0\})
\end{aligned}$$

and since P is continuous and $\{0\}$ is a closed set, $(\mathrm{id} - P)^{-1}(\{0\})$ is closed and the above line shows $P(\mathcal{H}) = (\mathrm{id} - P)^{-1}(\{0\})$ is closed.

Projection: In view of Corollary 26.14 we have to show that $Ph - h$ is for any $h \in \mathcal{H}$ orthogonal to $f \in P(\mathcal{H})$. But

$$\begin{aligned}\langle Ph - h, f\rangle &= \langle Ph, f\rangle - \langle h, f\rangle \\ &= \langle h, Pf\rangle - \langle h, f\rangle \\ &= \langle h, f\rangle - \langle h, f\rangle \;=\; 0.\end{aligned}$$

(iii) Since, by assumption, $\|Ph\| \leqslant \|h\|$, P is continuous and closedness follows just as in (ii). It is, therefore, enough to show that P is an orthogonal projection.

We will show that $\mathcal{N} := \{h \in \mathcal{H} : Ph = 0\}$ satisfies $\mathcal{N}^\perp = P(\mathcal{H})$.

For this we observe that if $h \in \mathcal{H}$, $P(Ph - h) = P^2h - Ph = Ph - Ph = 0$ so that $Ph - h \in \mathcal{N}$. In particular

$$\begin{aligned}h \in \mathcal{N}^\perp \;&\Rightarrow\; y = Ph - h \in \mathcal{N} \\ &\Rightarrow\; Ph = h + y \quad \text{with} \quad h \perp y. \end{aligned} \tag{*}$$

Thus,

$$\|h\|^2 + \|y\|^2 = \|Ph\|^2 \leqslant \|h\|^2 \;\Rightarrow\; \|y\|^2 \;\Rightarrow\; y = 0.$$

We conclude that

$$h \in \mathcal{N}^\perp \;\Rightarrow\; Ph - h = 0 \;\Rightarrow\; Ph = h \;\Rightarrow\; h \in P(\mathcal{H})$$

and we have shown that $\mathcal{N}^\perp \subset P(\mathcal{H})$.

To see the converse direction we pick $h \in P(\mathcal{H})$ and find $Ph = h$. Since $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$ we have $h = x + x^\perp$ with $x \in \mathcal{N}$ and $x^\perp \in \mathcal{N}^\perp$. Thus,

$$Ph = Px + P(x^\perp) = P(x^\perp) \overset{(*)}{=} x^\perp,$$

thus

$$h = Ph = x^\perp \;\Rightarrow\; P(\mathcal{H}) \subset \mathcal{N}^\perp.$$

We have seen that $P(\mathcal{H}) = \mathcal{N}^\perp \perp \mathcal{N} = \mathrm{kernel}(P)$. This means that

$$\langle Ph - h, Ph\rangle = 0$$

and we conclude that P is an orthogonal projection.

■■

Problem 26.18 Solution:

(i) Pick $u_j \in Y_j$ and $u_k \in Y_k$, $j \neq k$. Then

$$\begin{aligned}\int_{A_m} u_j u_k \, d\mu &\leqslant \sqrt{\int_{A_m} u_j^2 \, d\mu} \sqrt{\int_{A_m} u_k^2 \, d\mu} \\ &= \begin{cases} 0 \cdot 0 & \text{if } m \notin \{j, k\} \\ \sqrt{\cdots} \cdot 0 & \text{if } m = j, m \neq k \\ 0 \cdot \sqrt{\cdots} & \text{if } m \neq j, m = k \end{cases} \\ &= 0.\end{aligned}$$

(ii) Let $u \in L^2(\mu)$ and set $w_j := w \mathbb{1}_{A_1 \cup \cdots \cup A_j}$. Since $(A_1 \cup \cdots \cup A_j)^c = A_1^c \cap \cdots \cap A_j^c \downarrow \emptyset$ we get by dominated convergence

$$\|u - w_j\|_2^2 = \int_{(A_1 \cup \cdots \cup A_j)^c} u^2 \, d\mu = \int_{A_1^c \cap \cdots \cap A_j^c} u^2 \, d\mu \xrightarrow[j \to \infty]{} 0.$$

(iii) P is given by $P_j(u) = u \mathbb{1}_{A_j}$. Clearly, $P_j : L^2(\mu) \to Y_j$ is linear and $P^2 = P$, i.e. it is a projection. Orthogonality follows from

$$\langle u - u\mathbb{1}_{A_j}, u\mathbb{1}_{A_j} \rangle = \int u \mathbb{1}_{A_j^c} \cdot u \mathbb{1}_{A_j} \, d\mu = \int u \mathbb{1}_{\emptyset} \, d\mu = 0.$$

■■

Problem 26.19 Solution:

(i) See Lemma 27.1 in Chapter 27.

(ii) Set $u_n := E^{\mathscr{A}_n} u$. Then

$$u_n = \sum_{j=0}^{n} \alpha_j \cdot \mathbb{1}_{A_j}, \qquad \alpha_j := \frac{1}{\mu(A_j)} \int_{A_j} u \, d\mu, \quad 0 \leqslant j \leqslant n.$$

where $A_0 := (A_1 \cup \cdots \cup A_n)^c$ and $1/\infty := 0$. This follows simply from the consideration that u_n, as an element of $L^2(\mathscr{A}_n)$, must be of the form $\sum_{j=0}^n \alpha_j \cdot \mathbb{1}_{A_j}$ while the α_j's are calculated as

$$\langle E^{\mathscr{A}_j} u, \mathbb{1}_{A_j} \rangle = \langle u, E^{\mathscr{A}_j} \mathbb{1}_{A_j} \rangle = \langle u, \mathbb{1}_{A_j} \rangle = \int_{A_j} u \, d\mu$$

(resp. $= 0$ if $\mu(A_0) = \infty$) so that, because of disjointness,

$$\alpha_j \mu(A_j) = \left\langle \sum_{k=0}^{n} \alpha_k \cdot \mathbb{1}_{A_k}, \mathbb{1}_{A_j} \right\rangle = \langle E^{\mathscr{A}_j} u, \mathbb{1}_{A_j} \rangle = \int_{A_j} u \, d\mu.$$

Clearly this is a linear map and $u_n \in L^2(\mathscr{A}_n)$. Orthogonality follows because all the $A_0, \ldots, A_n$ are disjoint so that

$$\langle u - u_n, u_n \rangle = \left\langle u - \sum_{j=0}^{n} \alpha_j \mathbb{1}_{A_j}, \sum_{k=0}^{n} \alpha_k \mathbb{1}_{A_k} \right\rangle$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \int_{A_j} (u - \alpha_j)\alpha_j \, d\mu \\
&= \sum_{j=0}^{n} \left(\alpha_j \int_{A_j} u \, d\mu - \mu(A_j)\alpha_j^2 \right) \\
&= \sum_{j=0}^{n} 0 \;=\; 0.
\end{aligned}
$$

(iii) We have

$$
L^2(\mathscr{A}_n)^\perp = \left\{ u - \sum_{j=0}^{n} \alpha_j \mathbb{1}_{A_j} = \sum_{j=0}^{n} (u - \alpha_j)\mathbb{1}_{A_j} \;:\; u \in L^2(\mu) \right\}
$$

(iv) In view of Remark 23.2 we have to show that

$$
\int_{A_j} E^{\mathscr{A}_n} u \, d\mu = \int_{A_j} E^{\mathscr{A}_{n+1}} u \, d\mu, \quad \forall\, A_0, A_1, \dots, A_n.
$$

Thus

$$
\int_{A_j} E^{\mathscr{A}_n} u \, d\mu = \langle E^{\mathscr{A}_n} u, \mathbb{1}_{A_j} \rangle = \langle u, E^{\mathscr{A}_n} \mathbb{1}_{A_j} \rangle = \langle u, \mathbb{1}_{A_j} \rangle = \int_{A_j} u \, d\mu
$$

for all $0 \leqslant j \leqslant n$. The same argument shows also that

$$
\int_{A_j} E^{\mathscr{A}_{n+1}} u \, d\mu = \int_{A_j} u \, d\mu \quad \forall\, j = 1, 2, \dots, n.
$$

Since the $A_1, A_2, \dots$ are pairwise disjoint and $A_0 = (A_1 \cup \dots \cup A_n)^c$, we have $A_{n+1} \subset A_0$ and $A_j \cap A_0 = \emptyset$, $1 \leqslant j \leqslant n$; if $j = 0$ we get

$$
\begin{aligned}
&\int_{A_0} E^{\mathscr{A}_{n+1}} u \, d\mu \\
&\quad = \int_{A_0} \left(\mathbb{1}_{A_{n+1}} \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mathbb{1}_{A_0 \setminus A_{n+1}} \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \right) d\mu \\
&\quad = \mu(A_0 \cap A_{n+1}) \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mu(A_0 \setminus A_{n+1}) \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \\
&\quad = \mu(A_{n+1}) \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mu(A_0 \setminus A_{n+1}) \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \\
&\quad = \int_{A_{n+1}} u \, d\mu + \int_{A_0 \setminus A_{n+1}} u \, d\mu \\
&\quad = \int_{A_0} u \, d\mu.
\end{aligned}
$$

The claim follows.

Remark. It is, actually, better to show that for $u_n := E^{\mathscr{A}_n} u$ the sequence $(u_n^2)_n$ is a sub-Martingale. (The advantage of this is that we do not have to assume that $u \in L^1$ and that $u \in L^2$ is indeed enough....). O.k.:

We have

$$\begin{aligned} A_0^n &:= (A_1 \uplus \cdots \uplus A_n)^c = A_1^c \cap \cdots \cap A_n^c \\ A_0^{n+1} &:= (A_1 \uplus \cdots \uplus A_n \uplus A_{n+1})^c = A_0^n \cap A_{n+1}^c \end{aligned}$$

and

$$\begin{aligned} E^{\mathscr{A}_n} u &= \sum_{j=1}^n \mathbb{1}_{A_j} \int_{A_j} u \, \frac{d\mu}{\mu(A_j)} + \mathbb{1}_{A_0^n} \int_{A_0^n} u \, \frac{d\mu}{\mu(A_0^n)} \\ E^{\mathscr{A}_{n+1}} u &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \int_{A_j} u \, \frac{d\mu}{\mu(A_j)} + \mathbb{1}_{A_0^{n+1}} \int_{A_0^{n+1}} u \, \frac{d\mu}{\mu(A_0^{n+1})} \end{aligned}$$

with the convention that $1/\infty = 0$. Since the A_j's are mutually disjoint,

$$\begin{aligned} \left(E^{\mathscr{A}_n} u\right)^2 &= \sum_{j=1}^n \mathbb{1}_{A_j} \left[\int_{A_j} u \, \frac{d\mu}{\mu(A_j)}\right]^2 + \mathbb{1}_{A_0^n} \left[\int_{A_0^n} u \, \frac{d\mu}{\mu(A_0^n)}\right]^2 \\ \left(E^{\mathscr{A}_{n+1}} u\right)^2 &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \left[\int_{A_j} u \, \frac{d\mu}{\mu(A_j)}\right]^2 + \mathbb{1}_{A_0^{n+1}} \left[\int_{A_0^{n+1}} u \, \frac{d\mu}{\mu(A_0^{n+1})}\right]^2. \end{aligned}$$

We have to show that $\left(E^{\mathscr{A}_n} u\right)^2 = u_n^2 \leqslant u_{n+1}^2 = \left(E^{\mathscr{A}_{n+1}} u\right)^2$. If $\mu(A_0^{n+1}) = \infty$ this follows trivially since in this case

$$\begin{aligned} \left(E^{\mathscr{A}_n} u\right)^2 &= \sum_{j=1}^n \mathbb{1}_{A_j} \left[\int_{A_j} u \, \frac{d\mu}{\mu(A_j)}\right]^2 \\ \left(E^{\mathscr{A}_{n+1}} u\right)^2 &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \left[\int_{A_j} u \, \frac{d\mu}{\mu(A_j)}\right]^2. \end{aligned}$$

If $\mu(A_0^{n+1}) < \infty$ we get

$$\begin{aligned} &\left(E^{\mathscr{A}_n} u\right)^2 - \left(E^{\mathscr{A}_{n+1}} u\right)^2 \\ &= \mathbb{1}_{A_0^n} \left[\int_{A_0^n} u \, \frac{d\mu}{\mu(A_0^n)}\right]^2 - \mathbb{1}_{A_{n+1}} \left[\int_{A_{n+1}} u \, \frac{d\mu}{\mu(A_{n+1})}\right]^2 \\ &\qquad + \mathbb{1}_{A_0^{n+1}} \left[\int_{A_0^{n+1}} u \, \frac{d\mu}{\mu(A_0^{n+1})}\right]^2 \\ &= \mathbb{1}_{A_{n+1}} \left(\left[\int_{A_{n+1}} u \, \frac{d\mu}{\mu(A_0^n)}\right]^2 - \left[\int_{A_{n+1}} u \, \frac{d\mu}{\mu(A_{n+1})}\right]^2 \right) \\ &\qquad + \mathbb{1}_{A_0^{n+1}} \left(\left[\int_{A_0^{n+1}} u \, \frac{d\mu}{\mu(A_0^n)}\right]^2 - \left[\int_{A_0^{n+1}} u \, \frac{d\mu}{\mu(A_0^{n+1})}\right]^2 \right) \end{aligned}$$

and each of the expressions in the brackets is negative since

$$A_0^n \supset A_{n+1} \implies \mu(A_0^n) \geqslant \mu(A_{n+1}) \implies \frac{1}{\mu(A_0^n)} \leqslant \frac{1}{\mu(A_{n+1})}$$

and

$$A_0^n \supset A_0^{n+1} \implies \mu(A_0^n) \geqslant \mu(A_0^{n+1}) \implies \frac{1}{\mu(A_0^n)} \leqslant \frac{1}{\mu(A_0^{n+1})}.$$

(v) Set $u_n := E^{\mathscr{A}_n}u$. Since $(u_n)_n$ is a martingale, u_n^2 is a submartingale. In fact, $(u_n^2)_n$ is even uniformly integrable. For this we remark that

$$u_n = \sum_{j=1}^n \mathbb{1}_{A_j} \int_{A_j} u(x)\, \frac{\mu(dx)}{\mu(A_j)} + \mathbb{1}_{A_0^n} \int_{A_0^n} u(x)\, frac\mu(dx)\mu(A_0^n)$$

$(1/\infty := 0)$ and that the function

$$v := \sum_{j=1}^\infty \mathbb{1}_{A_j} \int_{A_j} u(x)\, \frac{\mu(dx)}{\mu(A_j)}$$

is in $L^2(\mathscr{A}_\infty)$. Only integrability is a problem: since the A_j's are mutually disjoint, the square of the series defining v factorizes, i.e.

$$\begin{aligned}
\int v^2(y)\,\mu(dy) &= \int \left(\sum_{j=1}^\infty \mathbb{1}_{A_j}(y) \int_{A_j} u(x)\, \frac{\mu(dx)}{\mu(A_j)} \right)^2 \mu(dy) \\
&= \sum_{j=1}^\infty \int \mathbb{1}_{A_j}(y)\,\mu(dy) \left(\int_{A_j} u(x)\, \frac{\mu(dx)}{\mu(A_j)} \right)^2 \\
&\leqslant \sum_{j=1}^\infty \int \mathbb{1}_{A_j}(y)\,\mu(dy) \int_{A_j} u^2(x)\, \frac{\mu(dx)}{\mu(A_j)} \\
&= \sum_{j=1}^\infty \int_{A_j} u^2(x)\,\mu(dx) \\
&= \int u^2(x)\,\mu(dx)
\end{aligned}$$

where we use Beppo Levi's theorem (twice) and Jensen's inequality. In fact,

$$v = E^{\mathscr{A}_\infty}u.$$

Since $u_n(x) = v(x)$ for all $x \in A_1 \cup \dots \cup A_n$, and since $A_0^n = (A_1 \cup \dots \cup A_n)^c \in \mathscr{A}_n$ we find by the submartingale property

$$\begin{aligned}
\int_{\{u_n^2 > (2v)^2\}} u_n^2\,d\mu &\leqslant \int_{A_0^n} u_n^2\,d\mu \\
&\leqslant \int_{A_0^n} u^2\,d\mu \\
&\xrightarrow[n\to\infty]{} 0
\end{aligned}$$

by dominated convergence since $A_0^n \to \emptyset$ and $u^2 \in L^1(\mu)$.

Using the convergence theorem for UI (sub)martingales, Theorem 24.6, we conclude that u_j^2 converges pointwise and in L^1-sense to some $u_\infty^2 \in L^1(\mathscr{A}_\infty)$ and that $(u_j^2)_{j\in\mathbb{N}\cup\{\infty\}}$ is again a submartingale. By Riesz's convergence theorem 13.10 we conclude that $u_j \to u_\infty$ in L^2-norm.

Remark: We can also identify u_∞ with v: since $E^{\mathscr{A}_j}v = u_j = E^{\mathscr{A}_j}u_\infty$ it follows that for $k = 1, 2, \dots, j$ and all j

$$0 = \langle E^{\mathscr{A}_j}v - E^{\mathscr{A}_j}u_\infty, \mathbb{1}_{A_k}\rangle = \langle v - u_\infty, E^{\mathscr{A}_j}\mathbb{1}_{A_k}\rangle = \langle v - u_\infty, \mathbb{1}_{A_k}\rangle$$

i.e. $v = u_\infty$ on all sets of the $\cap$-stable generator of $\mathscr{A}_\infty$ which can easily be extended to contain an exhausting sequence $A_1 \uplus \cdots \uplus A_n$ of sets of finite μ-measure.

(vi) The above considerations show that the functions

$$D := \left\{ \alpha_0 \mathbb{1}_{A_0^n} + \sum_{j=1}^n \alpha_j \mathbb{1}_{A_j} \;:\; n \in \mathbb{N}, \alpha_j \in \mathbb{R} \right\}$$

(if $\mu(A_0^n) = \infty$, then $\alpha_0 = 0$) are dense in $L^2(\mathscr{A}_\infty)$. It is easy to see that

$$E := \left\{ q_0 \mathbb{1}_{A_0^n} + \sum_{j=1}^n q_j \mathbb{1}_{A_j} \;:\; n \in \mathbb{N}, \alpha_j \in \mathbb{Q} \right\}$$

(if $\mu(A_0^n) = \infty$, then $q_0 = 0$) is countable and dense in D so that the claim follows.

27 Conditional expectations. Solutions to Problems 27.1–27.19

Problem 27.1 Solution: In Theorem 27.4(vii) we have seen that

$$\mathbb{E}^{\mathscr{H}}\mathbb{E}^{\mathscr{G}}u = \mathbb{E}^{\mathscr{H}}u.$$

Since, by 27.4(i) and 27.1 $\mathbb{E}^{\mathscr{H}}u \in L^2(\mathscr{H}) \subset L^2(\mathscr{G})$ we have, because of 27.4

$$\mathbb{E}^{\mathscr{G}}\mathbb{E}^{\mathscr{H}}u = \mathbb{E}^{\mathscr{H}}u.$$

∎∎

Problem 27.2 Solution: Note that by the Markov inequality $\mu\{u > 1\} \leqslant \int u^2\,d\mu < \infty$, i.e. $u\mathbb{1}_{\{u>1\}}$ is an integrable function (use Cauchy-Schwarz).

We have

$$1\mu\{u > 1\} = \int_{\{u>1\}} 1\,d\mu \overset{(*)}{<} \int_{\{u>1\}} u\,d\mu \overset{\text{assumption}}{\leqslant} \mu\{u > 1\}.$$

In the step marked (*) we really (!) need that $\mu\{u > 1\} > 0$ — otherwise we could not get a strict inequality. Thus, $\mu\{u > 1\} < \mu\{u > 1\}$ which is a contradiciton. Therefore, $\mu\{u > 1\} = 0$ and we have $u \leqslant 1$ a.e.

If you are unhappy with strict inequalities, you can extend the argument as follows: By assumption $\mu\{u > 1\} > 0$. Since $\{u > 1\} = \bigcup_{n\geqslant 1}\{u \geqslant 1+1/n\}$, there is some N such that $\mu\{u \geqslant 1+1/n\} > 0$ for all $n \geqslant N$ — use a continuity of measure argument. Now we get for all $n \geqslant N$

$$\begin{aligned}
\int_{\left\{u\geqslant 1+\frac{1}{n}\right\}} 1\,d\mu &= \mu\left\{u \geqslant 1 + \tfrac{1}{n}\right\} \\
&< \left(1 + \tfrac{1}{n}\right)\mu\left\{u \geqslant 1 + \tfrac{1}{n}\right\} \\
&= \int_{\left\{u\geqslant 1+\frac{1}{n}\right\}} \left(1 + \tfrac{1}{n}\right) d\mu \\
&\leqslant \int_{\left\{u\geqslant 1+\frac{1}{n}\right\}} u\,d\mu.
\end{aligned}$$

Observe that

$$\int_{\{u>1\}} 1\,d\mu = \sum_{n=N+1}^{\infty} \int_{\left\{1+1/n\leqslant u<1+1/(n-1)\right\}} 1\,d\mu + \int_{\left\{u\geqslant 1+\frac{1}{N}\right\}} 1\,d\mu$$

$$\begin{aligned}
&\leqslant \sum_{n=N+1}^{\infty} \int_{\left\{1+1/n\leqslant u<1+1/(n-1)\right\}} u\,d\mu + \int_{\left\{u\geqslant 1+\frac{1}{N}\right\}} 1\,d\mu \\
&< \sum_{n=N+1}^{\infty} \int_{\left\{1+1/n\leqslant u<1+1/(n-1)\right\}} u\,d\mu + \int_{\left\{u\geqslant 1+\frac{1}{N}\right\}} u\,d\mu \\
&= \int_{\{u>1\}} u\,d\mu.
\end{aligned}$$

With our assumption we thus get the contradiction $\mu\{u > 1\} < \mu\{u > 1\}$.

Alternative: From $\int_{\{u>1\}} u\,d\mu \leqslant \mu(u > 1)$ we get

$$\int_{\{u>1\}} (u-1)\,d\mu \leqslant 0.$$

Observe that $(u-1)1_{\{u>1\}} \geqslant 0$ implies

$$\int_{\{u>1\}} (u-1)\,d\mu \geqslant 0.$$

Therefore, $\int_{\{u>1\}}(u-1)\,d\mu = 0$ and we see that $(u-1)1_{\{u>1\}} = 0$ a.e., hence $1_{\{u>1\}} = 0$ a.e.

Problem 27.3 Solution: Note that, since $\mathbb{E}^{\mathscr{G}}$ is (currently...) only defined for L^2-functions the problem implicitly requires that $f \in L^2(\mathscr{A}, \mu)$. (A look at the next section reveals that this is not really necessary...). Below we will write $\langle\cdot,\cdot\rangle_{L^2(\mu)}$ resp. $\langle\cdot,\cdot\rangle_{L^2(\nu)}$ to indicate which scalar product is meant.

We begin with a general consideration: Let u, w be functions such that $u^2, v^2 \in L^2(\mu)$. Then we have $|u \cdot w| \leqslant \frac{1}{2}(u^2 + w^2) \in L^2(\mu)$ and, using again the elementary inequality

$$|xy| \leqslant \frac{x^2}{2} + \frac{y^2}{2}$$

for $x = |u|/\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(u^2)}$ and $y = |w|/\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}$ we conclude that on $G_n := \{\mathbb{E}^{\mathscr{G}}_\mu(u^2) > \frac{1}{n}\} \cap \{\mathbb{E}^{\mathscr{G}}_\mu(w^2) > \frac{1}{n}\}$

$$\frac{|u| \cdot |w|}{\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}} \mathbb{1}_{G_n} \leqslant \left[\frac{u^2}{2\,\mathbb{E}^{\mathscr{G}}_\mu(u^2)} + \frac{w^2}{2\,\mathbb{E}^{\mathscr{G}}_\mu(w^2)}\right] \mathbb{1}_{G_n}.$$

Taking conditional expectations on both sides yields, since $G_n \in \mathscr{G}$:

$$\frac{\mathbb{E}^{\mathscr{G}}_\mu\big(|u| \cdot |w|\big)}{\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}} \mathbb{1}_{G_n} \leqslant \mathbb{1}_{G_n}.$$

Multiplying through with the denominator of the lhS and letting $n \to \infty$ gives

$$\left|\mathbb{E}^{\mathscr{G}}_\mu(uw)\right| \mathbb{1}_{G^*} \leqslant \mathbb{E}^{\mathscr{G}}_\mu\big(|uw|\big)\mathbb{1}_{G^*} \leqslant \sqrt{\mathbb{E}^{\mathscr{G}}_\mu(u^2)}\sqrt{\mathbb{E}^{\mathscr{G}}_\mu(w^2)}$$

on the set $G^* := G_u \cap G_w := \{\mathbb{E}^{\mathscr{G}}_\mu u^2 > 0\} \cap \{\mathbb{E}^{\mathscr{G}}_\mu w^2 > 0\}$.

(i) Set $G^* := \{\mathbb{E}_\mu^{\mathscr{G}} f > 0\}$ and $G_n := \{\mathbb{E}_\mu^{\mathscr{G}} f > \frac{1}{n}\}$. Clearly, using the Markov inequality,

$$\mu(G_n) \leqslant n^2 \int (\mathbb{E}_\mu^{\mathscr{G}} f)^2 \, d\mu \leqslant n \int f^2 \, d\mu < \infty$$

so that by monotone convergence we find for all $G \in \mathscr{G} \cap G^*$

$$\begin{aligned}
\nu(G) &= \langle f, \mathbb{1}_G \rangle_{L^2(\mu)} \\
&= \sup_n \langle f, \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \sup_n \langle f, \mathbb{E}_\mu^{\mathscr{G}} \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \sup_n \langle \mathbb{E}_\mu^{\mathscr{G}} f, \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \langle \mathbb{E}_\mu^{\mathscr{G}} f, \mathbb{1}_G \rangle_{L^2(\mu)}
\end{aligned}$$

which means that $\nu|_{\mathscr{G} \cap G^*} = \mathbb{E}^{\mathscr{G}} f \cdot \mu|_{\mathscr{G} \cap G^*}$.

(ii) We define for *bounded* $u \in L^2(\nu)$

$$Pu := \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*}.$$

Let us show that $P \in L^2(\nu)$. Set $G_{\sqrt{f}u} := \{\mathbb{E}_\mu^{\mathscr{G}}\big(f \cdot u^2\big) > 0\}$. Then, for *bounded* $u \in L^2(\nu)$

$$\begin{aligned}
&\left\| \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \right\|_{L^2(\nu)}^2 \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\big[\mathbb{E}_\mu^{\mathscr{G}}(fu)\big]^2}{\big[\mathbb{E}_\mu^{\mathscr{G}} f\big]^2} \, d\nu \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\big[\mathbb{E}_\mu^{\mathscr{G}}(fu)\big]^2}{\big[\mathbb{E}_\mu^{\mathscr{G}} f\big]^2} \, f \, d\mu \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\big[\mathbb{E}_\mu^{\mathscr{G}}(fu)\big]^2}{\big[\mathbb{E}_\mu^{\mathscr{G}} f\big]^2} \, \mathbb{E}_\mu^{\mathscr{G}} f \, d\mu \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\big[\mathbb{E}_\mu^{\mathscr{G}}(fu)\big]^2}{\mathbb{E}_\mu^{\mathscr{G}} f} \, d\mu \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\big[\mathbb{E}_\mu^{\mathscr{G}}\big[\sqrt{f}(\sqrt{f}u)\big]\big]^2}{\mathbb{E}_\mu^{\mathscr{G}} f} \, d\mu \\
&\quad \leqslant \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\mathbb{E}_\mu^{\mathscr{G}} f \cdot \mathbb{E}_\mu^{\mathscr{G}}\big[fu^2\big]}{\mathbb{E}_\mu^{\mathscr{G}} f} \, d\mu \\
&\quad = \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \mathbb{E}_\mu^{\mathscr{G}}\big[fu^2\big] \, d\mu \\
&\quad = \sup_n \int \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \mathbb{E}_\mu^{\mathscr{G}}\big[fu^2\big] \, d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \sup_n \int \mathbb{E}_\mu^{\mathscr{G}} \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} f u^2 \, d\mu \\
&= \sup_n \int \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} f u^2 \, d\mu \\
&= \int \mathbb{1}_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} f u^2 \, d\mu \\
&\leqslant \int f u^2 \, d\mu = \left\| \sqrt{f} u \right\|_{L^2(\mu)}^2 = \|u\|_{L^2(\nu)}^2 < \infty.
\end{aligned}
$$

Still for bounded $u \in L^2(\nu)$,

$$
\begin{aligned}
&\int_{G_n \cap \{f<n\} \cap \{\mathbb{E}^{\mathscr{G}}(fu^2)=0\}} \mathbb{E}_\mu^{\mathscr{G}}(fu) \, d\mu \\
&= \int_{G_n \cap \{\mathbb{E}^{\mathscr{G}}(\sqrt{f}u)=0\}} f u \, d\mu \\
&\leqslant \sqrt{\int_{G_n \cap \{f<n\}} f \, d\mu} \sqrt{\int_{G_n \cap \{\mathbb{E}^{\mathscr{G}}(fu^2)=0\}} f u^2 \, d\mu} \\
&= \sqrt{\int_{G_n \cap \{f<n\}} f \, d\mu} \sqrt{\int_{G_n \cap \{\mathbb{E}^{\mathscr{G}}(fu^2)=0\}} \mathbb{E}_\mu^{\mathscr{G}} f u^2 \, d\mu} \\
&= 0
\end{aligned}
$$

and, using monotone convergence, we have

$$
\|Pu\|_{L^2(\nu)}^2 \leqslant \|u\|_{L^2(\nu)}^2
$$

for all bounded $u \in L^2(\nu)$, hence – through extension by continuity – for all $u \in L^2(\nu)$.

(iii) Since

$$
\begin{aligned}
&\left\langle u - Pu, Pu \right\rangle_{L^2(\nu)} \\
&= \left\langle fu - f Pu, Pu \right\rangle_{L^2(\mu)} \\
&= \left\langle fu - f \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathscr{G}} \left[fu - f \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right], \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathscr{G}}(fu) - \mathbb{E}_\mu^{\mathscr{G}} \left[f \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right], \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathscr{G}}(fu) - \mathbb{E}_\mu^{\mathscr{G}}(f) \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathscr{G}}(fu) - \mathbb{E}_\mu^{\mathscr{G}}(fu) \mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= 0
\end{aligned}
$$

which shows that P is the (uniquely determined) orthogonal projection onto $L^2(\nu, \mathscr{G})$, i.e. $P = \mathbb{E}_\nu^{\mathscr{G}}$.

(Note that we have, implicitly, extended $\mathbb{E}_\mu^{\mathscr{G}}$ onto L^1....)

(iv) The condition that $f\mathbb{1}_{G^*}$ is $\mathscr{G}$-measurable will do. Indeed, since $G^* \in \mathscr{G}$:

$$\mathbb{E}_\nu^{\mathscr{G}} u = \frac{\mathbb{E}_\mu^{\mathscr{G}}(fu)}{\mathbb{E}_\mu^{\mathscr{G}} f}\mathbb{1}_{G^*} = \frac{\mathbb{E}_\mu^{\mathscr{G}}((f\mathbb{1}_{G^*})u)}{\mathbb{E}_\mu^{\mathscr{G}}(f\mathbb{1}_{G^*})} = \frac{(f\mathbb{1}_{G^*})\mathbb{E}_\mu^{\mathscr{G}}(u)}{(f\mathbb{1}_{G^*})} = \mathbb{E}_\mu^{\mathscr{G}} u.$$

In fact, if $f \in L^4(\mu, \mathscr{A})$ this is also necessary:

$$\mathbb{E}_\mu^{\mathscr{G}} f = \mathbb{E}_\nu^{\mathscr{G}} f$$

implies, because of (i), that

$$\begin{aligned}\mathbb{E}_\mu^{\mathscr{G}} f = \frac{\mathbb{E}_\mu^{\mathscr{G}}(f^2)}{\mathbb{E}_\mu^{\mathscr{G}} f}\mathbb{1}_{\{\mathbb{E}_\mu^{\mathscr{G}} f>0\}} &\iff \left(\mathbb{E}_\mu^{\mathscr{G}} f\right)^2 = \mathbb{E}_\mu^{\mathscr{G}}(f^2)\mathbb{1}_{\{\mathbb{E}_\mu^{\mathscr{G}} f>0\}} \\ &\iff \left(\mathbb{E}_\mu^{\mathscr{G}} f\right)^2 = \mathbb{E}_\mu^{\mathscr{G}}(f^2).\end{aligned}$$

Thus,

$$\mathbb{E}_\mu^{\mathscr{G}}\left[\left(f - \mathbb{E}_\mu^{\mathscr{G}} f\right)^2\right] = 0,$$

which means that on the set $G^* = \bigcup_n G_n$ with $\mu(G_n) < \infty$, see above,

$$0 = \int_{G_n} \mathbb{E}_\mu^{\mathscr{G}}(f - \mathbb{E}_\mu^{\mathscr{G}} f)^2 \, d\mu = \int_{G_n} (f - \mathbb{E}_\mu^{\mathscr{G}} f)^2 \, d\mu$$

i.e. $f = \mathbb{E}_\mu^{\mathscr{G}} f$ on $G^* = \{\mathbb{E}_\mu^{\mathscr{G}} f > 0\}$

Problem 27.4 Solution: Since $\mathscr{G} = \{G_1, \dots, G_n\}$ such that the G_j's form a mutually disjoint partition of the whole space X, we have

$$L^2(\mathscr{G}) = \left\{ \sum_{j=1}^n \alpha_j \mathbb{1}_{G_j} \,:\, \alpha_j \in \mathbb{R} \right\}.$$

It is, therefore, enough to determine the values of the α_j. Using the symmetry and idempotency of the conditional expectation we get for $k \in \{1, 2, \dots, n\}$

$$\langle \mathbb{E}^{\mathscr{G}} u, \mathbb{1}_{G_k}\rangle = \langle u, \mathbb{E}^{\mathscr{G}}\mathbb{1}_{G_k}\rangle = \langle u, \mathbb{1}_{G_k}\rangle = \int_{G_k} u \, d\mu.$$

On the other hand, using that $\mathbb{E}^{\mathscr{G}} u \in L^2(\mathscr{G})$ we find

$$\langle \mathbb{E}^{\mathscr{G}} u, \mathbb{1}_{G_k}\rangle = \left\langle \sum_{j=1}^n \alpha_j \mathbb{1}_{G_j}, \mathbb{1}_{G_k}\right\rangle = \sum_{j=1}^n \alpha_j \langle \mathbb{1}_{G_j}, \mathbb{1}_{G_k}\rangle = \alpha_k \mu(G_k)$$

and we conclude that

$$\alpha_k = \frac{1}{\mu(G_k)} \int_{G_k} u \, d\mu = \int_{G_k} u(x) \, \frac{\mu(dx)}{\mu(G_k)}.$$

Problem 27.5 Solution: We follow the hint. Let $u \in L^p(\mu)$ and define $u_n = [(-n) \vee u \wedge n]\mathbb{1}_{\{|u| \geqslant 1/n\}}$. Clearly, u_n is bounded, and by the Markov inequality (11.4)

$$\mu\{|u| \geqslant 1/n\} = \mu\{|u|^p \geqslant 1/n^p\} \leqslant n^p \int |u|^p \, d\mu < \infty.$$

Therefore, $u_n \in L^r(\mu)$ for all $r \geqslant 1$:

$$\int |u_n|^r \, d\mu = \int_{\{|u_n| \geqslant 1/n\}} (|u| \wedge n)^r \leqslant n^r \mu\{|u_n| \geqslant 1/n\} \leqslant n^{r+p} \int |u|^p \, d\mu < \infty.$$

Since $u_n \to u$ a.e., dominated convergence (use the majorant $|u|^p$) shows that $u_n \to u$ in L^p. Thus, we see as in the remark before Theorem 27.5 that $(Tu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu)$, i.e. the limit L^p-$\lim_n Tu_n$ exists. If $(w_n)_n$ is a further approximating sequence such that $w_n \to u$ in $L^p(\mu)$, we get

$$\|Tu_n - Tw_n\|_p = \|T(u_n - w_n)\|_p \leqslant c\|u_n - w_n\|_p \leqslant c\|u_n - u\|_p + c\|u - w_n\|_p \xrightarrow[n\to\infty]{} 0$$

which shows that $\lim_n Tu_n = \lim_n Tw_n$, i.e. $\widetilde{T}u := \lim_n Tu_n$ (as an L^p-limit) is well-defined since it is independent of the approximating sequence. Linearity is clear from the linearity of the limit.

Assume now that $0 \leqslant u_n \uparrow u$ where $u_n \in L^p(\mu) \cap L^2(\mu)$. By the first part, $\widetilde{T}u = \lim_n Tu_n$ in L^p, so there is a subsequence such that $\widetilde{T}u = \lim_k Tu_{n_k}$ a.e. Because of monotonicity we have

$$Tu_{n_k} \leqslant Tu_n \quad \forall\, n \geqslant n(k) \implies 0 \leqslant \widetilde{T}u - Tu_n \leqslant \widetilde{T}u - Tu_{n_k}.$$

So,

$$0 \leqslant \limsup_{n\to\infty}(\widetilde{T}u - Tu_n) \leqslant \widetilde{T}u - Tu_{n_k} \xrightarrow[k\to\infty]{} 0,$$

which shows that $\lim_n(\widetilde{T}u - Tu_n) = 0$.

■■

Problem 27.6 Solution: Let $G_u := \{\mathbb{E}^{\mathscr{G}}|u|^p > 0\}$, $G_w := \{\mathbb{E}^{\mathscr{G}}|w|^q > 0\}$ and $G := G_u \cap G_w$. Following the hint we get

$$\frac{|u|}{\left[\mathbb{E}^{\mathscr{G}}(|u|^p)\right]^{1/p}} \frac{|w|}{\left[\mathbb{E}^{\mathscr{G}}(|w|^q)\right]^{1/q}} \mathbb{1}_G \leqslant \frac{|u|^p}{p\,\mathbb{E}^{\mathscr{G}}(|u|^p)}\mathbb{1}_G + \frac{|u|^q}{q\,\mathbb{E}^{\mathscr{G}}(|w|^q)}\mathbb{1}_G$$

Since $\mathbb{1}_G$ is bounded and $\mathscr{G}$-measurable, we can apply $\mathbb{E}^{\mathscr{G}}$ on both sides of the above inequality and get

$$\frac{\mathbb{E}^{\mathscr{G}}(|u||w|)}{\left[\mathbb{E}^{\mathscr{G}}(|u|^p)\right]^{1/p}\left[\mathbb{E}^{\mathscr{G}}(|w|^q)\right]^{1/q}}\mathbb{1}_G \leqslant \frac{\mathbb{E}^{\mathscr{G}}(|u|^p)}{p\,\mathbb{E}^{\mathscr{G}}(|u|^p)}\mathbb{1}_G + \frac{\mathbb{E}^{\mathscr{G}}(|u|^q)}{q\,\mathbb{E}^{\mathscr{G}}(|w|^q)}\mathbb{1}_G = \mathbb{1}_G$$

or

$$\begin{aligned}\mathbb{E}^{\mathscr{G}}(|u||w|)\mathbb{1}_G &\leqslant \left[\mathbb{E}^{\mathscr{G}}(|u|^p)\right]^{1/p}\left[\mathbb{E}^{\mathscr{G}}(|w|^q)\right]^{1/q}\mathbb{1}_G \\ &\leqslant \left[\mathbb{E}^{\mathscr{G}}(|u|^p)\right]^{1/p}\left[\mathbb{E}^{\mathscr{G}}(|w|^q)\right]^{1/q}.\end{aligned}$$

Denote by G_n an exhaustion of X such that $G_n \in \mathscr{G}$, $G_n \uparrow X$ and $\mu(G_n) < \infty$. Then

$$\begin{aligned}\int_{G_u^c} |u|^p \, d\mu &= \sup_n \int_{G_u^c \cap G_n} |u|^p \, d\mu \\ &= \sup_n \langle \mathbb{1}_{G_u^c \cap G_n}, |u|^p \rangle \\ &= \sup_n \langle \mathbb{E}^{\mathscr{G}} \mathbb{1}_{G_u^c \cap G_n}, |u|^p \rangle \\ &= \sup_n \langle \mathbb{1}_{G_u^c \cap G_n}, \mathbb{E}^{\mathscr{G}}(|u|^p) \rangle \\ &= 0\end{aligned}$$

which means that $\mathbb{1}_{G_u} u = u$ almost everywhere. Thus,

$$\mathbb{E}^{\mathscr{G}}(|u||w|)\mathbb{1}_G = \mathbb{E}^{\mathscr{G}}(|u||w|\mathbb{1}_G) = \mathbb{E}^{\mathscr{G}}(|u|\mathbb{1}_{G_u}|w|\mathbb{1}_{G_w}) = \mathbb{E}^{\mathscr{G}}(|u||w|)$$

and the inequality follows since

$$\left|\mathbb{E}^{\mathscr{G}}(uw)\right| \leqslant \mathbb{E}^{\mathscr{G}}(|uw|).$$

Problem 27.7 Solution: In this problem it is helpful to keep the distinction between $\mathbb{E}^{\mathscr{G}}$ defined on $L^2(\mathscr{A})$ and the extension $E^{\mathscr{G}}$ defined on $L^{\mathscr{G}}(\mathscr{A})$.

Since $\mu|_{\mathscr{A}}$ is σ-finite we can find an exhausting sequence of sets $A_n \uparrow X$ with $\mu(A_n) < \infty$. Setting for $u, w \in L^{\mathscr{G}}(\mathscr{A})$ with $uE^{\mathscr{G}} w \in L^1(\mathscr{A})$ $u_n := \big((-n) \vee u \wedge n\big) \cdot \mathbb{1}_{A_n}$ and $w_n := \big((-n) \vee w \wedge n\big) \cdot \mathbb{1}_{A_n}$ we have found approximating sequences such that $u_n, w_n \in L^1(\mathscr{A}) \cap L^\infty(\mathscr{A})$ and, in particular, $\in L^2(\mathscr{A})$.

(iii): For $u, w \geqslant 0$ we find by monotone convergence, using the properties listed in Theorem 27.4:

$$\begin{aligned}\langle E^{\mathscr{G}} u, w \rangle &= \lim_n \langle \mathbb{E}^{\mathscr{G}} u_n, w \rangle \\ &= \lim_n \lim_m \langle \mathbb{E}^{\mathscr{G}} u_n, w_m \rangle \\ &= \lim_n \lim_m \langle u_n, \mathbb{E}^{\mathscr{G}} w_m \rangle \\ &= \lim_n \langle u_n, E^{\mathscr{G}} w \rangle \\ &= \langle u, E^{\mathscr{G}} w \rangle.\end{aligned}$$

In the general case we write

$$\langle E^{\mathscr{G}} u, w \rangle = \langle E^{\mathscr{G}} u^+, w^+ \rangle - \langle E^{\mathscr{G}} u^-, w^+ \rangle - \langle E^{\mathscr{G}} u^+, w^- \rangle + \langle E^{\mathscr{G}} u^-, w^- \rangle$$

and consider each term separately.

The equality $\langle E^{\mathscr{G}} u, w \rangle = \langle E^{\mathscr{G}} u, E^{\mathscr{G}} w \rangle$ follows similarly.

(iv): we have

$$u = w \;\Rightarrow\; u_j = w_j \quad \forall j \;\Rightarrow\; \mathbb{E}^{\mathscr{G}} u_j = \mathbb{E}^{\mathscr{G}} w_j \quad \forall j$$

and we get

$$E^{\mathscr{G}}u = \lim_j \mathbb{E}^{\mathscr{G}}u_j = \lim_j \mathbb{E}^{\mathscr{G}}w_j = w.$$

(ix): we have

$$\begin{aligned} 0 \leqslant u \leqslant 1 \;&\Rightarrow\; 0 \leqslant u_n \leqslant 1 \quad \forall n \\ &\Rightarrow\; 0 \leqslant \mathbb{E}^{\mathscr{G}}u_n \leqslant 1 \quad \forall n \\ &\Rightarrow\; 0 \leqslant E^{\mathscr{G}}u = \lim_n \mathbb{E}^{\mathscr{G}}u_n \leqslant 1. \end{aligned}$$

(x):

$$u \leqslant w \;\Rightarrow\; 0 \leqslant w - u \;\Rightarrow\; 0 \leqslant E^{\mathscr{G}}(u-w) = E^{\mathscr{G}}u - E^{\mathscr{G}}w.$$

(xi):

$$\pm u \leqslant |u| \;\Rightarrow\; \pm E^{\mathscr{G}}u \leqslant E^{\mathscr{G}}|u| \;\Rightarrow\; \left|E^{\mathscr{G}}u\right| \leqslant E^{\mathscr{G}}|u|.$$

■■

Problem 27.8 Solution: (Mind the typo in the hint: $\mathbb{E}^{\mathscr{G}} = \mathbb{E}^{\mathscr{G}}$ should read $\mathbb{E}^{\mathscr{G}} = E^{\mathscr{G}}$.) Assume first that $\mu|_{\mathscr{G}}$ is σ-finite and denote by $G_k \in \mathscr{G}$, $G_k \uparrow X$ and $\mu(G_k) < \infty$ an exhausting sequence. Then $\mathbb{1}_{G_k} \in L^2(\mathscr{G})$, $\mathbb{1}_{G_k} \uparrow 1$ and

$$E^{\mathscr{G}}1 = \sup_k \mathbb{E}^{\mathscr{G}}\mathbb{1}_{G_k} = \sup_k \mathbb{1}_{G_k} = 1.$$

Conversely, let $E^{\mathscr{G}}1 = 1$. Because of Lemma 27.7 there is a sequence $(u_k)_k \subset L^2(\mathscr{A})$ with $u_k \uparrow 1$. By the very definition of $E^{\mathscr{G}}$ we have

$$E^{\mathscr{G}}1 = \sup_k \mathbb{E}^{\mathscr{G}}u_k = 1,$$

i.e. there is a sequence $g_k := \mathbb{E}^{\mathscr{G}}u_k \in L^2(\mathscr{G})$ such that $g_k \uparrow 1$. Set $G_k := \{g_k > 1 - 1/k\}$ and observe that $G_k \uparrow X$ as well as

$$\begin{aligned} \mu(G_k) &\leqslant \frac{1}{(1-\frac{1}{k})^2}\int g_k^2\,d\mu \\ &= \frac{1}{(1-\frac{1}{k})^2}\|\mathbb{E}^{\mathscr{G}}u_k\|_{L^2}^2 \\ &\leqslant \frac{1}{(1-\frac{1}{k})^2}\|u_k\|_{L^2}^2 \\ &< \infty. \end{aligned}$$

This shows that $\mu|_{\mathscr{G}}$ is σ-finite.

If $\mathscr{G}$ is not σ-finite, e.g. if $\mathscr{G} = \{\emptyset, G, G^c, X\}$ where $\mu(G) < \infty$ and $\mu(G^c) = \infty$ we find that

$$L^2(\mathscr{G}) = \{c\mathbb{1}_G : c \in \mathbb{R}\}$$

which means that $E^{\mathscr{G}} 1 = \mathbb{1}_G$ since for every $A \subset G^c$, $A \in \mathscr{A}$ and $\mu(A) < \infty$ we find

$$E^{\mathscr{G}} \mathbb{1}_{A \cup G} = E^{\mathscr{G}}(\mathbb{1}_A + \mathbb{1}_G) = E^{\mathscr{G}} \mathbb{1}_A + E^{\mathscr{G}} \mathbb{1}_G = E^{\mathscr{G}} \mathbb{1}_A + \mathbb{1}_G$$

Since this must be an element of $L^2(\mathscr{G})$, we have necessarily $E^{\mathscr{G}} \mathbb{1}_A = c\mathbb{1}_G$ or

$$\langle c\mathbb{1}_G, \mathbb{1}_G\rangle = \langle E^{\mathscr{G}} \mathbb{1}_A, \mathbb{1}_G\rangle = \langle \mathbb{1}_A, E^{\mathscr{G}} \mathbb{1}_G\rangle = \langle \mathbb{1}_A, \mathbb{1}_G\rangle = \mu(A \cap G) = 0,$$

hence $c = 0$ or $E^{\mathscr{G}} \mathbb{1}_A = 0$.

This shows that

$$E^{\mathscr{G}} 1 = \mathbb{1}_G \leqslant 1$$

is best possible.

Problem 27.9 Solution: For this problem it is helpful to distinguish between $\mathbb{E}^{\mathscr{G}}$ (defined on L^2) and the extension $E^{\mathscr{G}}$.

Without loss of generality we may assume that $g \geqslant 0$—otherwise we would consider positive and negative parts separately. Since $g \in L^p(\mathscr{G})$ we have that

$$\mu\{g > 1/j\} \leqslant j^p \int g^p \, d\mu < \infty$$

which means that the sequence $g_j := (j \wedge g)\mathbb{1}_{\{g>1/j\}} \in L^2(\mathscr{G})$. Obviously, $g_j \uparrow g$ pointwise as well as in L^p-sense. Using the results from Theorem 27.4 we get

$$\mathbb{E}^{\mathscr{G}} g_j = g_j \implies E^{\mathscr{G}} = \sup_j \mathbb{E}^{\mathscr{G}} g_j = \sup_j g_j = g.$$

Problem 27.10 Solution: For this problem it is helpful to distinguish between $\mathbb{E}^{\mathscr{G}}$ (defined on L^2) and the extension $E^{\mathscr{G}}$.

For $u \in L^p(\mathscr{A})$ we get $E^{\mathscr{H}} E^{\mathscr{G}} u = E^{\mathscr{H}} u$ because of Theorem 27.11(vi) while the other equality $E^{\mathscr{G}} E^{\mathscr{H}} u = E^{\mathscr{H}} u$ follows from Problem 27.9.

If $u \in M^+(\mathscr{A})$ (mind the misprint in the problem!) we get a sequence $u_j \uparrow u$ of functions $u_j \in L^2_+(\mathscr{A})$. From Theorem 27.4 we know that $\mathbb{E}^{\mathscr{G}} u_j \in L^2(\mathscr{G})$ increases and, by definition, it increases towards $E^{\mathscr{G}} u$. Thus,

$$\mathbb{E}^{\mathscr{H}} \mathbb{E}^{\mathscr{G}} u_j = \mathbb{E}^{\mathscr{H}} u_j \uparrow E^{\mathscr{H}} u$$

while

$$\mathbb{E}^{\mathscr{H}}\mathbb{E}^{\mathscr{G}}u_j \uparrow E^{\mathscr{H}}\big(\sup_j \mathbb{E}^{\mathscr{G}}u_j\big) = E^{\mathscr{H}}E^{\mathscr{G}}u.$$

The other equality is similar.

Problem 27.11 Solution: We know that

$$L^p(\mathscr{A}_n) = \left\{ \sum_{j=1}^{n} c_j \mathbb{1}_{[j-1,j)} \;:\; c_j \in \mathbb{R} \right\}$$

since $c_0\mathbb{1}_{[n,\infty)} \in L^p$ if, and only if, $c_0 = 0$. Thus, $E^{\mathscr{A}_n}u$ is of the form

$$E^{\mathscr{A}_n}u(x) = \sum_{j=1}^{n} c_j \mathbb{1}_{[j-1,j)}(x)$$

and integrating over $[k-1,k)$ yields

$$\int_{[k-1,k)} E^{\mathscr{A}_n}u(x)\,dx = c_k.$$

Since

$$\begin{aligned}
\int_{[k-1,k)} E^{\mathscr{A}_n}u(x)\,dx &= \langle E^{\mathscr{A}_n}u, \mathbb{1}_{[k-1,k)}\rangle \\
&= \langle u, E^{\mathscr{A}_n}\mathbb{1}_{[k-1,k)}\rangle \\
&= \langle u, \mathbb{1}_{[k-1,k)}\rangle \\
&= \int_{[k-1,k)} u(x)\,dx
\end{aligned}$$

we get

$$E^{\mathscr{A}_n}u(x) = \sum_{j=1}^{n} \int_{[j-1,j)} u(t)\,dt\mathbb{1}_{[j-1,j)}(x).$$

Problem 27.12 Solution: For this problem it is helpful to distinguish between $\mathbb{E}^{\mathscr{G}}$ (defined on L^2) and the extension $E^{\mathscr{G}}$.

If $\mu(X) = \infty$ and if $\mathscr{G} = \{\emptyset, X\}$, then $L^1(\mathscr{G}) = \{0\}$ which means that $E^{\mathscr{G}}u = 0$ for any $u \in L^1(\mathscr{A})$. Thus for integrable functions $u > 0$ and $\mu|_{\mathscr{G}}$ not σ-finite we can only have '$\leqslant$'.

If $\mu|_{\mathscr{G}}$ is σ-finite and if $G_j \uparrow X$, $G_j \in \mathscr{G}$, $\mu(G_j) < \infty$ is an exhausting sequence, we find for any $u \in L^1_+(\mathscr{A})$

$$\begin{aligned}
\int E^{\mathscr{G}}u\,d\mu &= \sup_j \int_{G_j} E^{\mathscr{G}}u\,d\mu \\
&= \sup_j \langle E^{\mathscr{G}}u, \mathbb{1}_{G_j}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sup_j \langle u, E^{\mathscr{G}} \mathbb{1}_{G_j} \rangle \\
&= \sup_j \langle u, \mathbb{1}_{G_j} \rangle \\
&= \langle u, 1 \rangle \\
&= \int u \, d\mu.
\end{aligned}
$$

If $\mu|_{\mathscr{G}}$ is not σ-finite and if $u \geqslant 0$, we perform a similar calculation with an exhausting sequence $A_j \in \mathscr{A}$, $A_j \uparrow X$, $\mu(A_j) < \infty$ (it is implicit that $\mu|_{\mathscr{A}}$ is σ-finite as otherwise the conditional expectation would not be defined!):

$$
\begin{aligned}
\int E^{\mathscr{G}} u \, d\mu &= \sup_j \int_{A_j} E^{\mathscr{G}} u \, d\mu \\
&= \sup_j \langle E^{\mathscr{G}} u, \mathbb{1}_{A_j} \rangle \\
&= \sup_j \langle u, E^{\mathscr{G}} \mathbb{1}_{A_j} \rangle \\
&\leqslant \langle u, 1 \rangle \\
&= \int u \, d\mu.
\end{aligned}
$$

Problem 27.13 Solution:

Proof of Corollary 27.14: Since

$$\liminf_{j\to\infty} u_j = \sup_k \inf_{j\geqslant k} u_j$$

we get

$$\mathbb{E}^{\mathscr{G}} \big(\inf_{j\geqslant k} u_j \big) \leqslant \mathbb{E}^{\mathscr{G}} u_m \quad \forall\, m \geqslant k$$

thus

$$\mathbb{E}^{\mathscr{G}} \big(\inf_{j\geqslant k} u_j \big) \leqslant \inf_{m\geqslant k} \mathbb{E}^{\mathscr{G}} u_m \leqslant \sup_k \inf_{m\geqslant k} \mathbb{E}^{\mathscr{G}} u_m = \liminf_{m\to\infty} \mathbb{E}^{\mathscr{G}} u_m.$$

Since on the other hand the sequence $\inf_{j\geqslant k} u_j$ increases, as $k \to \infty$, towards $\sup_k \inf_{j\geqslant k} u_j$ we can use the conditional Beppo Levi theorem 27.13 on the left-hand side and find

$$\mathbb{E}^{\mathscr{G}} \big(\liminf_{j\to\infty} u_j \big) = \mathbb{E}^{\mathscr{G}} \big(\sup_k \inf_{j\geqslant k} u_j \big) = \sup_k \mathbb{E}^{\mathscr{G}} \big(\inf_{j\geqslant k} u_j \big) \leqslant \liminf_{m\to\infty} \mathbb{E}^{\mathscr{G}} u_m.$$

The Corollary is proved.

Proof of Corollary 27.15: Since $|u_j| \leqslant w$ we conclude that $|u| = \lim_j |u_j| \leqslant w$ and that $2w - |u - u_j| \geqslant 0$. Applying the conditional Fatou lemma 27.14 we find

$$\mathbb{E}^{\mathscr{G}}(2w) = \mathbb{E}^{\mathscr{G}} \big(\liminf_j 2w - |u - u_j| \big)$$

$$\leqslant \liminf_j \mathbb{E}^{\mathscr{G}}\big(2w - |u - u_j|\big)$$
$$= \mathbb{E}^{\mathscr{G}}(2w) - \limsup_j \mathbb{E}^{\mathscr{G}}(|u - u_j|)$$

which shows that

$$\limsup_j \mathbb{E}^{\mathscr{G}}(|u - u_j|) = 0 \implies \lim_j \mathbb{E}^{\mathscr{G}}(|u - u_j|) = 0.$$

Since, however,

$$\left|\mathbb{E}^{\mathscr{G}} u_j - \mathbb{E}^{\mathscr{G}} u\right| = \left|\mathbb{E}^{\mathscr{G}}(u_j - u)\right| \leqslant \mathbb{E}^{\mathscr{G}}|u_j - u| \xrightarrow[j\to\infty]{} 0$$

the claim follows.

Problem 27.14 Solution: (i) $\Rightarrow$ (ii): Let $A \in \mathscr{A}_\infty$ be such that $\mu(A) < \infty$. Then, by Hölder's inequality with $1/p + 1/q = 1$,

$$\left|\int_A u_j\,d\mu - \int_A u\,d\mu\right| \leqslant \int_A |u_j - u|\,d\mu \leqslant \|u_j - u\|_p\,\mu(A)^{1/q} \xrightarrow[j\to\infty]{} 0.$$

Thus, if $u_\infty := \mathbb{E}^{\mathscr{A}_\infty} u$, we find by the martingale property for all $k > j$ and $A \in \mathscr{A}_j$ such that $\mu(A) < \infty$

$$\int_A u_j\,d\mu = \int_A u_k\,d\mu = \lim_{k\to\infty}\int_A u_k\,d\mu = \int_A u\,d\mu = \int_A u_\infty\,d\mu,$$

and since we are in a σ-finite setting, we can apply Theorem 27.12(i) and find that $u_j = \mathbb{E}^{\mathscr{A}_j} u_\infty$.

(ii) $\Rightarrow$ (iii): Assume first that $u_\infty \in L^1 \cap L^p$. Then $u_j = \mathbb{E}^{\mathscr{A}_j} u_\infty \in L^1 \cap L^p$ and Theorem 27.19(i) shows that $u_j \xrightarrow[j\to\infty]{} u_\infty$ both in L^1 and a.e. In particular, we get

$$\langle u_\infty - u_j, \phi\rangle \leqslant \|u_\infty - u_j\|_1 \|\phi\|_\infty \to 0 \quad \forall\,\phi \in L^\infty.$$

In the general case where $u_\infty \in L^p(\mathscr{A}_\infty)$ we find for every $\epsilon > 0$ an element $u_\infty^\epsilon \in L^1(\mathscr{A}_\infty) \cap L^p(\mathscr{A}_\infty)$ such that

$$\|u_\infty - u_\infty^\epsilon\|_p \leqslant \epsilon$$

(indeed, since we are working in a σ-finite filtered measure space, there is an exhaustion $A_k \uparrow X$ such that $A_k \in \mathscr{A}_\infty$ and for large enough $k = k_\epsilon$ the function $u_\infty^\epsilon := u_\infty \mathbb{1}_{A_k}$ will to the job). Similarly, we can approximate any fixed $\phi \in L^q$ by $\phi^\epsilon \in L^q \cap L^1$ such that $\|\phi - \phi^\epsilon\|_q \leqslant \epsilon$.

Now we set $u_j^\epsilon := \mathbb{E}^{\mathscr{A}_j} u_\infty^\epsilon$ and observe that

$$\|u_j - u_j^\epsilon\|_p = \|\mathbb{E}^{\mathscr{A}_j} u_\infty - \mathbb{E}^{\mathscr{A}_j} u_\infty^\epsilon\|_p \leqslant \|u_\infty - u_\infty^\epsilon\|_p \leqslant \epsilon.$$

Thus, for any $\phi \in L^q$,

$$\langle u_j - u_\infty, \phi\rangle$$

$$
\begin{aligned}
&= \langle u_j - u_j^\epsilon - u_\infty + u_\infty^\epsilon, \phi\rangle + \langle u_j^\epsilon - u_\infty^\epsilon, \phi\rangle \\
&= \langle u_j - u_j^\epsilon - u_\infty + u_\infty^\epsilon, \phi\rangle + \langle u_j^\epsilon - u_\infty^\epsilon, \phi - \phi^\epsilon\rangle + \langle u_j^\epsilon - u_\infty^\epsilon, \phi^\epsilon\rangle \\
&\leqslant \big(\|u_j - u_j^\epsilon\|_p + \|u_\infty - u_\infty^\epsilon\|_p\big)\|\phi\|_q \\
&\qquad + \|u_j^\epsilon - u_\infty^\epsilon\|_p\|\phi - \phi^\epsilon\|_q + \langle u_j^\epsilon - u_\infty^\epsilon, \phi^\epsilon\rangle \\
&\leqslant 2\epsilon\,\|\phi\|_q + \epsilon\,\underbrace{\|u_j^\epsilon - u_\infty^\epsilon\|_p}_{\leqslant 2\|u_\infty^\epsilon\|_p \leqslant 2(\epsilon + \|u_\infty\|_p)} + \underbrace{\langle u_j^\epsilon - u_\infty^\epsilon, \phi^\epsilon\rangle}_{\xrightarrow[j\to\infty]{} 0} \qquad\qquad \leqslant \text{const.}\epsilon
\end{aligned}
$$

for sufficiently large j's, and the claim follows.

(iii) $\Rightarrow$ (ii): Let $u_{n(j)}$ be a subsequence converging weakly to some $u \in L^p$, i.e.,

$$\lim_k \langle u_{n(k)} - u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q.$$

Then, in particular,

$$\lim_k \langle u_{n(k)} - u, \mathbb{E}^{\mathscr{A}_n}\phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}$$

or

$$\lim_k \langle \mathbb{E}^{\mathscr{A}_n} u_{n(k)} - \mathbb{E}^{\mathscr{A}_n} u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}.$$

Since u_j is a martingale, we find that $\mathbb{E}^{\mathscr{A}_n} u_{n(k)}$ if $n < n(k)$, i.e.,

$$\langle u_n - \mathbb{E}^{\mathscr{A}_n} u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}.$$

and we conclude that $u_n = \mathbb{E}^{\mathscr{A}_n} u$. Because of the tower property we can always replace u by $u_\infty := \mathbb{E}^{\mathscr{A}_\infty} u$:

$$u_n = \mathbb{E}^{\mathscr{A}_n} u = \mathbb{E}^{\mathscr{A}_n}\mathbb{E}^{\mathscr{A}_\infty} u = \mathbb{E}^{\mathscr{A}_n} u_\infty$$

and the claim follows.

(ii) $\Rightarrow$ (i): We show that we can take $u = u_\infty$. First, if $u_\infty \in L^1 \cap L^\infty$ we find by the closability of martingales, Theorem 27.19(i), that

$$\lim_j \|u_j - u\|_1 = 0.$$

Moreover, using that $|a - b|^r \leqslant (|a| + |b|)^r \leqslant 2^r(|a|^r + |b|^r)$, we find

$$
\begin{aligned}
\|u_j - u\|_p^p &= \int |u_j - u|^p \, d\mu \\
&= \int |u_j - u| \cdot |u_j - u|^{p-1} \, d\mu \\
&\leqslant 2^{p-1}(\|u_j\|_\infty^{p-1} + \|u\|_\infty^{p-1}) \int |u_j - u| \, d\mu \\
&\leqslant 2^p \|u\|_\infty^{p-1} \cdot \|u_j - u\|_1
\end{aligned}
$$

$$\xrightarrow[j\to\infty]{} 0$$

where we use that

$$\|u_j\|_\infty = \|\mathbb{E}_j^{\mathscr{A}} u\|_\infty \leqslant \mathbb{E}_j^{\mathscr{A}}\big(\|u\|_\infty\big) \leqslant \|u\|_\infty.$$

Now for the general case where $u_\infty \in L^p$. Since we are in a σ-finite setting, we can set $u^\epsilon := (u \cdot \mathbb{1}_{A_j}) \wedge j$, $j = j(\epsilon)$ sufficiently large and $A_j \to X$ an exhausting sequence of sets from $\mathscr{A}_\infty$, and can guarantee that

$$\|u - u^\epsilon\|_p \leqslant \epsilon.$$

At the same time, we get for $u_j^\epsilon := \mathbb{E}^{\mathscr{A}_j} u^\epsilon \in L^1 \cap L^\infty$ that

$$\|u_j - u_j^\epsilon\|_p = \|\mathbb{E}^{\mathscr{A}_j} u - \mathbb{E}^{\mathscr{A}_j} u^\epsilon\|_p \leqslant \|u - u^\epsilon\|_p \leqslant \epsilon.$$

Thus, by the consideration for the special case where $u^\epsilon \in L^1 \cap L^\infty$,

$$\begin{aligned}
\|u_j - u\|_p &\leqslant \|u_j - u_j^\epsilon\|_p + \|u_j^\epsilon - u^\epsilon\|_p + \|u_\epsilon - u\|_p \\
&\leqslant \epsilon + \|u_j^\epsilon - u^\epsilon\|_p + \epsilon \\
&\xrightarrow[j\to\infty]{} 2\epsilon \xrightarrow[\epsilon\to 0]{} 0.
\end{aligned}$$

■■

Problem 27.15 Solution: Obviously,

$$m_k = m_{k-1} + (u_k - E^{\mathscr{A}_{k-1}} u_k).$$

Since $m_1 = u_1 \in L^1\mathscr{A}_1$, this shows, by induction, that $m_k \in L^1(\mathscr{A}_k)$. Applying $E^{\mathscr{A}_{k-1}}$ to both sides of the displayed equality yields

$$\begin{aligned}
E^{\mathscr{A}_{k-1}} m_k &= E^{\mathscr{A}_{k-1}} m_{k-1} + E^{\mathscr{A}_{k-1}}(u_k - E^{\mathscr{A}_{k-1}} u_k) \\
&= m_{k-1} + E^{\mathscr{A}_{k-1}} u_k - E^{\mathscr{A}_{k-1}} u_k \\
&= m_{k-1}
\end{aligned}$$

which shows that m_k is indeed a martingale.

■■

Problem 27.16 Solution: Problem 27.15 shows that s_k is a martingale, so that s_k^2 is a sub-martingale (use Jensen's inequality for conditional expectations). Now

$$\int s_k^2\,d\mu = \sum_j \int u_k^2\,d\mu + 2\sum_{j<k} \int u_j u_k\,d\mu$$

and if $j < k$

$$\int u_j u_k\,d\mu = \int E^{\mathscr{A}_j}(u_j u_k)\,d\mu = \int u_j \underbrace{E^{\mathscr{A}_j}(u_k)}_{=0}\,d\mu = 0.$$

■■

Problem 27.17 Solution: Problem 27.15 shows that m_j is a martingale.

Since $a_1 = E^{\mathscr{A}_0}u_1 - u_0 = E^{\{\emptyset,X\}}u_1 = \int u_1\,d\mu$ is constant, i.e., $\mathscr{A}_0$-measurable, the recursion formula

$$a_{j+1} = a_j + E^{\mathscr{A}_j}u_{j+1} - u_j$$

implies that a_{j+1} is $\mathscr{A}_j$-measurable.

Since u_j is a submartingale, we get

$$E^{\mathscr{A}_j}u_{j+1} \geqslant u_j \implies a_{j+1} - a_j \geqslant 0$$

i.e., the sequence a_j increases.

Finally, if $m_j + a_j = u_j = \widetilde{m}_j + \widetilde{a}_j$ are two such decompositions we find that $m_j - \widetilde{m}_j = a_j - \widetilde{a}_j$ is $\mathscr{A}_{j-1}$ measurable. Using the martingale property we find

$$m_j - \widetilde{m}_j = E^{\mathscr{A}_{j-1}}(m_j - \widetilde{m}_j) \overset{\text{Martingale}}{=} m_{j-1} - \widetilde{m}_{j-1}$$

and applying this recursively for $j = 1, 2, 3, \dots$ yields

$$m_1 - \widetilde{m}_1 = 0, \quad m_2 - \widetilde{m}_2 = 0, \quad m_3 - \widetilde{m}_3 = 0, \dots$$

so that $m_j = \widetilde{m}_j$ and, consequently, $a_j = \widetilde{a}_j$.

■■

Problem 27.18 Solution: Assume that $M_k = E^{\mathscr{A}_k}M$. Then we know from Theorem 27.19 that $\widetilde{M} = \lim_k M_k$ exists a.e. and in L^1. Moreover, $\int M_k\,dP = 1$ so that $\widetilde{M}$ cannot be trivial. On the other hand,

$$P(\widetilde{M} > 0) \leqslant P(M_k > 0) = P(X_j > 0 \quad \forall j = 1, 2, \dots, k) = 2^{-k} \xrightarrow[k\to\infty]{} 0$$

which yields a contradiction.

■■

Problem 27.19 Solution: (Compare this problem with Problem 22.16.) Recall that in finite measure spaces uniform integrability follows from (and is actually equivalent to)

$$\lim_{R\to\infty} \sup_n \int_{\{|u_n|>R\}} |u_n|\,d\mu = 0;$$

this is true since in a finite measure space the constant function $w \equiv R$ is integrable.

Observe now that

$$\begin{aligned}\int_{\{|u_n|>R\}} |u_n|\,d\mu &\leqslant \int_{\{|u_n|>R\}} E^{\mathscr{A}_n} f\,d\mu \\ &= \int_{\{|u_n|>R\}} f\,d\mu\end{aligned}$$

$$
\begin{aligned}
&= \int_{\{|u_n|>R\}\cap\{f\leqslant R/2\}} f\,d\mu + \int_{\{|u_n|>R\}\cap\{f>R/2\}} f\,d\mu \\
&\leqslant \int_{\{|u_n|>R\}\cap\{f\leqslant R/2\}} \frac{1}{2}|u_n|\,d\mu + \int_{\{|u_n|>R\}\cap\{f>R/2\}} f\,d\mu \\
&\leqslant \int_{\{|u_n|>R\}} \frac{1}{2}|u_n|\,d\mu + \int_{\{f>R/2\}} f\,d\mu
\end{aligned}
$$

This shows that

$$
\frac{1}{2}\int_{\{|u_n|>R\}} |u_n|\,d\mu \leqslant \int_{\{f>R/2\}} f\,d\mu \xrightarrow[\text{uniformly for all } n]{R\to\infty} 0.
$$

■■

28 Orthonormal systems and their convergence behaviour. Solutions to Problems 28.1–28.11

Problem 28.1 Solution: Since $J_k^{(\alpha,\beta)}$ is a polynomial of degree k, it is enough to show that $J_k^{(\alpha,\beta)}$ is orthogonal in $L^2(I, \rho(x)\,dx)$ to any polynomial $p(x)$ of degree $j < k$. We write ∂^k for $\frac{d^k}{dx^k}$ and $u(x) = (x-1)^{k+\alpha}(x+1)^{k+\beta}$. Then we get by repeatedly integrating by parts

$$\begin{aligned}
&\int_{-1}^{1} J_k^{(\alpha,\beta)}(x)p(x)(x-1)^\alpha(x+1)^\beta\,dx \\
&= \frac{(-1)^k}{k!\,2^k}\int_{-1}^{1} p(x)\partial^k u(x)\,dx \\
&= \Big[p(x)\cdot\partial^{k-1}u(x) - \partial^1 p(x)\cdot\partial^{k-2}u(x) + \cdots + (-1)^{k-1}\partial^{k-1}p(x)\cdot u(x)\Big]_{-1}^{1} \\
&\qquad + (-1)^k\int_{-1}^{1} u(x)\partial^k p(x)\,dx.
\end{aligned}$$

Obviously, $\partial^\ell u(-1) = \partial^\ell u(1) = 0$ for all $0 \leqslant \ell \leqslant k-1$ and $\partial^k p \equiv 0$ since p is a polynomial of degree $j < k$.

Problem 28.2 Solution: It is pretty obvious how to go about this problem. The calculations themselves are quite tedious and therefore omitted.

Problem 28.3 Solution: Theorem 28.6: *The polynomials are dense in $C[a,b]$ with respect to uniform convergence.*

Proof 1: mimic the proof of 28.6 with the obvious changes;

Proof 2: Let $f \in C[a,b]$. Then $\widetilde{f}(y) := f(a+(b-a)y)$, $y \in [0,1]$ satisfies $\widetilde{f} \in C[0,1]$ and, because of Theorem 28.6, there is a sequence of polynomials $\widetilde{p}_n$ such that

$$\lim_{n\to\infty}\sup_{y\in[0,1]}|\widetilde{f}(y) - \widetilde{p}_n(y)| = 0.$$

Define $p_n(x) := \widetilde{p}_n\left(\frac{x-a}{b-a}\right)$, $x \in [a,b]$. Clearly p_n is a polynomial and we have

$$\sup_{x\in[a,b]}|p_n(x) - f(x)| = \sup_{y\in[0,1]}|\widetilde{p}_n(y) - \widetilde{f}(y)|.$$

Corollary 28.8: *The monomials are complete in $L^1([a,b],dt)$.*

Proof 1: mimic the proof of 28.8 with the obvious changes;

Proof 2: assume that for all $j \in \mathbb{N}_0$ we have

$$\int_a^b u(x)x^j\,dx = 0.$$

Since

$$\begin{aligned}\int_0^1 u((b-a)t+a)t^j\,dx &= \int_a^b u(x)\left[\frac{x-a}{b-a}\right]^j dx\\ &= \sum_{k=0}^j c_k \int_a^b u(x)x^k\,dx\\ &= 0\end{aligned}$$

we get from Corollary 28.8 that

$$u((b-a)t+a) = 0 \qquad \text{Lebesgue almost everywhere on } [0,1]$$

and since the map $[0,1] \ni t \mapsto x = (b-a)t + a \in [a,b]$ is continuous, bijective and with a continuous inverse, we also get

$$u(x) = 0 \qquad \text{Lebesgue almost everywhere on } [a,b].$$

Problem 28.4 Solution: Observe that

$$\begin{aligned}\operatorname{Re}\left(e^{i(x-y)} - e^{i(x+y)}\right) &= \operatorname{Re}\left[e^{ix}\left(e^{-iy} - e^{iy}\right)\right]\\ &= \operatorname{Re}\left[-2ie^{ix}\sin y\right]\\ &= 2\sin x\sin y,\end{aligned}$$

and that

$$\begin{aligned}\operatorname{Re}\left(e^{i(x+y)} + e^{i(x-y)}\right) &= \operatorname{Re}\left[e^{ix}\left(e^{iy} + e^{-iy}\right)\right]\\ &= \operatorname{Re}\left[2e^{ix}\cos y\right]\\ &= 2\cos x\cos y.\end{aligned}$$

Moreover, we see that for $N \in \mathbb{N}_0$

$$\int_{-\pi}^{\pi} e^{iNx}\,dx = \begin{cases}\frac{e^{iNx}}{iN}\Big|_{-\pi}^{\pi} = 0, & \text{if } N \neq 0;\\ 2\pi, & \text{if } N = 0.\end{cases}$$

Thus, if $k \neq \ell$

$$\int_{-\pi}^{\pi} 2\cos kx\cos \ell x\,dx = \operatorname{Re}\left(\int_{-\pi}^{\pi} e^{i(k+\ell)x}\,dx + \int_{-\pi}^{\pi} e^{i(k+\ell)x}\,dx\right) = 0$$

and if $k = \ell \geqslant 1$

$$\int_{-\pi}^{\pi} 2\cos kx \cos kx \, dx = \mathrm{Re}\left(\int_{-\pi}^{\pi} e^{2ikx}\, dx + \int_{-\pi}^{\pi} 1 \, dx \right) = 2\pi$$

and if $k = \ell = 0$,

$$\int_{-\pi}^{\pi} 2\cos kx \cos kx \, dx = \int 2\, dx = 4\pi.$$

The proof for the pure sines integral is similar while for the mixed sine-cosine integrals the integrand

$$x \mapsto \cos kx \sin \ell x$$

is always an odd function, the integral over the symmetric (w.r.t. the origin) interval $(-\pi, \pi)$ is always zero.

Problem 28.5 Solution:

(i) We have

$$\begin{aligned} 2^k \cos^k(x) &= 2^k \Big(\frac{e^{ix} + e^{-ix}}{2} \Big)^k \\ &= \Big(\frac{e^{ix} + e^{-ix}}{2} \Big)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} e^{ijx} e^{-i(k-j)x} \\ &= \sum_{j=0}^{k} \binom{k}{j} e^{i(2j-k)x} \end{aligned}$$

Adding the first and last terms, second and penultimate terms, term no. j and $k-j$, etc. under the sum gives, since the binomial coefficients satisfy $\binom{k}{j} = \binom{k}{k-j}$,

- if $k = 2n$ is even

$$\begin{aligned} 2^{2n} \cos^{2n}(x) &= \sum_{j=0}^{n-1} \binom{2n}{j} (e^{i(2j-2n)x} + e^{i(2n-2j)x}) + \binom{2n}{n} \\ &= \sum_{j=0}^{n} \binom{2n}{j} 2\cos(2j - 2n) + \binom{2n}{n} \end{aligned}$$

- if $k = 2n-1$ is odd

$$\begin{aligned} 2^{2n-1} \cos^{2n-1}(x) &= \sum_{j=0}^{n-1} \binom{2n-1}{j} (e^{i(2j-2n+1)x} + e^{i(2n-2j-1)x}) \\ &= \sum_{j=0}^{n-1} \binom{2n-1}{j} 2\cos(2n - 2j - 1)x. \end{aligned}$$

In a similar way we compute $\sin^k x$:

$$\begin{aligned}
2^k \sin^k(x) &= 2^k \Big(\frac{e^{ix} - e^{-ix}}{2i} \Big)^k \\
&= i^{-k} \Big(\frac{e^{ix} - e^{-ix}}{2} \Big)^k \\
&= i^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{ijx} e^{-i(k-j)x} \\
&= i^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{i(2j-k)x}.
\end{aligned}$$

Adding the first and last terms, second and penultimate terms, term no. j and $k-j$, etc. under the sum gives, since the binomial coefficients satisfy $\binom{k}{j} = \binom{k}{k-j}$,

- if $k = 2n$ is even

$$\begin{aligned}
&2^{2n} \sin^{2n}(x) \\
&= (-1)^n \sum_{j=0}^{n-1} \binom{2n}{j} \big((-1)^{2n-j} e^{i(2j-2n)x} + (-1)^j e^{i(2n-2j)} \big) + \binom{2n}{n} \\
&= \sum_{j=0}^{n-1} \binom{2n}{j} (-1)^{n-j} \big(e^{i(2j-2n)x} + e^{i(2n-2j)} \big) + \binom{2n}{n} \\
&= \sum_{j=0}^{n-1} \binom{2n}{j} (-1)^{n-j} 2 \cos(2n-2j)x + \binom{2n}{n}
\end{aligned}$$

- if $k = 2n-1$ is odd

$$\begin{aligned}
&2^{2n-1} \sin^{2n-1}(x) \\
&= i(-1)^n \sum_{j=0}^{n-1} \binom{2n-1}{j} \big((-1)^{2n-1-j} e^{i(2j-2n+1)x} + (-1)^{-j} e^{i(2n-2j-1)} \big) \\
&= i \sum_{j=0}^{n-1} \binom{2n-1}{j} (-1)^{n-j} \big(- e^{i(2j-2n+1)x} + e^{i(2n-2j-1)} \big) \\
&= i \sum_{j=0}^{n-1} \binom{2n-1}{j} (-1)^{n-j} 2i \sin(2n-2j+1)x \\
&= \sum_{j=0}^{n-1} \binom{2n-1}{j} (-1)^{n-j-1} 2 \sin(2n-2j+1)x.
\end{aligned}$$

(ii) We have

$$\cos kx + i \sin kx = e^{ikx} = \big(e^{ix} \big)^k = \big(\cos x + i \sin x \big)^k$$

and we find, using the binomial formula,

$$\cos kx + i \sin kx = \sum_{j=0}^{k} \binom{k}{j} \cos^j x \cdot i^{k-j} \sin^{k-j} x$$

and the claim follows by separating real and imaginary parts.

(iii) Since a trigonometric polynomial is of the form

$$T_n(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right)$$

it is a matter of double summation and part (ii) to see that $T_n(x)$ can be written like $U_n(x)$.

Conversely, part (i) enables us to rewrite any expression of the form $U_n(x)$ as $T_n(x)$.

Problem 28.6 Solution: By definition,

$$D_N(x) = \frac{1}{2} + \sum_{j=1}^{N} \cos jx.$$

Multiplying both sides by $\sin \frac{x}{2}$ and using the formula

$$\cos ax \sin bx = \frac{1}{2}\left(\sin \frac{(a+b)x}{2} - \sin \frac{(a-b)x}{2}\right)$$

where $j = (a+b)/2$ and $1/2 = (a-b)/2$, i.e. $a = (2j+1)/2$ and $b = (2j-1)/2$ we arrive at

$$D_N(x) \sin \tfrac{x}{2} = \tfrac{1}{2} \sin \tfrac{x}{2} + \tfrac{1}{2} \sum_{j=1}^{N} \left(\sin \tfrac{(2j+1)x}{2} - \sin \tfrac{(2j-1)x}{2}\right) = \sin \tfrac{(2N+1)x}{2}.$$

Problem 28.7 Solution: We have

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right).$$

Indeed, let us calculate the Fourier coefficients 28.8. First,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \sin kx \, dx = 0, \quad k \in \mathbb{N},$$

since the integrand is an odd function. So no sines appear in the Fourier series expansion. Further, using the symmetry properties of the sine function

$$\begin{aligned} a_0/2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sin x| \, dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} |\sin x| \, dx \\ &= \frac{1}{\pi} (-\cos x)\Big|_0^{\pi} \\ &= \frac{2}{\pi} \end{aligned}$$

and using the elementary formula $2 \sin a \cos b = \sin(a-b) + \sin(a+b)$ we get

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos jx \, dx$$

$$\begin{aligned}
&= \frac{2}{\pi}\int_0^\pi \sin x \cos jx\,dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\big(\sin((j+1)x) - \sin((j-1)x)\big)\,dx \\
&= \frac{1}{\pi}\left[\frac{\cos((j-1)x)}{j-1} - \frac{\cos((j+1)x)}{j+1}\right]_0^\pi \\
&= \frac{1}{\pi}\left[\frac{\cos((j-1)\pi)}{j-1} - \frac{\cos((j+1)\pi)}{j+1} - \frac{1}{j-1} + \frac{1}{j+1}\right].
\end{aligned}$$

If j is odd, we get $a_j = 0$ and if j is even, we have

$$a_j = \frac{1}{\pi}\left[\frac{-1}{j-1} - \frac{-1}{j+1} - \frac{1}{j-1} + \frac{1}{j+1}\right] = -\frac{4}{\pi}\,\frac{1}{(j-1)(j+1)}.$$

This shows that we have only evenly indexed cosines in the Fourier series.

Problem 28.8 Solution: This is not as trivial as it looks in the first place! Since u is itself a Haar function, we have

$$s_N(u,x) = u(x) \qquad \forall N \in \mathbb{N}$$

(it is actually the first Haar function) so that s_N converges in any L^p-norm, $1 \leqslant p < \infty$ to u.

The same applies to the *right tail* of the Haar wavelet expansion. The left tail, however, converges only for $1 < p < \infty$ in L^p. The reason is the calculation of Step 5 in the proof of Theorem 28.20 which goes in the case $p = 1$:

$$\begin{aligned}
\mathbb{E}^{\mathscr{A}^\Delta_{-M}} u &= 2^{-M}\int_{[-2^M,0)} u(x)\,dx\,\mathbb{1}_{[-2^M,0)} + 2^{-M}\int_{[0,2^M)} u(x)\,dx\,\mathbb{1}_{[0,2^M)} \\
&= 2^{-M}\mathbb{1}_{[0,2^M)},
\end{aligned}$$

but this is not L^1-convergent to 0 as it would be required. For $p > 1$ all is fine, though....

Problem 28.9 Solution: Assume that u is uniformly continuous (C_c and C_∞-functions are!). Since

$$s_n(u;x) = \mathbb{E}^{\mathscr{A}^H_n} u(x)$$

is the projection onto the sets in $\mathscr{A}^H_n$, see e.g. Step 2 in the proof of Theorem 28.17, we have

$$s_n(u;x) = \frac{1}{\lambda(I)}\int_I u(y)\,dx\,\mathbb{1}_I(x)$$

where I is an dyadic interval from the generator of $\mathscr{A}^H_n$ as in Step 2 of the proof of Theorem 28.17. Thus, if x is from I we get

$$\begin{aligned}
|s_n(u;x) - u(x)| &= \left|\frac{1}{\lambda(I)}\int_I (u(y) - u(x))\,dx\right| \\
&\leqslant \frac{1}{\lambda(I)}\int_I |u(y) - u(x)|\,dx
\end{aligned}$$

$$\leqslant \frac{1}{\lambda(I)} \int_I \epsilon \, dx$$
$$= \epsilon$$

if $\lambda(I) < \delta$ for small enough $\delta > 0$. This follows from uniform continuity: for given $\epsilon > 0$ there is some $\delta > 0$ such that for $x, y \in I$ (this entails $|x - y| \leqslant \delta$!) we have $|u(x) - u(y)| \leqslant \epsilon$.

The above calculation holds uniformly for all x and we are done.

Problem 28.10 Solution: The calculation for the right tail is more or less the same as in Problem 28.9. Only the left tail differs. Here we argue as in Step 5 of the proof of Theorem 28.20: if $u \in C_c(\mathbb{R})$ we can assume that $\operatorname{supp} u \subset [-R, R]$ and we see

$$\begin{aligned}
\mathbb{E}^{\mathscr{A}^\Delta_{-M}} u(x) &= 2^{-M} \int_{[-R,0]} u(x)\, dx \mathbb{1}_{[-2^M,0)} + 2^{-M} \int_{[0,R]} u(x)\, dx \mathbb{1}_{[0,2^M)} \\
&\leqslant 2^{-M} R \, \|u\|_\infty \mathbb{1}_{[-2^M,0)} + 2^{-M} R \, \|u\|_\infty \mathbb{1}_{[0,2^M)} \\
&= 2^{-M} R \, \|u\|_\infty \mathbb{1}_{[-2^M,2^M)} \\
&\leqslant 2^{-M} R \, \|u\|_\infty \xrightarrow[\text{uniformly for all } x]{M\to\infty} 0
\end{aligned}$$

If $u \in C_\infty$ we can use the fact that C_c is dense in C_∞, i.e. we can find for every $\epsilon > 0$ functions $v = v_\epsilon \in C_c$ and $w = w_\epsilon \in C_\infty$ such that

$$u = v + w \quad \text{and} \quad \|w\|_\infty \leqslant \epsilon.$$

Then

$$\begin{aligned}
\left|\mathbb{E}^{\mathscr{A}^\Delta_{-M}} u(x)\right| &\leqslant \left|\mathbb{E}^{\mathscr{A}^\Delta_{-M}} v(x)\right| + \left|\mathbb{E}^{\mathscr{A}^\Delta_{-M}} w(x)\right| \\
&\leqslant \left|\mathbb{E}^{\mathscr{A}^\Delta_{-M}} v(x)\right| + \mathbb{E}^{\mathscr{A}^\Delta_{-M}} \|w\|_\infty \\
&\leqslant \left|\mathbb{E}^{\mathscr{A}^\Delta_{-M}} v(x)\right| + \epsilon
\end{aligned}$$

and, by the first calculation for C_c-functions, the right-hand side converges, since $v \in C_c$, to $0 + \epsilon$ uniformly for all x, and letting $\epsilon \to 0$ we conclude the proof.

Problem 28.11 Solution: See the picture at the end of this solution. Since the function $u(x) := \mathbb{1}_{[0,1/3)}(x)$ is piecewise constant, and since for each Haar function $\int \chi_{k,j}\, dx = 0$ unless $j = k = 1$, we see that only a single Haar function contributes to the value of $s_N(u; \frac{1}{3})$, namely where $\frac{1}{3} \in \operatorname{supp} \chi_{n,j}$.

The idea of the proof is now pretty clear: take values N where $x = \frac{1}{3}$ is in the left 'half' of $\chi_{N,k}$, i.e. where $\chi_{N,k}(\frac{1}{3}) = 1$ and values M such that $x = \frac{1}{3}$ is in the opposite, negative 'half' of $\chi_{M,\ell}$, i.e. where $\chi_{M,\ell}(\frac{1}{3}) = -1$. Of course, k, ℓ depend on x, N and M respectively. One should expect that the partial sums for these different positions lead to different limits, hence different upper and lower limits.

The problem is to pick N's and M's. We begin with the simple observation that the dyadic (i.e. base-2) representation of $1/3$ is the periodic, infinite dyadic fraction

$$\frac{1}{3} = 0.01010101\cdots = \sum_{k=1}^{\infty} \frac{1}{2^{2k}}$$

and that the finite fractions

$$d_n := 0.\underbrace{0101\cdots 01}_{2n} = \sum_{k=1}^{n} \frac{1}{2^{2k}}$$

approximate $1/3$ from the left in such a way that

$$\frac{1}{3} - d_n = \sum_{k=n+1}^{\infty} \frac{1}{2^{2k}} < \sum_{\ell=2n+2}^{\infty} \frac{1}{2^{\ell}} = \frac{1}{2^{2n+2}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{2n+1}}$$

Now consider those Haar functions whose support consists of intervals of the length 2^{-2n}, i.e. the $\chi_{2n,j}$'s and agree that $j = j(1/3, n)$ is the one value where $\frac{1}{3} \in \operatorname{supp}\chi_{2n,j}$. By construction $\operatorname{supp}\chi_{2n,j} = [d_n, d_n + 1/2^{2n}]$ and we get for the Haar-Fourier partial sum

$$\begin{aligned}
s_{2n}(u, \tfrac{1}{3}) - \frac{1}{3} &= \int_{d_n}^{1/3} 2^n dx \cdot \chi_{2n,j}(\tfrac{1}{3}) \\
&= 2^{2n}\big(\tfrac{1}{3} - d_n\big) \\
&= 4^n \sum_{k=n+1}^{\infty} \frac{1}{2^{2k}} \\
&= 4^n \sum_{k=n+1}^{\infty} \frac{1}{4^k} \\
&= 4^n 4^{-n-1} \frac{1}{1-\frac{1}{4}} \\
&= \frac{1}{3}.
\end{aligned}$$

The shift by $-1/3$ comes from the starting 'atypical' Haar function $\chi_{0,0}$ since $\langle u, \chi_{0,0}\rangle = \int_0^{1/3} dx = \frac{1}{3}$.

Using the next smaller Haar functions with support of length 2^{-2n-1}, i.e. the $\chi_{2n+1,k}$'s, we see that with j as above $\chi_{2n+1,2j-1}(\frac{1}{3}) = -1$ (since twice as many Haar functions appear in the run-up to d_n) and that

$$\begin{aligned}
&s_{2n+1}(u, \tfrac{1}{3}) - \frac{1}{3} \\
&= \left[\int_{d_n}^{d_n+1/2^{2n+2}} 2^{n+1} dx - \int_{d_n+1/2^{2n+2}}^{1/3} 2^{n+1} dx\right] \cdot \chi_{2n+1,2j-1}(\tfrac{1}{3}) \\
&= \left[d_n + \frac{1}{2^{2n+2}} - d_n - \frac{1}{3} + d_n + \frac{1}{2^{2n+2}}\right] 2^{n+1} \cdot (-2^{n+1}) \\
&= \left[d_n - \frac{1}{3} + \frac{2}{2^{2n+2}}\right] \cdot (-2^{2n+2})
\end{aligned}$$

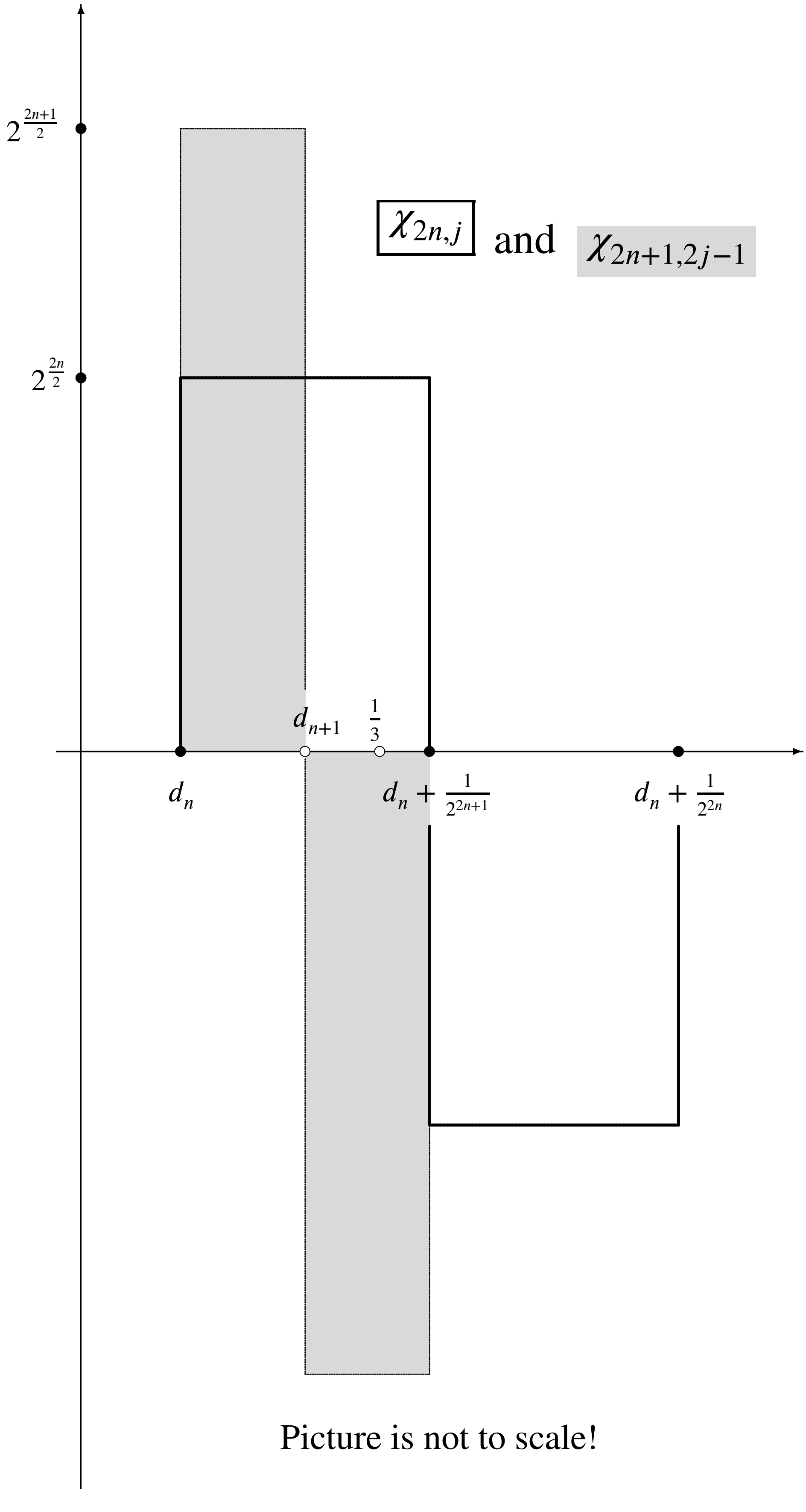
$$
\begin{aligned}
&= 4 \cdot 2^{2n}\left(\frac{1}{3} - d_n\right) - 2 \\
&= 4 \cdot \frac{1}{3} - 2 && \text{(using the result above)} \\
&= -\frac{2}{3}
\end{aligned}
$$

This shows that

$$s_{2n}(u; \tfrac{1}{3}) = \frac{2}{3} > -\frac{1}{3} = s_{2n+1}(u, \tfrac{1}{3})$$

and the claim follows since because of the above inequality,

$$\liminf_N s_N(u; \tfrac{1}{3}) \leqslant -\frac{1}{3} \leqslant \frac{2}{3} \leqslant \limsup_N s_N(u; \tfrac{1}{3}).$$



Picture is not to scale!