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Abbreviations and symbols

Here I list the most common abbreviations and symbolic notation used throughout the text.

	this indicates that you should check it by yourself
	this indicates a warning
	this indicates further information
a.a.	almost all
a.e.	almost every(where)
a.s.	almost surely
bdd	bounded
BL, cBL	(conditional) Beppo Levi theorem
BM	Brownian motion
BV	bounded variation
b/o	because of
c.f.	characteristic function
cf.	confer, see
CLT	central limit theorem
d -convergence	convergence in distribution
DCT, cDCT	(conditional) dominated convergence theorem
e.g.	exempli gratia, for example
fdd	finite dimensional distributions
iid	independent identically distributed
LLN	law of large numbers
mble	measurable
MC	Markov chain
MCT	monotone class theorem <i>or</i> martingale convergence theorem
mg	martingale
MP	Markov property
\mathbb{P} -convergence	convergence in probability
PP, cPP	Poisson process, compound Poisson process

rv, rvs	random variable, random variables
RW	random walk
SLLN	strong law of large numbers
SMP	strong Markov property
SRW	simple random walk
ui	uniformly integrable
WLLN	weak law of large numbers
wlog	without loss of generality
positive	always used in the sense $\gg 0$
negative	always used in the sense $\ll 0$
increasing	$x \leq y \implies f(x) \leq f(y)$ (f may have »flat« pieces)
decreasing	$x \leq y \implies f(x) \geq f(y)$ (f may have »flat« pieces)
\mathbb{N}	natural numbers $1, 2, 3, \dots$
\mathbb{N}_0	natural numbers $0, 1, 2, 3, \dots$
$X \sim \mu$	the rv X is distributed like μ
$X \sim Y$	the rv X is distributed like the rv Y
$X \perp\!\!\!\perp Y$	X and Y are independent
$x \gg 1, \epsilon \ll 1$	x is sufficiently large, $\epsilon \in (0, 1)$ is sufficiently small
δ_{ij}	Kronecker's delta: $\delta_{ij} = \mathbb{1}_{\{i\}}(j) = \delta_j(\{i\}) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
$\mathcal{L}^p(\dots), \mathbb{L}^p(\dots)$	Lebesgue spaces of integrable functions $1 \leq p \leq \infty$
$\mathcal{L}^0(\mathcal{A})$	\mathcal{A} -measurable functions
$\mathcal{E}(\mathcal{A})$	\mathcal{A} -measurable simple (»step«) functions

A »+« as sub- or superscript, such as \mathcal{E}^+ or \mathcal{L}_+^p means the positive (≥ 0) elements of \mathcal{E} or \mathcal{L}^p etc.