Preface

A **counterexample** /ˈkaʊntərɪɡˌzɑːmpl/ is *an example that opposes or contradicts an idea or theory.*[1] It is fair to say that the word 'counterexample' is not too common in everyday language, but rather a concept from philosophy and, of course, mathematics. In mathematics, there are proofs and examples, and while an example, say, of some $x \in A$ satisfying $x \in B$ does not prove $A \subseteq B$, the *counter*example of some $x_0 \in B$ such that $x_0 \notin A$ disproves $A \subseteq B$; in other words, it proves that $A \subseteq B$ does not hold. This observation shows that there is no sharp distinction between example and counterexample, and we do not give a definition of what a counterexample should or could be (you may want to consult Lakatos [94] instead), but assume the more pragmatic point of view of a working mathematician. If we want to solve a problem, we look at the same time for a proof and for counterexamples which help us to capture and delineate the subject matter.

The same is also true for the student of mathematics, who will gain a better understanding of a theorem or theory if he knows its limitations – which may be expressed in the form of counterexamples. The present collection of (counter-)examples grew out of our own experience, in the classroom and on `stackexchange.com`, where we are often asked after the 'how' and 'why' of many a result. This explains the wide range of examples, from the fairly obvious to rather intricate constructions. The choice of the examples reflects, naturally, our own taste. We decided to include only those counterexamples which could be dealt with in a couple of pages (or less) and which are not too pathological – one can, indeed, destroy almost anything by the choice of the underlying topology. We intend the present volume as a companion to our textbook *Measures, Integrals and Martingales* [MIMS], which means that most examples are from elementary measure and integration, not touching on integration on

[1] Oxford dictionaries `https://en.oxforddictionaries.com/definition/counterexample`, accessed 11-May-2019.

groups (Haar measure) or on really deep axiomatic issues (e.g. as in descriptive set theory, see Kechris [89], and the advanced constructive theory of functions, see Kharazishvili [91, 92]).

This book is intended as supplementary reading for a course in measure and integration theory, or for seminars and reading courses where students can explore certain aspects of the theory by themselves. Where appropriate, we have added comments putting the example into context and pointing the reader to further literature. We think that this book will also be useful for lecturers and tutors in teaching measure and integration, and for researchers who may discover new and sometimes unexpected phenomena. Readers are assumed to have basic knowledge of functional analysis, point-set topology and, of course, measure and integration theory. For novices, there is a panorama of measure and integration which gives a non-technical overview on the subject and can serve, to some extent, as a first introduction. The overall presentation is as self-contained as possible; in order to make the text easy to access, we use only a few standard references – Schilling [MIMS] and Bogachev [19] for measure and integration, Rudin [151] and Yosida [202] for functional analysis, and Willard [200] and Engelking [53] for topology.

Some of the counterexamples are famous, many are more or less well known, and a few are of our own making. When we could trace the origin of an example, we have given references and attached names, but most entries are 'standard' examples which seem to have been in the public domain for ages; having said this, we acknowledge a huge debt to many anonymous authors and we do apologize if we have failed to give proper credit. The three classic counterexample books by Gelbaum & Olmsted [65], Steen & Seebach [172], and Stoyanov [180] were both inspiration and encouragement. We hope that this book lives up to their high standards.

It is a pleasure to acknowledge the interest and skill of our editor, Roger Astley, in the preparation of this book and Cambridge University Press for the excellent book design. Many colleagues have contributed to this text with comments and suggestions, in particular M. Auer, R. Baumgarth, G. Berschneider, N.H. Bingham – for the famous full red-ink treatment, C.-S. Deng, D.E. Edmunds and C. Goldie – for most helpful discussions, Y. Ishikawa, N. Jacob – for access to his legendary library, Y. Mishura and N. Sandrić. We thank our colleagues and friends who suffered for quite a while from our destructive search for counterexamples (*Do you know an example of a measure which fails to … ?*), strange functions and many outer-worldly excursions – and our families who have us back in real life.