

Brownian Motion (3rd edition)

An Guide to Random Processes and Stochastic Calculus

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Solution Manual

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1 Robert Brown's new thing

Problem 1.1. Solution:

a) We show the result for \mathbb{R}^d -valued random variables. Let $\xi, \eta \in \mathbb{R}^d$. By assumption,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \exp \left[i \left\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \right\rangle \right] &= \mathbb{E} \exp \left[i \left\langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \right\rangle \right] \\ \iff \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \xi, X_n \rangle + i \langle \eta, Y_n \rangle] &= \mathbb{E} \exp [i \langle \xi, X \rangle + i \langle \eta, Y \rangle] \end{aligned}$$

If we take $\xi = 0$ and $\eta = 0$, respectively, we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \eta, Y_n \rangle] &= \mathbb{E} \exp [i \langle \eta, Y \rangle] \quad \text{or} \quad Y_n \xrightarrow{d} Y \\ \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \xi, X_n \rangle] &= \mathbb{E} \exp [i \langle \xi, X \rangle] \quad \text{or} \quad X_n \xrightarrow{d} X. \end{aligned}$$

Since $X_n \perp\!\!\!\perp Y_n$ we find

$$\begin{aligned} \mathbb{E} \exp [i \langle \xi, X \rangle + i \langle \eta, Y \rangle] &= \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \xi, X_n \rangle + i \langle \eta, Y_n \rangle] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \xi, X_n \rangle] \mathbb{E} \exp [i \langle \eta, Y_n \rangle] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \xi, X_n \rangle] \lim_{n \rightarrow \infty} \mathbb{E} \exp [i \langle \eta, Y_n \rangle] \\ &= \mathbb{E} \exp [i \langle \xi, X \rangle] \mathbb{E} \exp [i \langle \eta, Y \rangle] \end{aligned}$$

and this shows that $X \perp\!\!\!\perp Y$.

b) We have

$$\begin{aligned} X_n &= X + \frac{1}{n} \xrightarrow[n \rightarrow \infty]{\text{almost surely}} X \implies X_n \xrightarrow{d} X \\ Y_n = 1 - X_n &= 1 - \frac{1}{n} - X \xrightarrow[n \rightarrow \infty]{\text{almost surely}} 1 - X \implies Y_n \xrightarrow{d} 1 - X \\ X_n + Y_n &= 1 \xrightarrow[n \rightarrow \infty]{\text{almost surely}} 1 \implies X_n + Y_n \xrightarrow{d} 1. \end{aligned}$$

A simple direct calculation shows that $1 - X \sim \frac{1}{2}(\delta_0 + \delta_1) \sim Y$. Thus,

$$X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{d} Y \sim 1 - X, \quad X_n + Y_n \xrightarrow{d} 1.$$

Assume that $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Since $X \perp\!\!\!\perp Y$, we find for the distribution of $X + Y$:

$$X + Y \sim \frac{1}{2}(\delta_0 + \delta_1) * \frac{1}{2}(\delta_0 + \delta_1) = \frac{1}{4}(\delta_0 * \delta_0 + 2\delta_1 * \delta_0 + \delta_1 * \delta_1) = \frac{1}{4}(\delta_0 + 2\delta_1 + \delta_2).$$

Thus, $X + Y \not\sim \delta_0 \sim 1 = \lim_n (X_n + Y_n)$ and this shows that we cannot have that $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

- c) If $X_n \perp\!\!\!\perp Y_n$ and $X \perp\!\!\!\perp Y$, then we have $X_n + Y_n \xrightarrow{d} X + Y$: this follows since we have for all $\xi \in \mathbb{R}$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi(X_n + Y_n)} &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \mathbb{E} e^{i\xi Y_n} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi Y_n} \\ &= \mathbb{E} e^{i\xi X} \mathbb{E} e^{i\xi Y} \\ &\stackrel{a)}{=} \mathbb{E} [e^{i\xi X} e^{i\xi Y}] \\ &= \mathbb{E} e^{i\xi(X+Y)}. \end{aligned}$$

A similar (even easier) argument works if $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Then we have

$$f(x, y) := e^{i\xi(x+y)}$$

is bounded and continuous, i.e. we get directly

$$\lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi(X_n + Y_n)} = \lim_{n \rightarrow \infty} \mathbb{E} f(X_n, Y_n) = \mathbb{E} f(X, Y) = \mathbb{E} e^{i\xi(X+Y)}.$$

For a counterexample (if X_n and Y_n are not independent), see part b).

Notice that the independence and d -convergence of the sequences X_n, Y_n already implies $X \perp\!\!\!\perp Y$ and the d -convergence of the bivariate sequence (X_n, Y_n) . This is a consequence of the following

Lemma. *Let $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ be sequences of random variables (or random vectors) on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If*

$$X_n \perp\!\!\!\perp Y_n \text{ for all } n \geq 1 \text{ and } X_n \xrightarrow[n \rightarrow \infty]{d} X \text{ and } Y_n \xrightarrow[n \rightarrow \infty]{d} Y,$$

then $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$ and $X \perp\!\!\!\perp Y$.

Proof. Write $\phi_X, \phi_Y, \phi_{X,Y}$ for the characteristic functions of X, Y and the pair (X, Y) . By assumption

$$\lim_{n \rightarrow \infty} \phi_{X_n}(\xi) = \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} = \mathbb{E} e^{i\xi X} = \phi_X(\xi).$$

A similar statement is true for Y_n and Y . For the pair we get, because of independence

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_{X_n, Y_n}(\xi, \eta) &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n + i\eta Y_n} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \mathbb{E} e^{i\eta Y_n} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \lim_{n \rightarrow \infty} \mathbb{E} e^{i\eta Y_n} \\ &= \mathbb{E} e^{i\xi X} \mathbb{E} e^{i\eta Y} \\ &= \phi_X(\xi) \phi_Y(\eta). \end{aligned}$$

Thus, $\phi_{X_n, Y_n}(\xi, \eta) \rightarrow h(\xi, \eta) = \phi_X(\xi)\phi_Y(\eta)$. Since h is continuous at the origin $(\xi, \eta) = 0$ and $h(0, 0) = 1$, we conclude from Lévy's continuity theorem that h is a (bivariate) characteristic function and that $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Moreover,

$$h(\xi, \eta) = \phi_{X, Y}(\xi, \eta) = \phi_X(\xi)\phi_Y(\eta)$$

which shows that $X \perp\!\!\!\perp Y$. □

Problem 1.2. Solution: Using the elementary estimate

$$|e^{iz} - 1| = \left| \int_0^{iz} e^\zeta d\zeta \right| \leq \sup_{|y| \leq |z|} |e^{iy}| |z| = |z| \quad (*)$$

we see that the function $t \mapsto e^{i\langle \xi, t \rangle}$, $\xi, t \in \mathbb{R}^d$ is locally Lipschitz continuous:

$$\left| e^{i\langle \xi, t \rangle} - e^{i\langle \xi, s \rangle} \right| = \left| e^{i\langle \xi, t-s \rangle} - 1 \right| \leq |\langle \xi, t-s \rangle| \leq |\xi| \cdot |t-s| \quad \text{for all } \xi, t, s \in \mathbb{R}^d,$$

Thus,

$$\begin{aligned} \mathbb{E} e^{i\langle \xi, Y_n \rangle} &= \mathbb{E} \left[e^{i\langle \xi, Y_n - X_n \rangle} e^{i\langle \xi, X_n \rangle} \right] \\ &= \mathbb{E} \left[\left(e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right) e^{i\langle \xi, X_n \rangle} \right] + \mathbb{E} e^{i\langle \xi, X_n \rangle}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \mathbb{E} e^{i\langle \xi, X_n \rangle} = \mathbb{E} e^{i\langle \xi, X \rangle}$, we are done if we can show that the first term in the last line of the displayed formula tends to zero. To see this, we use the Lipschitz continuity of the exponential function. Fix $\xi \in \mathbb{R}^d$.

$$\begin{aligned} &\left| \mathbb{E} \left[\left(e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right) e^{i\langle \xi, X_n \rangle} \right] \right| \\ &\leq \mathbb{E} \left| \left(e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right) e^{i\langle \xi, X_n \rangle} \right| \\ &= \mathbb{E} \left| e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right| \\ &= \int_{|Y_n - X_n| \leq \delta} \left| e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right| d\mathbb{P} + \int_{|Y_n - X_n| > \delta} \left| e^{i\langle \xi, Y_n - X_n \rangle} - 1 \right| d\mathbb{P} \\ &\stackrel{(*)}{\leq} \delta |\xi| + \int_{|Y_n - X_n| > \delta} 2 d\mathbb{P} \\ &= \delta |\xi| + 2 \mathbb{P}(|Y_n - X_n| > \delta) \\ &\xrightarrow[n \rightarrow \infty]{} \delta |\xi| \xrightarrow[\delta \rightarrow 0]{} 0, \end{aligned}$$

where we used in the last step the fact that $X_n - Y_n \xrightarrow{\mathbb{P}} 0$. ■ ■

Problem 1.3. Solution: Recall that $Y_n \xrightarrow{d} Y$ with $Y = c$ a.s., i.e. where $Y \sim \delta_c$ for some constant $c \in \mathbb{R}$. Since the d -limit is trivial, this implies $Y_n \xrightarrow{\mathbb{P}} Y$. This means that both “is this still true”-questions can be answered in the affirmative.

We will show that $(X_n, Y_n) \xrightarrow{d} (X_n, c)$ holds – without assuming anything on the joint distribution of the random vector (X_n, Y_n) , i.e. we do not make assumption on the correlation structure of X_n and Y_n . Since the maps $x \mapsto x + y$ and $x \mapsto x \cdot y$ are continuous, we see that

$$\lim_{n \rightarrow \infty} \mathbb{E} f(X_n, Y_n) = \mathbb{E} f(X, c) \quad \forall f \in C_b(\mathbb{R} \times \mathbb{R})$$

implies both

$$\lim_{n \rightarrow \infty} \mathbb{E} g(X_n Y_n) = \mathbb{E} g(X c) \quad \forall g \in C_b(\mathbb{R})$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} h(X_n + Y_n) = \mathbb{E} h(X + c) \quad \forall h \in C_b(\mathbb{R}).$$

This proves (a) and (b).

In order to show that (X_n, Y_n) converges in distribution, we use Lévy's characterization of distributional convergence, i.e. the pointwise convergence of the characteristic functions. This means that we take $f(x, y) = e^{i(\xi x + \eta y)}$ for any $\xi, \eta \in \mathbb{R}$:

$$\begin{aligned} \left| \mathbb{E} e^{i(\xi X_n + \eta Y_n)} - \mathbb{E} e^{i(\xi X + \eta c)} \right| &\leq \left| \mathbb{E} e^{i(\xi X_n + \eta Y_n)} - \mathbb{E} e^{i(\xi X_n + \eta c)} \right| + \left| \mathbb{E} e^{i(\xi X_n + \eta c)} - \mathbb{E} e^{i(\xi X + \eta c)} \right| \\ &\leq \mathbb{E} \left| e^{i(\xi X_n + \eta Y_n)} - e^{i(\xi X_n + \eta c)} \right| + \left| \mathbb{E} e^{i(\xi X_n + \eta c)} - \mathbb{E} e^{i(\xi X + \eta c)} \right| \\ &\leq \mathbb{E} \left| e^{i\eta Y_n} - e^{i\eta c} \right| + \left| \mathbb{E} e^{i\xi X_n} - \mathbb{E} e^{i\xi X} \right|. \end{aligned}$$

The second expression on the right-hand side converges to zero as $X_n \xrightarrow{d} X$. For fixed η we have that $y \mapsto e^{i\eta y}$ is uniformly continuous. Therefore, the first expression on the right-hand side becomes, with any $\epsilon > 0$ and a suitable choice of $\delta = \delta(\epsilon) > 0$

$$\begin{aligned} \mathbb{E} \left| e^{i\eta Y_n} - e^{i\eta c} \right| &= \mathbb{E} \left[\left| e^{i\eta Y_n} - e^{i\eta c} \right| \mathbf{1}_{\{|Y_n - c| > \delta\}} \right] + \mathbb{E} \left[\left| e^{i\eta Y_n} - e^{i\eta c} \right| \mathbf{1}_{\{|Y_n - c| \leq \delta\}} \right] \\ &\leq 2 \mathbb{E} \left[\mathbf{1}_{\{|Y_n - c| > \delta\}} \right] + \mathbb{E} \left[\epsilon \mathbf{1}_{\{|Y_n - c| \leq \delta\}} \right] \\ &\leq 2 \mathbb{P}(|Y_n - c| > \delta) + \epsilon \\ &\xrightarrow[n \rightarrow \infty]{\text{P-convergence as } \delta, \epsilon \text{ are fixed}} \epsilon \xrightarrow{\epsilon \downarrow 0} 0. \end{aligned}$$

Remark. The direct approach to (a) is possible but relatively ugly. Part (b) has a relatively simple direct proof:

Fix $\xi \in \mathbb{R}$.

$$\mathbb{E} e^{i\xi(X_n + Y_n)} - \mathbb{E} e^{i\xi X} = \left(\mathbb{E} e^{i\xi(X_n + Y_n)} - \mathbb{E} e^{i\xi X_n} \right) + \underbrace{\left(\mathbb{E} e^{i\xi X_n} - \mathbb{E} e^{i\xi X} \right)}_{\xrightarrow[n \rightarrow \infty]{} 0 \text{ by } d\text{-convergence}}.$$

For the first term on the right we find with the uniform-continuity argument from Problem 1.2 and any $\epsilon > 0$ and suitable $\delta = \delta(\epsilon, \xi)$ that

$$\begin{aligned} \left| \mathbb{E} e^{i\xi(X_n + Y_n)} - \mathbb{E} e^{i\xi X_n} \right| &\leq \mathbb{E} \left| e^{i\xi X_n} (e^{i\xi Y_n} - 1) \right| \\ &= \mathbb{E} \left| e^{i\xi Y_n} - 1 \right| \end{aligned}$$

$$\leq \epsilon + \mathbb{P}(|Y_n| > \delta)$$

$$\xrightarrow[n \rightarrow \infty]{\substack{\epsilon \text{ fixed} \\ \epsilon \rightarrow 0}} \epsilon \xrightarrow{\epsilon \rightarrow 0} 0$$

where we use \mathbb{P} -convergence in the penultimate step.

Problem 1.4. Solution: Let $\xi, \eta \in \mathbb{R}$ and note that $f(x) = e^{i\xi x}$ and $g(y) = e^{i\eta y}$ are bounded and continuous functions. Thus we get

$$\begin{aligned} \mathbb{E} e^{i\langle (\xi), (X) \rangle} &= \mathbb{E} e^{i\xi X} e^{i\eta Y} \\ &= \mathbb{E} f(X)g(Y) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} f(X_n)g(Y) \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} e^{i\eta Y} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\langle (\xi), (X_n) \rangle} \end{aligned}$$

and we see that $(X_n, Y) \xrightarrow{d} (X, Y)$.

Assume now that $X = \phi(Y)$ for some Borel function ϕ . Let $f \in \mathcal{C}_b$ and pick $g := f \circ \phi$. Clearly, $f \circ \phi \in \mathcal{B}_b$ and we get

$$\begin{aligned} \mathbb{E} f(X_n)f(X) &= \mathbb{E} f(X_n)f(\phi(Y)) \\ &= \mathbb{E} f(X_n)g(Y) \\ &\xrightarrow[n \rightarrow \infty]{} \mathbb{E} f(X)g(Y) \\ &= \mathbb{E} f(X)f(X) \\ &= \mathbb{E} f^2(X). \end{aligned}$$

Now observe that $f \in \mathcal{C}_b \implies f^2 \in \mathcal{C}_b$ and $g \equiv 1 \in \mathcal{B}_b$. By assumption

$$\mathbb{E} f^2(X_n) \xrightarrow[n \rightarrow \infty]{} \mathbb{E} f^2(X).$$

Thus,

$$\begin{aligned} \mathbb{E} (|f(X) - f(X_n)|^2) &= \mathbb{E} f^2(X_n) - 2\mathbb{E} f(X_n)f(X) + \mathbb{E} f^2(X) \\ &\xrightarrow[n \rightarrow \infty]{} \mathbb{E} f^2(X) - 2\mathbb{E} f(X)f(X) + \mathbb{E} f^2(X) = 0, \end{aligned}$$

i.e. $f(X_n) \xrightarrow{L^2} f(X)$.

Now fix $\epsilon > 0$ and $R > 0$ and set $f(x) = -R \vee x \wedge R$. Clearly, $f \in \mathcal{C}_b$. Then

$$\begin{aligned} &\mathbb{P}(|X_n - X| > \epsilon) \\ &\leq \mathbb{P}(|X_n - X| > \epsilon, |X| \leq R, |X_n| \leq R) + \mathbb{P}(|X| \geq R) + \mathbb{P}(|X_n| \geq R) \\ &= \mathbb{P}(|f(X_n) - f(X)| > \epsilon, |X| \leq R, |X_n| \leq R) + \mathbb{P}(|X| \geq R) + \mathbb{P}(|f(X_n)| \geq R) \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geq R) + \mathbb{P}(|f(X_n)| \geq R) \\ &\leq \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geq R) + \mathbb{P}(|f(X)| \geq R/2) + \mathbb{P}(|f(X_n) - f(X)| \geq R/2) \end{aligned}$$

where we used that $\{|f(X_n)| \geq R\} \subset \{|f(X)| \geq R/2\} \cup \{|f(X_n) - f(X)| \geq R/2\}$ because of the triangle inequality: $|f(X_n)| \leq |f(X)| + |f(X) - f(X_n)|$

$$\begin{aligned} &= \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geq R/2) + \mathbb{P}(|X| \geq R/2) + \mathbb{P}(|f(X_n) - f(X)| \geq R/2) \\ &= \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + 2\mathbb{P}(|X| \geq R/2) + \mathbb{P}(|f(X_n) - f(X)| \geq R/2) \\ &\leq \left(\frac{1}{\epsilon^2} + \frac{4}{R^2}\right) \mathbb{E}(|f(X) - f(X_n)|^2) + 2\mathbb{P}(|X| \geq R/2) \\ &\xrightarrow[n \rightarrow \infty]{\epsilon, R \text{ fixed and } f=f_R \in \mathcal{C}_b} 2\mathbb{P}(|X| \geq R/2) \xrightarrow[R \rightarrow \infty]{X \text{ is a.s. } \mathbb{R}\text{-valued}} 0. \end{aligned}$$

■ ■

Problem 1.5. Solution: Note that $\mathbb{E} \delta_j = 0$ and $\mathbb{V} \delta_j = \mathbb{E} \delta_j^2 = 1$. Thus, $\mathbb{E} S_{[nt]} = 0$ and $\mathbb{V} S_{[nt]} = [nt]$.

a) We have, by the central limit theorem (CLT)

$$\frac{S_{[nt]}}{\sqrt{n}} = \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{S_{[nt]}}{\sqrt{[nt]}} \xrightarrow[n \rightarrow \infty]{\text{CLT}} \sqrt{t} G_1$$

where $G_1 \sim N(0, 1)$, hence $G_t := \sqrt{t} G_1 \sim N(0, t)$.

b) Let $s < t$. Since the δ_j are iid, we have, $S_{[nt]} - S_{[ns]} \sim S_{[nt]-[ns]}$, and by the central limit theorem (CLT)

$$\frac{S_{[nt]-[ns]}}{\sqrt{n}} = \frac{\sqrt{[nt]-[ns]}}{\sqrt{n}} \frac{S_{[nt]-[ns]}}{\sqrt{[nt]-[ns]}} \xrightarrow[n \rightarrow \infty]{\text{CLT}} \sqrt{t-s} G_1 \sim G_{t-s}.$$

If we know that the bivariate random variable $(S_{[ns]}, S_{[nt]} - S_{[ns]})$ converges in distribution, we do get $G_t \sim G_s + G_{t-s}$ because of Problem 1.1. But this follows again from the lemma which we prove in part d). This lemma shows that the limit has independent coordinates, see also part c). This is as close as we can come to $G_t - G_s \sim G_{t-s}$, unless we have a realization of ALL the G_t on a good space. It is Brownian motion which will achieve just this.

c) We know that the entries of the vector $(X_{t_m}^n - X_{t_{m-1}}^n, \dots, X_{t_2}^n - X_{t_1}^n, X_{t_1}^n)$ are independent (they depend on different blocks of the δ_j and the δ_j are iid) and, by the one-dimensional argument of b) we see that

$$X_{t_k}^n - X_{t_{k-1}}^n \xrightarrow[n \rightarrow \infty]{d} \sqrt{t_k - t_{k-1}} G_1^k \sim G_{t_k - t_{k-1}}^k \quad \text{for all } k = 1, \dots, m$$

where the G_1^k , $k = 1, \dots, m$ are standard normal random vectors.

By the lemma in part d) of the solution we even see that

$$(X_{t_m}^n - X_{t_{m-1}}^n, \dots, X_{t_2}^n - X_{t_1}^n, X_{t_1}^n) \xrightarrow[n \rightarrow \infty]{d} (\sqrt{t_1} G_1^1, \dots, \sqrt{t_m - t_{m-1}} G_1^m)$$

and the G_1^k , $k = 1, \dots, m$ are independent. Thus, by the second assertion of part b)

$$(\sqrt{t_1}G_1^1, \dots, \sqrt{t_m - t_{m-1}}G_1^m) \sim (G_{t_1}^1, \dots, G_{t_m - t_{m-1}}^m) \sim (G_{t_1}, \dots, G_{t_m} - G_{t_{m-1}}).$$

d) We have the following

Lemma. Let $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ be sequences of random variables (or random vectors) on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If

$$X_n \perp\!\!\!\perp Y_n \text{ for all } n \geq 1 \text{ and } X_n \xrightarrow[n \rightarrow \infty]{d} X \text{ and } Y_n \xrightarrow[n \rightarrow \infty]{d} Y,$$

then $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$ and $X \perp\!\!\!\perp Y$ (for suitable versions of the rv's).

Proof. Write $\phi_X, \phi_Y, \phi_{X,Y}$ for the characteristic functions of X, Y and the pair (X, Y) . By assumption

$$\lim_{n \rightarrow \infty} \phi_{X_n}(\xi) = \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} = \mathbb{E} e^{i\xi X} = \phi_X(\xi).$$

A similar statement is true for Y_n and Y . For the pair we get, because of independence

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_{X_n, Y_n}(\xi, \eta) &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n + i\eta Y_n} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \mathbb{E} e^{i\eta Y_n} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} e^{i\xi X_n} \lim_{n \rightarrow \infty} \mathbb{E} e^{i\eta Y_n} \\ &= \mathbb{E} e^{i\xi X} \mathbb{E} e^{i\eta Y} \\ &= \phi_X(\xi) \phi_Y(\eta). \end{aligned}$$

Thus, $\phi_{X_n, Y_n}(\xi, \eta) \rightarrow h(\xi, \eta) = \phi_X(\xi) \phi_Y(\eta)$. Since h is continuous at the origin $(\xi, \eta) = 0$ and $h(0, 0) = 1$, we conclude from Lévy's continuity theorem that h is a (bivariate) characteristic function and that $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Moreover,

$$h(\xi, \eta) = \phi_{X,Y}(\xi, \eta) = \phi_X(\xi) \phi_Y(\eta)$$

which shows that $X \perp\!\!\!\perp Y$. □

Problem 1.6. Solution: Necessity is clear. For sufficiency write

$$\frac{B(t) - B(s)}{\sqrt{t-s}} = \frac{1}{\sqrt{2}} \left(\frac{B(t) - B(\frac{s+t}{2})}{\sqrt{\frac{t-s}{2}}} + \frac{B(\frac{s+t}{2}) - B(s)}{\sqrt{\frac{t-s}{2}}} \right) =: \frac{1}{\sqrt{2}} (X + Y).$$

By assumption $X \sim Y$, $X \perp\!\!\!\perp Y$ and $X \sim \frac{1}{\sqrt{2}}(X + Y)$. This is already enough to guarantee that $X \sim N(0, 1)$, since $\mathbb{V} X = 1$, cf. Rényi [14, Chapter VI.5, Theorem 2, pp. 324–325].

Alternative Solution: Fix $s < t$ and define $t_j := s + \frac{j}{n}(t - s)$ for $j = 0, \dots, n$. Then

$$B_t - B_s = \sqrt{t_j - t_{j-1}} \sum_{j=1}^n \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}} = \sqrt{\frac{t-s}{n}} \sum_{j=1}^n \underbrace{\frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}}_{=: G_j^n}$$

By assumption, the random variables $(G_j^n)_{j,n}$ are identically distributed (for all j, n) and independent (in j). Moreover, $\mathbb{E}(G_j^n) = 0$ and $\mathbb{V}(G_j^n) = 1$. Applying the central limit theorem (for triangular arrays) we obtain

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n G_j^n \xrightarrow{d} G_1$$

where $G_1 \sim \mathbf{N}(0, 1)$. Thus, $B_t - B_s \sim \mathbf{N}(0, t - s)$. ■ ■

Problem 1.7. Solution: Let $\xi, \eta \in \mathbb{R}$. Then

$$\begin{aligned} \mathbb{E}\left(e^{i\xi\Gamma^-} e^{i\eta\Gamma^+}\right) &= \mathbb{E}\left(e^{i\frac{\xi}{\sqrt{2}}(G-G')} e^{i\frac{\eta}{\sqrt{2}}(G+G')}\right) \\ &= \mathbb{E}\left(e^{i\frac{\xi+\eta}{\sqrt{2}}G} e^{i\frac{\eta-\xi}{\sqrt{2}}G'}\right) \\ &\stackrel{G \perp G'}{=} \mathbb{E}\left(e^{i\frac{\xi+\eta}{\sqrt{2}}G}\right) \mathbb{E}\left(e^{i\frac{\eta-\xi}{\sqrt{2}}G'}\right) \\ &\stackrel{G, G' \sim \mathbf{N}(0,1)}{=} e^{-\frac{1}{2}\left[\frac{\xi+\eta}{\sqrt{2}}\right]^2} e^{-\frac{1}{2}\left[\frac{\eta-\xi}{\sqrt{2}}\right]^2} \\ &= e^{-\frac{1}{2}\xi^2} e^{-\frac{1}{2}\eta^2}. \end{aligned}$$

Taking $\eta = 0$ or $\xi = 0$ we find that $\Gamma^- \sim \mathbf{N}(0, 1)$ and $\Gamma^+ \sim \mathbf{N}(0, 1)$, respectively. Moreover, since ξ, η are arbitrary, we conclude

$$\mathbb{E}\left(e^{i\xi\Gamma^-} e^{i\eta\Gamma^+}\right) = \mathbb{E}\left(e^{i\xi\Gamma^-}\right) \mathbb{E}\left(e^{i\eta\Gamma^+}\right) \implies \Gamma^- \perp \Gamma^+.$$

In the last implication we used Kac's characterization of independence by characteristic functions. ■ ■

Problem 1.8. Solution:

- a) Since the conditions (B0)–(B3) involve only finitely many of the random variables X_t , it is clear from

$$\begin{aligned} &\mathbb{P}(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) \\ &= \mathbb{P}(\{X_{t_1} \in A_1\} \cap \dots \cap \{X_{t_n} \in A_n\}) \\ &= \mathbb{P}(\{X_{t_1} \in A_1\} \cap \dots \cap \{X_{t_n} \in A_n\} \cap \Omega_0) \quad (\text{use } \mathbb{P}(\Omega_0) = 1) \\ &= \mathbb{P}(\{B_{t_1} \in A_1\} \cap \dots \cap \{B_{t_n} \in A_n\} \cap \Omega_0) \\ &= \mathbb{P}(\{B_{t_1} \in A_1\} \cap \dots \cap \{B_{t_n} \in A_n\}) \quad (\text{use } \mathbb{P}(\Omega_0) = 1) \end{aligned}$$

that they remain valid for $(B_t)_{t \geq 0}$, too. Moreover, (B0) and (B4) are trivially satisfied for B_0 and $t \mapsto B_t$, respectively.

- b) We have to show that $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$ is a probability space. Since Ω_0 is measurable and $\mathbb{P}(\Omega_0) = 1$, this is a special case of Problem 9. The argument of the previous part then shows that $(X_t)_{t \geq 0}$ satisfies the axioms of a Brownian motion.

Remark: Method (b) also works if Ω_0 is not measurable, see Exercise 9.

Problem 1.9. Solution: The family \mathcal{A}_0 is the trace σ -algebra and it is not difficult to see that \mathbb{P}_0 is a measure, once we know that \mathbb{P}_0 is well-defined. In order to see this, assume that $\Omega_0 \cap A = \Omega_0 \cap B \neq \emptyset$ for $A, B \in \mathcal{A}$. This entails that

$$\Omega_0 \subset A^c \cup B = (A \cap B^c)^c = (A \setminus B)^c$$

i.e. we have because of the monotonicity of the outer measure

$$1 - \mathbb{P}(A \setminus B) = \mathbb{P}((A \setminus B)^c) = \mathbb{P}(A^c \cup B) = \mathbb{P}^*(A^c \cup B) \geq \mathbb{P}^*(\Omega_0) = 1;$$

this shows that $\mathbb{P}(A \setminus B) = 0$. Switching the roles of A and B also gives $\mathbb{P}(B \setminus A) = 0$, and so

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}((A \cap B) \cup (A \setminus B)) = \mathbb{P}((A \cap B)) + \underbrace{\mathbb{P}(A \setminus B)}_{=0} \\ &= \mathbb{P}((B \cap A)) + \underbrace{\mathbb{P}(B \setminus A)}_{=0} \\ &= \mathbb{P}((B \cap A) \cup (B \setminus A)) = \mathbb{P}(B). \end{aligned}$$

Thus, \mathbb{P}_0 is well-defined.

Problem 1.10. Solution: The argument of Problem 9 shows that \mathbb{P}' is well-defined, i.e. independent of the choice of A_2 . Indeed, if

$$A_1 \subset A' \subset A_2, \mathbb{P}(A_2 \setminus A_1) = 0 \quad \text{and} \quad B_1 \subset A' \subset B_2, \mathbb{P}(B_2 \setminus B_1) = 0.$$

Then we have $B_2 \supset A_1$ and so

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 \setminus B_2) + \mathbb{P}(A_2 \cap B_2) \leq \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_2 \cap B_2) \leq 0 + \mathbb{P}(B_2).$$

Switching the roles of the A_i and B_i gives the opposite inequality, so $\mathbb{P}(A_2) = \mathbb{P}(B_2)$, showing that \mathbb{P}' is well-defined.

All that remains is to show that \mathcal{A}' is a σ -algebra. Clearly, $\emptyset \in \mathcal{A}'$. If $A' \in \mathcal{A}'$ and if $A \subset A' \subset B$ satisfy $A, B \in \mathcal{A}$ and $\mathbb{P}(B \setminus A) = 0$, then we get

$$A^c, B^c \in \mathcal{A}, \quad B^c \subset (A')^c \subset A^c, \quad A \setminus B = A \cap B^c = B^c \cap (A^c)^c = B^c \setminus A^c$$

which shows that $(A')^c \in \mathcal{A}'$. Finally, let $A'_n \in \mathcal{A}'$ with $A_n \subset A'_n \subset B_n$ and $A_n, B_n \in \mathcal{A}$ and $\mathbb{P}(B_n \setminus A_n) = 0$. Then

$$\bigcup_n A_n \subset \bigcup_n A'_n \subset \bigcup_n B_n, \quad \bigcup_n A_n, \bigcup_n B_n \in \mathcal{A}$$

and

$$\left(\bigcup_n B_n \right) \setminus \left(\bigcup_n A_n \right) = \bigcup_n \left(B_n \setminus \bigcup_k A_k \right) \subset \bigcup_n (B_n \setminus A_n)$$

is a countable union of null sets, hence a null set. This shows that $\bigcup_n A'_n \in \mathcal{A}'$.

■ ■

2 Brownian motion as a Gaussian process

Problem 2.1. Solution: Let us check first that $f(u, v) := g(u)g(v)(1 - \sin u \sin v)$ is indeed a probability density. Clearly, $f(u, v) \geq 0$. Since $g(u) = (2\pi)^{-1/2} e^{-u^2/2}$ is even and $\sin u$ is odd, we get

$$\iint f(u, v) du dv = \int g(u) du \int g(v) dv - \int g(u) \sin u du \int g(v) \sin v dv = 1 - 0.$$

Moreover, the density $f_U(u)$ of U is

$$f_U(u) = \int f(u, v) dv = g(u) \int g(v) dv - g(u) \sin u \int g(v) \sin v dv = g(u).$$

This, and an analogous argument show that $U, V \sim \mathbf{N}(0, 1)$.

Let us show that (U, V) is not a normal random variable. Assume that (U, V) is normal, then $U + V \sim \mathbf{N}(0, \sigma^2)$, i.e.

$$\mathbb{E} e^{i\xi(U+V)} = e^{-\frac{1}{2}\xi^2\sigma^2}. \quad (*)$$

On the other hand we calculate with $f(u, v)$ that

$$\begin{aligned} \mathbb{E} e^{i\xi(U+V)} &= \iint e^{i\xi u + i\xi v} f(u, v) du dv \\ &= \left(\int e^{i\xi u} g(u) du \right)^2 - \left(\int e^{i\xi u} g(u) \sin u du \right)^2 \\ &= e^{-\xi^2} - \left(\frac{1}{2i} \int e^{i\xi u} (e^{iu} - e^{-iu}) g(u) du \right)^2 \\ &= e^{-\xi^2} - \left(\frac{1}{2i} \int (e^{i(\xi+1)u} - e^{i(\xi-1)u}) g(u) du \right)^2 \\ &= e^{-\xi^2} - \left(\frac{1}{2i} \left(e^{-\frac{1}{2}(\xi+1)^2} - e^{-\frac{1}{2}(\xi-1)^2} \right) \right)^2 \\ &= e^{-\xi^2} + \frac{1}{4} \left(e^{-\frac{1}{2}(\xi+1)^2} - e^{-\frac{1}{2}(\xi-1)^2} \right)^2 \\ &= e^{-\xi^2} + \frac{1}{4} e^{-1} e^{-\xi^2} (e^{-\xi} - e^{\xi})^2, \end{aligned}$$

and this contradicts (*).

Problem 2.2. Solution: We have because of (2.4)

$$\mathbb{E} e^{i\langle \xi, AG \rangle} = \mathbb{E} e^{i\langle A^\top \xi, G \rangle} = e^{i\langle A^\top \xi, m \rangle - \frac{1}{2} \langle A^\top \xi, CA^\top \xi \rangle} = e^{i\langle \xi, Am \rangle - \frac{1}{2} \langle \xi, ACA^\top \xi \rangle}.$$

Problem 2.3. Solution: We have

$$X \perp\!\!\!\perp \Gamma \implies \begin{pmatrix} X \\ \Gamma \end{pmatrix} \sim \text{normal} \xrightarrow{\text{Ex. 2}} \begin{pmatrix} X - \Gamma \\ X + \Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ \Gamma \end{pmatrix} \sim \text{normal}.$$

Thus, $X \pm \Gamma$ is normal as well as

$$\mathbb{E}(X \pm \Gamma) = \mathbb{E}X \pm \mathbb{E}\Gamma = 0 \quad \text{and} \quad \mathbb{V}(X \pm \Gamma) \stackrel{X \perp\!\!\!\perp \Gamma}{=} \mathbb{V}X + \mathbb{V}\Gamma = 2t.$$

Since we also have

$$\text{Cov}(X - \Gamma, X + \Gamma) = \mathbb{E}(X - \Gamma)(X + \Gamma) = \mathbb{E}(X^2 - \Gamma^2) = t - t = 0$$

we can use the fact that uncorrelated Gaussian vectors already have independent coordinates: $X - \Gamma \perp\!\!\!\perp X + \Gamma$.

Problem 2.4. Solution: We have for all $\xi \in \mathbb{R}$

$$G_n \xrightarrow{d} G \iff \mathbb{E}e^{i\xi G_n} \rightarrow \mathbb{E}e^{i\xi G} \iff e^{-\frac{1}{2}t_n \xi^2} \rightarrow \mathbb{E}e^{i\xi G} \iff t_n \rightarrow t.$$

In particular, $\mathbb{E}e^{i\xi G} = e^{-\frac{1}{2}t\xi^2}$.

Problem 2.5. Show that the covariance matrix $C = (t_j \wedge t_k)_{j,k=1,\dots,n}$ appearing in Theorem 2.6 is positive definite. **Solution:** Let $(\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$ and set $t_0 = 0$. Then we find from (2.12)

$$\sum_{j=1}^n \sum_{k=1}^n (t_j \wedge t_k) \xi_j \xi_k = \sum_{j=1}^n \underbrace{(t_j - t_{j-1})}_{>0} (\xi_j + \dots + \xi_n)^2 \geq 0. \quad (2.1)$$

Equality ($= 0$) occurs if, and only if, $(\xi_j + \dots + \xi_n)^2 = 0$ for all $j = 1, \dots, n$. This implies that $\xi_1 = \dots = \xi_n = 0$.

Abstract alternative: Let $(X_t)_{t \in I}$ be a real-valued stochastic process which has a second moment (such that the covariance is defined!), set $\mu_t = \mathbb{E}X_t$. For any finite set $S \subset I$ we pick $\lambda_s \in \mathbb{C}$, $s \in S$. Then

$$\begin{aligned} \sum_{s,t \in S} \text{Cov}(X_s, X_t) \lambda_s \bar{\lambda}_t &= \sum_{s,t \in S} \mathbb{E}((X_s - \mu_s)(X_t - \mu_t)) \lambda_s \bar{\lambda}_t \\ &= \mathbb{E} \left(\sum_{s,t \in S} (X_s - \mu_s) \lambda_s \overline{(X_t - \mu_t) \lambda_t} \right) \\ &= \mathbb{E} \left(\sum_{s \in S} (X_s - \mu_s) \lambda_s \overline{\sum_{t \in S} (X_t - \mu_t) \lambda_t} \right) \\ &= \mathbb{E} \left(\left| \sum_{s \in S} (X_s - \mu_s) \lambda_s \right|^2 \right) \geq 0. \end{aligned}$$

Remark: Note that this alternative does not prove that the covariance is *strictly* positive definite. A standard counterexample is to take $X_s \equiv X$.

Problem 2.6. Solution: These are direct & straightforward calculations. ■■

Problem 2.7. Solution: Let $e_i = \underbrace{(0, \dots, 0, 1, 0 \dots)}_i \in \mathbb{R}^n$ be the i^{th} standard unit vector. Then

$$a_{ii} = \langle Ae_i, e_i \rangle = \langle Be_i, e_i \rangle = b_{ii}.$$

Moreover, for $i \neq j$, we get by the symmetry of A and B

$$\langle A(e_i + e_j), e_i + e_j \rangle = a_{ii} + a_{jj} + 2b_{ij}$$

and

$$\langle B(e_i + e_j), e_i + e_j \rangle = b_{ii} + b_{jj} + 2b_{ij}$$

which shows that $a_{ij} = b_{ij}$. Thus, $A = B$.

We have

Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices. If $\langle Ax, x \rangle = \langle Bx, x \rangle$ for all $x \in \mathbb{R}^n$, then $A = B$. ■■

Problem 2.8. Solution:

a) $X_t = 2B_{t/4}$ is a BM^1 : scaling property with $c = 1/4$, cf. 2.16.

b) $Y_t = B_{2t} - B_t$ is not a BM^1 , the independent increments is clearly violated:

$$\begin{aligned} \mathbb{E}(Y_{2t} - Y_t)Y_t &= \mathbb{E}(Y_{2t}Y_t) - \mathbb{E}Y_t^2 \\ &= \mathbb{E}(B_{4t} - B_{2t})(B_{2t} - B_t) - \mathbb{E}(B_{2t} - B_t)^2 \\ &\stackrel{\text{(B1)}}{=} \mathbb{E}(B_{4t} - B_{2t}) \mathbb{E}(B_{2t} - B_t) - \mathbb{E}(B_{2t} - B_t)^2 \\ &\stackrel{\text{(B1)}}{=} -\mathbb{E}(B_t^2) = -t \neq 0. \end{aligned}$$

c) $Z_t = \sqrt{t}B_1$ is not a BM^1 , the independent increments property is violated:

$$\mathbb{E}(Z_t - Z_s)Z_s = (\sqrt{t} - \sqrt{s})\sqrt{s} \mathbb{E}B_1^2 = (\sqrt{t} - \sqrt{s})\sqrt{s} \neq 0. \quad \blacksquare$$

Problem 2.9. Solution: We use formula (2.10b).

a) $f_{B(s), B(t)}(x, y) = \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s}\right)\right].$

b)

$$\begin{aligned} f_{B(s), B(t)|B(1)}(x, y|B(1) = z) \\ = \frac{f_{B(s), B(t), B(1)}(x, y, z)}{f_{B(1)}(z)} \end{aligned}$$

$$= \frac{1}{(2\pi)^{3/2} \sqrt{s(t-s)(1-t)}} \exp \left[-\frac{1}{2} \left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{(z-y)^2}{1-t} \right) \right] (2\pi)^{1/2} \exp \left[\frac{z^2}{2} \right].$$

Thus,

$$f_{B(s),B(t)|B(1)}(x, y | B(1) = 0) = \frac{1}{2\pi \sqrt{s(t-s)(1-t)}} \exp \left[-\frac{1}{2} \left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{y^2}{1-t} \right) \right].$$

Note that

$$\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{y^2}{1-t} = \frac{t}{s(t-s)} \left(x - \frac{s}{t} y \right)^2 + \frac{y^2}{t} + \frac{y^2}{1-t} = \frac{t}{s(t-s)} \left(x - \frac{s}{t} y \right)^2 + \frac{y^2}{t(1-t)}.$$

Therefore,

$$\begin{aligned} \mathbb{E}(B(s)B(t) | B(1) = 0) &= \iint xy f_{B(s),B(t)|B(1)}(x, y | B(1) = 0) dx dy \\ &= \frac{1}{2\pi \sqrt{s(t-s)(1-t)}} \int_{y=-\infty}^{\infty} y \exp \left[-\frac{1}{2} \frac{y^2}{t(1-t)} \right] \times \\ &\quad \times \underbrace{\int_{x=-\infty}^{\infty} x \exp \left[-\frac{1}{2} \frac{t}{s(t-s)} \left(x - \frac{s}{t} y \right)^2 \right] dx}_{= \frac{\sqrt{s(t-s)}}{\sqrt{t}} \sqrt{2\pi} \frac{s}{t} y} dy \\ &= \frac{1}{\sqrt{2\pi} \sqrt{t(1-t)}} \int_{y=-\infty}^{\infty} y^2 \frac{s}{t} \exp \left[-\frac{1}{2} \frac{y^2}{t(1-t)} \right] dy \\ &= \frac{s}{t} t(1-t) = s(1-t). \end{aligned}$$

c) In analogy to part b) we get

$$\begin{aligned} &f_{B(t_2),B(t_3)|B(t_1),B(t_4)}(x, y | B(t_1) = u, B(t_4) = z) \\ &= \frac{f_{B(t_1),B(t_2),B(t_3),B(t_4)}(u, x, y, z)}{f_{B(t_1),B(t_4)}(u, z)} \\ &= \frac{1}{2\pi} \left[\frac{t_1(t_4 - t_1)}{t_1(t_2 - t_1)(t_3 - t_2)(t_4 - t_3)} \right]^{\frac{1}{2}} \exp \left[-\frac{1}{2} \left(\frac{u^2}{t_1} + \frac{(x-u)^2}{t_2 - t_1} + \frac{(y-x)^2}{t_3 - t_2} + \frac{(z-y)^2}{t_4 - t_3} \right) \right] \times \\ &\quad \times \exp \left[\frac{1}{2} \left(\frac{u^2}{t_1} + \frac{(z-u)^2}{t_4 - t_1} \right) \right]. \end{aligned}$$

Thus,

$$\begin{aligned} &f_{B(t_2),B(t_3)|B(t_1),B(t_4)}(x, y | B(t_1) = B(t_4) = 0) \\ &= \frac{1}{2\pi} \left[\frac{t_1(t_4 - t_1)}{t_1(t_2 - t_1)(t_3 - t_2)(t_4 - t_3)} \right]^{\frac{1}{2}} \exp \left[-\frac{1}{2} \left(\frac{x^2}{t_2 - t_1} + \frac{(y-x)^2}{t_3 - t_2} + \frac{y^2}{t_4 - t_3} \right) \right]. \end{aligned}$$

Observe that

$$\frac{x^2}{t_2 - t_1} + \frac{(y-x)^2}{t_3 - t_2} + \frac{y^2}{t_4 - t_3} = \frac{t_3 - t_1}{(t_2 - t_1)(t_3 - t_2)} \left(x - \frac{t_2 - t_1}{t_3 - t_1} y \right)^2 + \frac{t_4 - t_1}{(t_3 - t_1)(t_4 - t_3)} y^2.$$

Therefore, we get (using physicists' notation: $\int dy h(y) := \int h(y) dy$ for easier readability)

$$\begin{aligned}
 & \iint xy f_{B(t_2), B(t_3) | B(t_1), B(t_4)}(x, y | B(t_1) = B(t_4) = 0) dx dy \\
 &= \frac{1}{2\pi(t_4 - t_3)} \int_{y=-\infty}^{\infty} dy \exp\left[-\frac{1}{2} \frac{t_4 - t_1}{(t_3 - t_1)(t_4 - t_3)} y^2\right] \times \\
 & \quad \times \underbrace{\frac{y}{\sqrt{2\pi(t_2 - t_1)(t_3 - t_2)}} \int_{x=-\infty}^{\infty} x \exp\left[-\frac{1}{2} \left(x - \frac{t_2 - t_1}{t_3 - t_1} y\right)^2 \frac{t_3 - t_1}{(t_2 - t_1)(t_3 - t_2)}\right] dx}_{= \frac{y^2}{\sqrt{t_3 - t_1}} \frac{t_2 - t_1}{t_3 - t_1}} \\
 &= \frac{t_2 - t_1}{t_3 - t_1} \frac{(t_4 - t_3)(t_3 - t_1)}{t_4 - t_1} = \frac{(t_2 - t_1)(t_4 - t_3)}{t_4 - t_1}.
 \end{aligned}$$

Problem 2.10. Solution: Let $s \leq t$. Then

$$\begin{aligned}
 C(s, t) &= \mathbb{E}(X_s X_t) \\
 &= \mathbb{E}(B_s^2 - s)(B_t^2 - t) \\
 &= \mathbb{E}(B_s^2 - s)([B_t - B_s + B_s]^2 - t) \\
 &= \mathbb{E}(B_s^2 - s)(B_t - B_s)^2 + 2\mathbb{E}(B_s^2 - s)B_s(B_t - B_s) + \mathbb{E}(B_s^2 - s)B_s^2 - \mathbb{E}(B_s^2 - s)t \\
 &\stackrel{(B1)}{=} \mathbb{E}(B_s^2 - s)\mathbb{E}(B_t - B_s)^2 + 2\mathbb{E}(B_s^2 - s)B_s\mathbb{E}(B_t - B_s) + \mathbb{E}(B_s^2 - s)B_s^2 - \mathbb{E}(B_s^2 - s)t \\
 &= 0 \cdot (t - s) + 2\mathbb{E}(B_s^2 - s)B_s \cdot 0 + \mathbb{E}B_s^4 - s\mathbb{E}B_s^2 - 0 \\
 &= 2s^2 = 2(s^2 \wedge t^2) = 2(s \wedge t)^2.
 \end{aligned}$$

Problem 2.11. Solution:

a) We have for $s, t \geq 0$

$$\begin{aligned}
 m(t) &= \mathbb{E} X_t = e^{-\alpha t/2} \mathbb{E} B_{e^{\alpha t}} = 0. \\
 C(s, t) &= \mathbb{E}(X_s X_t) = e^{-\frac{\alpha}{2}(s+t)} \mathbb{E} B_{e^{\alpha s}} B_{e^{\alpha t}} = e^{-\frac{\alpha}{2}(s+t)} (e^{\alpha s} \wedge e^{\alpha t}) = e^{-\frac{\alpha}{2}|t-s|}.
 \end{aligned}$$

b) We have

$$\mathbb{P}(X(t_1) \leq x_1, \dots, X(t_n) \leq x_n) = \mathbb{P}(B(e^{\alpha t_1}) \leq e^{\alpha t_1/2} x_1, \dots, B(e^{\alpha t_n}) \leq e^{\alpha t_n/2} x_n)$$

Thus, the density is

$$\begin{aligned}
 & f_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n) \\
 &= \prod_{k=1}^n e^{\alpha t_k/2} f_{B(e^{\alpha t_1}), \dots, B(e^{\alpha t_n})}(e^{\alpha t_1/2} x_1, \dots, e^{\alpha t_n/2} x_n) \\
 &= \prod_{k=1}^n e^{\alpha t_k/2} (2\pi)^{-n/2} \left(\prod_{k=1}^n (e^{\alpha t_k} - e^{\alpha t_{k-1}}) \right)^{-1/2} e^{-\frac{1}{2} \sum_{k=1}^n (e^{\alpha t_k/2} x_k - e^{\alpha t_{k-1}/2} x_{k-1})^2 / (e^{\alpha t_k} - e^{\alpha t_{k-1}})}
 \end{aligned}$$

$$= (2\pi)^{-n/2} \left(\prod_{k=1}^n (1 - e^{-\alpha(t_k - t_{k-1})}) \right)^{-1/2} e^{-\frac{1}{2} \sum_{k=1}^n (x_k - e^{-\alpha(t_k - t_{k-1})/2} x_{k-1})^2 / (1 - e^{-\alpha(t_k - t_{k-1})})}$$

(we use the convention $t_0 = -\infty$ and $x_0 = 0$).

Remark: the form of the density shows that the Ornstein–Uhlenbeck is strictly stationary, i.e.

$$(X(t_1 + h), \dots, X(t_n + h)) \sim (X(t_1), \dots, X(t_n)) \quad \forall h > 0.$$

Problem 2.12. Solution: Set

$$\Sigma := \bigcup_{J \subset [0, \infty), J \text{ countable}} \sigma(B(t) : t \in J)$$

Clearly,

$$\bigcup_{t \geq 0} \sigma(B_t) \subset \Sigma \subset \sigma(B_t : t \geq 0) \stackrel{\text{def}}{=} \mathcal{F}_\infty^B \quad (*)$$

The first inclusion follows from the fact that each B_t is measurable with respect to Σ .

Let us show that Σ is a σ -algebra. Obviously,

$$\emptyset \in \Sigma \quad \text{and} \quad F \in \Sigma \implies F^c \in \Sigma.$$

Let $(A_n)_n \subset \Sigma$. Then, for every n there is a countable set J_n such that $A_n \in \sigma(B(t) : t \in J_n)$. Since $J = \bigcup_n J_n$ is still countable we see that $A_n \in \sigma(B(t) : t \in J)$ for all n . Since the latter family is a σ -algebra, we find

$$\bigcup_n A_n \in \sigma(B(t) : t \in J) \subset \Sigma.$$

Since $\bigcup_t \sigma(B_t) \subset \Sigma$, we get—note: \mathcal{F}_∞^B is, by definition, the smallest σ -algebra for which all B_t are measurable—that

$$\mathcal{F}_\infty^B \subset \Sigma.$$

This shows that $\Sigma = \mathcal{F}_\infty^B$.

Problem 2.13. Solution: Assume that the indices t_1, \dots, t_m and s_1, \dots, s_n are given. Let $\{u_1, \dots, u_p\} := \{s_1, \dots, s_n\} \cup \{t_1, \dots, t_m\}$. By assumption,

$$(X(u_1), \dots, X(u_p)) \perp\!\!\!\perp (Y(u_1), \dots, Y(u_p)).$$

Thus, we may thin out the indices on each side without endangering independence: $\{s_1, \dots, s_n\} \subset \{u_1, \dots, u_p\}$ and $\{t_1, \dots, t_m\} \subset \{u_1, \dots, u_p\}$, and so

$$(X(s_1), \dots, X(s_n)) \perp\!\!\!\perp (Y(t_1), \dots, Y(t_m)).$$

Problem 2.14. Solution: Since $\mathcal{F}_t \subset \mathcal{F}_\infty$ and $\mathcal{G}_t \subset \mathcal{G}_\infty$ it is clear that

$$\mathcal{F}_\infty \perp \mathcal{G}_\infty \implies \mathcal{F}_t \perp \mathcal{G}_t.$$

Conversely, since $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{G}_t)_{t \geq 0}$ are filtrations we find

$$\forall F \in \bigcup_{t \geq 0} \mathcal{F}_t, \quad \forall G \in \bigcup_{t \geq 0} \mathcal{G}_t, \quad \exists t_0 : F \in \mathcal{F}_{t_0}, G \in \mathcal{G}_{t_0}.$$

By assumption: $\mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G)$. Thus,

$$\bigcup_{t \geq 0} \mathcal{F}_t \perp \bigcup_{t \geq 0} \mathcal{G}_t.$$

Since the families $\bigcup_{t \geq 0} \mathcal{F}_t$ and $\bigcup_{t \geq 0} \mathcal{G}_t$ are \cap -stable (use again the argument that we have filtrations to find for $F, F' \in \bigcup_{t \geq 0} \mathcal{F}_t$ some t_0 with $F, F' \in \mathcal{F}_{t_0}$ etc.), the σ -algebras generated by these families are independent:

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right) \perp \sigma\left(\bigcup_{t \geq 0} \mathcal{G}_t\right) = \mathcal{G}_\infty.$$

■

Problem 2.15. Solution: Let $U \in \mathbb{R}^{d \times d}$ be an orthogonal matrix: $UU^\top = \text{id}$ and set $X_t := UB_t$ for a BM^d $(B_t)_{t \geq 0}$. Then

$$\begin{aligned} \mathbb{E}\left(\exp\left[i \sum_{j=1}^n \langle \xi_j, X(t_j) - X(t_{j-1}) \rangle\right]\right) &= \mathbb{E}\left(\exp\left[i \sum_{j=1}^n \langle \xi_j, UB(t_j) - UB(t_{j-1}) \rangle\right]\right) \\ &= \mathbb{E}\left(\exp\left[i \sum_{j=1}^n \langle U^\top \xi_j, B(t_j) - B(t_{j-1}) \rangle\right]\right) \\ &= \exp\left[-\frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1}) \langle U^\top \xi_j, U^\top \xi_j \rangle\right] \\ &= \exp\left[-\frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1}) |\xi_j|^2\right]. \end{aligned}$$

(Observe $\langle U^\top \xi_j, U^\top \xi_j \rangle = \langle UU^\top \xi_j, \xi_j \rangle = \langle \xi_j, \xi_j \rangle = |\xi_j|^2$). The claim follows.

■

Problem 2.16. Solution: Note that the coordinate processes b and β are independent BM¹.

a) Since $b \perp \beta$, the process $W_t = (b_t + \beta_t)/\sqrt{2}$ is a Gaussian process with continuous sample paths. We determine its mean and covariance functions:

$$\begin{aligned} \mathbb{E}W_t &= \frac{1}{\sqrt{2}}(\mathbb{E}b_t + \mathbb{E}\beta_t) = 0; \\ \text{Cov}(W_s, W_t) &= \mathbb{E}(W_s W_t) \\ &= \frac{1}{2} \mathbb{E}(b_s + \beta_s)(b_t + \beta_t) \\ &= \frac{1}{2} \left(\mathbb{E}b_s b_t + \mathbb{E}\beta_s \beta_t + \mathbb{E}b_s \beta_t + \mathbb{E}\beta_s b_t \right) \end{aligned}$$

$$= \frac{1}{2}(s \wedge t + 0 + 0 + s \wedge t) = s \wedge t$$

where we used that, by independence, $\mathbb{E} b_u \beta_v = \mathbb{E} b_u \mathbb{E} \beta_v = 0$. Now the claim follows from Corollary 2.7.

b) The process $X_t = (W_t, \beta_t)$ has the following properties

- W and β are BM^1
- $\mathbb{E}(W_t \beta_t) = 2^{-1/2} \mathbb{E}(b_t + \beta_t) \beta_t = 2^{-1/2} (\mathbb{E} b_t \mathbb{E} \beta_t + \mathbb{E} \beta_t^2) = t/\sqrt{2} \neq 0$, i.e. W and β are NOT independent.

This means that X is not a BM^2 , as its coordinates are not independent.

The process Y_t can be written as

$$\frac{1}{\sqrt{2}} \begin{pmatrix} b_t + \beta_t \\ b_t - \beta_t \end{pmatrix} = U \begin{pmatrix} b_t \\ \beta_t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_t \\ \beta_t \end{pmatrix}.$$

Clearly, $UU^\top = \text{id}$, i.e. Problem 2.15 shows that $(Y_t)_{t \geq 0}$ is a BM^2 .

Problem 2.17. Solution: Observe that $b \perp \beta$ since B is a BM^2 . Since

$$\begin{aligned} \mathbb{E} X_t &= 0 \\ \text{Cov}(X_t, X_s) &= \mathbb{E} X_t X_s \\ &= \mathbb{E}(\lambda b_s + \mu \beta_s)(\lambda b_t + \mu \beta_t) \\ &= \lambda^2 \mathbb{E} b_s b_t + \lambda \mu \mathbb{E} b_s \beta_t + \lambda \mu \mathbb{E} b_t \beta_s + \mu^2 \mathbb{E} \beta_s \beta_t \\ &= \lambda^2 \mathbb{E} b_s b_t + \lambda \mu \mathbb{E} b_s \mathbb{E} \beta_t + \lambda \mu \mathbb{E} b_t \mathbb{E} \beta_s + \mu^2 \mathbb{E} \beta_s \beta_t \\ &= \lambda^2 (s \wedge t) + 0 + 0 + \mu^2 s \wedge t = (\lambda^2 + \mu^2)(s \wedge t). \end{aligned}$$

Thus, by Corollary 2.7, X is a BM^1 if, and only if, $\lambda^2 + \mu^2 = 1$.

Problem 2.18. Solution: $X_t = (b_t, \beta_{s-t} - \beta_t)$, $0 \leq t \leq s$, is NOT a Brownian motion: $X_0 = (0, \beta_s) \neq (0, 0)$.

On the other hand, $Y_t = (b_t, \beta_{s-t} - \beta_s)$, $0 \leq t \leq s$, IS a Brownian motion, since b_t and $\beta_{s-t} - \beta_s$ are independent BM^1 , cf. Time inversion 2.15 and Theorem 2.10.

Problem 2.19. Solution: We have

$$W_t = U B_t^\top = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} b_t \\ \beta_t \end{pmatrix}.$$

The matrix U is a rotation, hence orthogonal and we see from Problem 2.15 that W is a Brownian motion.

Generalization: take U orthogonal. ■■

Problem 2.20. Solution: If $G \sim N(0, Q)$ then Q is the covariance matrix, i.e. $\text{Cov}(G^j, G^k) = q_{jk}$. Thus, we get for $s < t$

$$\begin{aligned} \text{Cov}(X_s^j, X_t^k) &= \mathbb{E}(X_s^j X_t^k) \\ &= \mathbb{E} X_s^j (X_t^k - X_s^k) + \mathbb{E}(X_s^j X_s^k) \\ &= \mathbb{E} X_s^j \mathbb{E}(X_t^k - X_s^k) + s q_{jk} \\ &= (s \wedge t) q_{jk}. \end{aligned}$$

The characteristic function is

$$\mathbb{E} e^{i\langle \xi, X_t \rangle} = \mathbb{E} e^{i\langle \Sigma^\top \xi, B_t \rangle} = e^{-\frac{t}{2} |\Sigma^\top \xi|^2} = e^{-\frac{t}{2} \langle \xi, \Sigma \Sigma^\top \xi \rangle},$$

and the transition probability is, if Q is non-degenerate,

$$f_Q(x) = \frac{1}{\sqrt{(2\pi t)^n \det Q}} \exp\left(-\frac{1}{2t} \langle x, Qx \rangle\right).$$

If Q is degenerate, there is an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that

$$UX_t = (Y_t^1, \dots, Y_t^k, \underbrace{0, \dots, 0}_{n-k})^\top$$

where $k < n$ is the rank of Q . The k -dimensional vector has a nondegenerate normal distribution in \mathbb{R}^k . ■■

Problem 2.21. Solution:

“ \Rightarrow ” Assume that we have (B1). Observe that the family of sets

$$\bigcup_{0 \leq u_1 < \dots < u_n \leq s, n \geq 1} \sigma(B_{u_1}, \dots, B_{u_n})$$

is a \cap -stable family. This means that it is enough to show that

$$B_t - B_s \perp (B_{u_1}, \dots, B_{u_n}) \quad \text{for all } t \geq s \geq 0.$$

By (B1) we know that

$$B_t - B_s \perp (B_{u_1}, B_{u_2} - B_{u_1}, \dots, B_{u_n} - B_{u_{n-1}})$$

and so

$$B_t - B_s \perp \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} B_{u_1} \\ B_{u_2} - B_{u_1} \\ B_{u_3} - B_{u_2} \\ \vdots \\ B_{u_n} - B_{u_{n-1}} \end{pmatrix} = \begin{pmatrix} B_{u_1} \\ B_{u_2} \\ B_{u_3} \\ \vdots \\ B_{u_n} \end{pmatrix}$$

“ \Leftarrow ” Let $0 = t_0 \leq t_1 < t_2 < \dots < t_n < \infty$, $n \geq 1$. Then we find for all $\xi_1, \dots, \xi_n \in \mathbb{R}^d$

$$\begin{aligned} \mathbb{E} \left(e^{i \sum_{k=1}^n \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle} \right) &= \mathbb{E} \left(e^{i \langle \xi_n, B(t_n) - B(t_{n-1}) \rangle} \cdot \underbrace{e^{i \sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle}}_{\mathcal{F}_{t_{n-1}} \text{ mble., hence } \perp B(t_n) - B(t_{n-1})} \right) \\ &= \mathbb{E} \left(e^{i \langle \xi_n, B(t_n) - B(t_{n-1}) \rangle} \right) \cdot \mathbb{E} \left(e^{i \sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle} \right) \\ &\vdots \\ &= \prod_{k=1}^n \mathbb{E} \left(e^{i \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle} \right). \end{aligned}$$

This shows (B1). ■ ■

Problem 2.22. Solution: Reflection invariance of BM, cf. 2.12, shows

$$\tau_a = \inf\{s \geq 0 : B_s = a\} \sim \inf\{s \geq 0 : -B_s = a\} = \inf\{s \geq 0 : B_s = -a\} = \tau_{-a}.$$

The scaling property 2.16 of BM shows for $c = 1/a^2$

$$\begin{aligned} \tau_a &= \inf\{s \geq 0 : B_s = a\} \sim \inf\{s \geq 0 : aB_{s/a^2} = a\} \\ &= \inf\{a^2 r \geq 0 : aB_r = a\} \\ &= a^2 \inf\{r \geq 0 : B_r = 1\} = a^2 \tau_1. \end{aligned}$$

Problem 2.23. Solution:

a) **Not stationary:**

$$\mathbb{E} W_t^2 = C(t, t) = \mathbb{E}(B_t^2 - t)^2 = \mathbb{E}(B_t^4 - 2tB_t^2 + t^2) = 3t^2 - 2t^2 + t^2 = 2t^2 \neq \text{const.}$$

b) **Stationary.** We have $\mathbb{E} X_t = 0$ and

$$\mathbb{E} X_s X_t = e^{-\alpha(t+s)/2} \mathbb{E} B_{e^{\alpha s}} B_{e^{\alpha t}} = e^{-\alpha(t+s)/2} (e^{\alpha s} \wedge e^{\alpha t}) = e^{-\alpha|t-s|/2},$$

i.e. it is stationary with $g(r) = e^{-\alpha|r|/2}$.

c) **Stationary.** We have $\mathbb{E} Y_t = 0$. Let $s \leq t$. Then we use $\mathbb{E} B_s B_t = s \wedge t$ to get

$$\begin{aligned} \mathbb{E} Y_s Y_t &= \mathbb{E}(B_{s+h} - B_s)(B_{t+h} - B_t) \\ &= \mathbb{E} B_{s+h} B_{t+h} - \mathbb{E} B_{s+h} B_t - \mathbb{E} B_s B_{t+h} + \mathbb{E} B_s B_t \\ &= (s+h) \wedge (t+h) - (s+h) \wedge t - s \wedge (t+h) + s \wedge t \\ &= (s+h) - (s+h) \wedge t = \begin{cases} 0, & \text{if } t > s+h \iff h < t-s \\ h - (t-s), & \text{if } t \leq s+h \iff h \geq t-s. \end{cases} \end{aligned}$$

Swapping the roles of s and t finally gives: the process is stationary with $g(t) = (h - |t|)^+ = (h - |t|) \vee 0$.

d) **Not stationary.** Note that

$$\mathbb{E} Z_t^2 = \mathbb{E} B_{e^t}^2 = e^t \neq \text{const.}$$

■ ■

Problem 2.24. Solution: Clearly, $t \mapsto W_t$ is continuous for $t \neq 1$. If $t = 1$ we get

$$\lim_{t \downarrow 1} W_t(\omega) = W_1(\omega) = B_1(\omega)$$

and

$$\lim_{t \downarrow 1} W_t(\omega) = B_1(\omega) - \lim_{t \downarrow 1} t\beta_{1/t}(\omega) - \beta_1(\omega) = B_1(\omega);$$

this proves continuity for $t = 1$.

Let us check that W is a Gaussian process with $\mathbb{E} W_t = 0$ and $\mathbb{E} W_s W_t = s \wedge t$. By Corollary 2.7, W is a BM¹.

Pick $n \geq 1$ and $t_0 = 0 < t_1 < \dots < t_n$.

Case 1: If $t_n \leq 1$, there is nothing to show since $(B_t)_{t \in [0,1]}$ is a BM¹.

Case 2: Assume that $t_n > 1$. Then we have

$$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & t_1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & t_2 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & 0 & t_3 & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & t_n & -1 \end{pmatrix} \begin{pmatrix} B_1 \\ \beta_{1/t_1} \\ \vdots \\ \beta_{1/t_n} \\ \beta_1 \end{pmatrix}$$

and since

$$B_1 \perp (\beta_{1/t_1}, \dots, \beta_{1/t_n}, \beta_1)^\top$$

are both Gaussian, we see that $(W_{t_1}, \dots, W_{t_n})$ is Gaussian.

Further, let $t \geq 1$ and $1 \leq t_i < t_j$:

$$\begin{aligned} \mathbb{E} W_t &= \mathbb{E} B_1 + t \mathbb{E} \beta_{1/t} - \mathbb{E} \beta_1 = 0 \\ \mathbb{E} W_{t_i} W_{t_j} &= \mathbb{E} (B_1 + t_i \beta_{1/t_i} - \beta_1) (B_1 + t_j \beta_{1/t_j} - \beta_1) \\ &= 1 + t_i t_j t_j^{-1} - t_i t_i^{-1} - t_j t_j^{-1} + 1 = t_i = t_i \wedge t_j. \end{aligned}$$

Case 3: Assume that $0 < t_1 < \dots < t_k \leq 1 < t_{k+1} < \dots < t_n$. Then we have

$$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_k} \\ \vdots \\ W_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ & & & 1 & t_{k+1} & 0 & \cdots & 0 & -1 \\ & & & 1 & 0 & t_{k+2} & & 0 & -1 \\ & & & \vdots & \vdots & & \ddots & \vdots & \\ & & & 1 & 0 & \cdots & & t_n & -1 \end{pmatrix} \begin{pmatrix} B_{t_1} \\ \vdots \\ \vdots \\ B_{t_k} \\ B_1 \\ \beta_{1/t_{k+1}} \\ \vdots \\ \beta_{1/t_n} \\ \beta_1 \end{pmatrix}.$$

Since

$$(B_{t_1}, \dots, B_{t_k}, B_1) \perp (\beta_{1/t_{k+1}}, \dots, \beta_{1/t_n}, \beta_1)$$

are Gaussian vectors, $(W_{t_1}, \dots, W_{t_n})$ is also Gaussian and we find

$$\mathbb{E} W_t = 0$$

$$\mathbb{E} W_{t_i} W_{t_j} = \mathbb{E} B_{t_i} (B_1 + t_j \beta_{1/t_j} - \beta_1) = t_i = t_i \wedge t_j$$

for $i \leq k < j$.

■ ■

Problem 2.25. Solution: The process $X(t) = B(e^t)$ has no memory since (cf. Problem 2.21)

$$\sigma(B(s) : s \leq e^a) \perp \sigma(B(s) - B(e^a) : s \geq e^a)$$

and, therefore,

$$\begin{aligned} \sigma(X(t) : t \leq a) &= \sigma(B(s) : 1 \leq s \leq e^a) \perp \sigma(B(e^{a+s}) - B(e^a) : s \geq 0) \\ &= \sigma(X(t+a) - X(a) : t \geq 0). \end{aligned}$$

The process $X(t) := e^{-t/2} B(e^t)$ is not memoryless. For example, $X(a+a) - X(a)$ is not independent of $X(a)$:

$$\mathbb{E}(X(2a) - X(a))X(a) = \mathbb{E}(e^{-a} B(e^{2a}) - e^{-a/2} B(e^a))e^{-a/2} B(e^a) = e^{-3a/2} e^a - e^{-a} e^a \neq 0.$$

■ ■

Problem 2.26. Solution: The process $W_t = B_{a-t} - B_a$, $0 \leq t \leq a$ clearly satisfies (B0) and (B4).

For $0 \leq s \leq t \leq a$ we find

$$W_t - W_s = B_{a-t} - B_{a-s} \sim B_{a-s} - B_{a-t} \sim B_{t-s} \sim \mathbf{N}(0, (t-s) \text{id})$$

and this shows (B2) and (B3).

For $0 = t_0 < t_1 < \dots < t_n \leq a$ we have

$$W_{t_j} - W_{t_{j-1}} = B_{a-t_j} - B_{a-t_{j-1}} \sim B_{a-t_{j-1}} - B_{a-t_j} \quad \forall j$$

and this proves that W inherits (B1) from B .

■ ■

Problem 2.27. Solution: We know from Paragraph 2.17 that

$$\lim_{t \downarrow 0} tB(1/t) = 0 \implies \lim_{s \uparrow \infty} \frac{B(s)}{s} = 0 \quad \text{a.s.}$$

Moreover,

$$\mathbb{E} \left(\frac{B(s)}{s} \right)^2 = \frac{s}{s^2} = \frac{1}{s} \xrightarrow{s \rightarrow \infty} 0$$

i.e. we get also convergence in mean square.

Remark: a direct proof of the SLLN is a bit more tricky. Of course we have by the classical SLLN that

$$\frac{B_n}{n} = \frac{\sum_{j=1}^n (B_j - B_{j-1})}{n} \xrightarrow[n \rightarrow \infty]{\text{SLLN}} 0 \quad \text{a.s.}$$

But then we have to make sure that B_s/s converges. This can be done in the following way: fix $s > 0$. Then there is a unique interval $(n, n+1]$ such that $s \in (n, n+1]$. Thus,

$$\left| \frac{B_s}{s} \right| \leq \left| \frac{B_s - B_{n+1}}{s} \right| + \left| \frac{B_{n+1}}{n+1} \right| \cdot \frac{n+1}{s} \leq \frac{\sup_{n \leq s \leq n+1} |B_s - B_{n+1}|}{n} + \frac{n+1}{n} \left| \frac{B_n}{n} \right|$$

and we have to show that the expression with the sup tends to zero. This can be done by showing, e.g., that the L^2 -limit of this expression goes to zero (using the reflection principle) and with a subsequence argument. ■ ■

Problem 2.28. Solution: Throughout this solution we suppress the superscript α .

a) Assume that the first condition holds. Then

$$\begin{aligned} \mathbb{E}(B_s - B_t)^2 &= \mathbb{E}(B_t B_t) + \mathbb{E}(B_s B_s) - 2\mathbb{E}(B_t B_s) \\ &= |t|^\alpha + |s|^\alpha - |s|^\alpha - |t|^\alpha + |s - t|^\alpha. \end{aligned}$$

Moreover $\mathbb{E}(B_0^2) = \mathbb{E}(B_0 B_0) = 0$, so $B_0 = 0$.

Conversely, assume that the second set of conditions holds. Using $B_0 = 0$ we have

$$\begin{aligned} 2\mathbb{E}(B_s B_t) &= \mathbb{E}(B_s B_s) + \mathbb{E}(B_t B_t) - \mathbb{E}[(B_s - B_t)(B_s - B_t)] \\ &= \mathbb{E}[(B_s - B_0)(B_s - B_0)] + \mathbb{E}[(B_t - B_0)(B_t - B_0)] - \mathbb{E}[(B_s - B_t)(B_s - B_t)] \\ &= |s - 0|^\alpha + |t - 0|^\alpha - |s - t|^\alpha \end{aligned}$$

and the first condition follows.

b) Since $(B_{t_1}, \dots, B_{t_n})$ has to be a Gaussian vector, it is enough to specify its mean and its covariance function. We have

$$m = 0 \in \mathbb{R}^n \quad \text{and} \quad C = \frac{1}{2} (|t_i|^\alpha + |t_k|^\alpha - |t_i - t_k|^\alpha)_{i,k=1,\dots,n}.$$

In order to get the density, we have to invert C and then the density is given by the formula (2.10a).

In order to get the existence of a fBM we need to appeal to Kolmogorov's existence theorem, see Theorem 4.8. In order to apply it, we have to show a compatibility condition for the vectors, namely that the law of $(B_{t_1}, \dots, B_{t_n})$ can be obtained by projection from the law of $(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})$. Note that t_1, \dots, t_n, t_{n+1} are arbitrary and do not have any ordering.

To do so, denote by C_n and C_{n+1} the respective covariance matrices and write $(\xi, \xi_{n+1}) = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$. Then we have to show that

$$\langle (\xi, 0)^\top, C_{n+1}(\xi, 0)^\top \rangle = \langle \xi^\top, C_n \xi^\top \rangle$$

since these are the exponents (up to the factor $\frac{1}{2}$) of the characteristic functions of the vectors $(B_{t_1}, \dots, B_{t_n}, B_{t_{n+1}})$ and $(B_{t_1}, \dots, B_{t_n})$; taking $\xi_{n+1} = 0$ just means projection. The latter identity is, however, clear as C_n appears as the largest principal minor matrix (the top left $n \times n$ block) of C_{n+1} —just have a look at the definition.

- c) If we take $\alpha = 1$, we get $\mathbb{E}(B_t B_s) = s \wedge t$, i.e. we have a Brownian motion with two-sided index set: $B_t = W'_t$ if $t \geq 0$ and $B_t = W''_{-t}$ if $t \leq 0$ where W', W'' are two independent Brownian motions which are glued together back-to-back. In view of the time inversion property (Paragraph 2.15) it is easy to see that $(B_t - B_{-a})_{t \geq -a}$ is a BM for every $a > 0$.

If we take $\alpha = 2$, we end up with a degenerate process $B_t = tB_1$. This follows from the fact that

$$\mathbb{E}[(B_t - tB_1)^2] = \mathbb{E}(B_t^2) - 2t \mathbb{E}(B_t B_1) + t^2 \mathbb{E}(B_1^2) = t^2 - 2\frac{t}{2}(t^2 + 1 - (t-1)^2) + t^2 = 0.$$

- d) Let us check the second criterion for $W_t := c^{-H} B_{ct}$. $\mathbb{E} W_t = 0$ and $W_0 = 0$ are clear. We have

$$\mathbb{E}[(W_t - W_s)^2] = c^{-2H} \mathbb{E}[(B_{ct} - B_{cs})^2] = c^{-2H} |ct - cs|^\alpha = |t - s|^\alpha$$

since $2H = \alpha$.

- e) The law of $B_t - B_s$ depends only on $\mathbb{E}(B_t - B_s) = 0$ and $\mathbb{E}[(B_t - B_s)^2] = |t - s|^\alpha$ and both quantities depend only on the time-difference. Hence, $B_t - B_s \sim B_{t+h} - B_{s+h}$.
- f) Consider the increments $B_t - B_s$ and $B_s - B_0$ if $0 < s < t$. Then

$$\begin{aligned} \mathbb{E}[(B_t - B_s)B_s] &= \mathbb{E}(B_t B_s) - \mathbb{E}(B_s^2) \\ &= \frac{1}{2}(t^\alpha + s^\alpha - (t-s)^\alpha) - s^\alpha \\ &= \frac{1}{2}(t^\alpha - s^\alpha - (t-s)^\alpha) \end{aligned}$$

and if you compare this with

$$\mathbb{E}(B_t - B_s) \mathbb{E} B_s = 0$$

then you see that these two quantities coincide if, and only if, $\alpha = 1$. Thus if $\alpha \neq 1$, the increments are not even uncorrelated, hence not independent. (In the case $\alpha = 1$, uncorrelated gives independent, since $(B_t - B_s, B_s)$ is jointly Gaussian!).



3 Constructions of Brownian motion

Problem 3.1. Solution: The partial sums

$$W_N(t, \omega) = \sum_{n=0}^{N-1} G_n(\omega) S_n(t), \quad t \in [0, 1],$$

converge as $N \rightarrow \infty$ \mathbb{P} -a.s. uniformly for t towards $B(t, \omega), t \in [0, 1]$ —cf. Problem 3.3. Therefore, the random variables

$$\int_0^1 W_N(t) dt = \sum_{n=0}^{N-1} G_n \int_0^1 S_n(t) dt \xrightarrow[N \rightarrow \infty]{\mathbb{P}\text{-a.s.}} X = \int_0^1 B(t) dt.$$

This shows that $\int_0^1 W_N(t) dt$ is the sum of independent $N(0, 1)$ -random variables, hence itself normal and so is its limit X .

From the definition of the Schauder functions (cf. Figure 3.2) we find

$$\begin{aligned} \int_0^1 S_0(t) dt &= \frac{1}{2} \\ \int_0^1 S_{2^j+k}(t) dt &= \frac{1}{4} 2^{-\frac{3}{2}j}, \quad k = 0, 1, \dots, 2^j - 1, j \geq 0. \end{aligned}$$

and this shows

$$\int_0^1 W_{2^{n+1}}(t) dt = \frac{1}{2} G_0 + \frac{1}{4} \sum_{j=0}^n \sum_{l=0}^{2^j-1} 2^{-\frac{3}{2}j} G_{2^j+l}.$$

Consequently, since the G_j are iid $N(0, 1)$ random variables,

$$\begin{aligned} \mathbb{E} \int_0^1 W_{2^{n+1}}(t) dt &= 0, \\ \mathbb{V} \int_0^1 W_{2^{n+1}}(t) dt &= \frac{1}{4} + \frac{1}{16} \sum_{j=0}^n \sum_{l=0}^{2^j-1} 2^{-3j} \\ &= \frac{1}{4} + \frac{1}{16} \sum_{j=0}^n 2^{-2j} \\ &= \frac{1}{4} + \frac{1}{16} \frac{1 - 2^{-2(n+1)}}{1 - \frac{1}{4}} \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{4} + \frac{1}{16} \frac{4}{3} = \frac{1}{3}. \end{aligned}$$

This means that

$$X = \frac{1}{2} G_0 + \sum_{j=0}^{\infty} \frac{1}{4} 2^{-\frac{3}{2}j} \underbrace{\sum_{l=0}^{2^j-1} G_{2^j+l}}_{\sim N(0, 2^j)}$$

where the series converges \mathbb{P} -a.s. and in mean square, and $X \sim N(0, \frac{1}{3})$.



Problem 3.2. Solution: Denote by λ Lebesgue measure on $[0, 1]$.

- a) By the independence of the random variables $G_n \sim \mathbf{N}(0, 1)$ and Parseval's identity, we have for $M < N$

$$\begin{aligned} \mathbb{E}(|W_N(B) - W_M(B)|^2) &= \mathbb{E} \left[\sum_{m,n=M}^{N-1} G_m G_n \langle \mathbf{1}_B, \phi_m \rangle_{L^2} \langle \mathbf{1}_B, \phi_n \rangle_{L^2} \right] \\ &= \sum_{m,n=M}^{N-1} \underbrace{\mathbb{E}(G_m G_n)}_{=0 \text{ (} n \neq m), \text{ or } =1 \text{ (} n=m)} \langle \mathbf{1}_B, \phi_m \rangle_{L^2} \langle \mathbf{1}_B, \phi_n \rangle_{L^2} \\ &= \sum_{n=M}^{N-1} \langle \mathbf{1}_B, \phi_n \rangle_{L^2}^2 \xrightarrow{M, N \rightarrow \infty} 0. \end{aligned}$$

This shows that $W(B) = L^2(\mathbb{P})\text{-}\lim_{N \rightarrow \infty} W_N(B)$ exists.

- b) We have $\mathbb{E} W(A)W(B) = \lambda(A \cap B)$. This can be seen as follows: Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} &\mathbb{E}(|W_N(A)W_N(B) - W(A)W(B)|) \\ &\leq \mathbb{E}(|W_N(A)(W_N(B) - W(B))|) + \mathbb{E}(|W(B)(W_N(A) - W(A))|) \\ &\leq \sqrt{\mathbb{E}(|W_N(A)|^2)} \sqrt{\mathbb{E}(|W_N(B) - W(B)|^2)} + \sqrt{\mathbb{E}(|W(B)|^2)} \sqrt{\mathbb{E}(|W_N(A) - W(A)|^2)} \end{aligned}$$

By part a), $W(A) = L^2(\mathbb{P})\text{-}\lim_{N \rightarrow \infty} W_N(A)$ and $W(B) = L^2(\mathbb{P})\text{-}\lim_{N \rightarrow \infty} W_N(B)$, and therefore this calculation shows $W(A)W(B) = L^1(\mathbb{P})\text{-}\lim_{N \rightarrow \infty} W_N(A)W_N(B)$.

A similar calculation as in the first part yields

$$\begin{aligned} \mathbb{E}(W(A)W(B)) &= \lim_{N \rightarrow \infty} \mathbb{E}(W_N(A)W_N(B)) \\ &= \lim_{N \rightarrow \infty} \sum_{m,n=0}^{N-1} \underbrace{\mathbb{E}(G_m G_n)}_{=0 \text{ (} n \neq m), \text{ or } =1 \text{ (} n=m)} \langle \mathbf{1}_A, \phi_m \rangle_{L^2} \langle \mathbf{1}_B, \phi_n \rangle_{L^2} \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \langle \mathbf{1}_A, \phi_n \rangle_{L^2} \langle \mathbf{1}_B, \phi_n \rangle_{L^2} \\ &= \lambda(A \cap B) \end{aligned}$$

where we used Parseval's identity for the last step.

- c) We have seen in part b) that

$$\mu(B) := \mathbb{E}(|W(B)|^2) = \lambda(B).$$

Consequently, μ is a measure. In contrast, the mapping $B \mapsto W(B)$ is not non-negative (and random), hence it does not define a measure on $[0, 1]$. In fact, it is not even a *signed* measure: Since $W(B)$ is a random variable (and an element of the space L^2 consisting of equivalence classes), it is only defined up to a null set, and the null set can (and will) depend on B . This means that we run into difficulties when we consider σ -additivity, since there are more than countably many ways to represent a set B as a countable union of disjoint sets B_j . Thus, the exceptional null sets may become uncontrollable ...

d) Let

$$f(t) = \sum_{i=1}^l f_i \mathbb{1}_{A_i}(t) = \sum_{j=1}^m g_j \mathbb{1}_{B_j}(t)$$

representations of a step function $f \in \mathcal{S}$. Without loss of generality we may assume $[0, 1] = \bigcup_{i=1}^l A_i = \bigcup_{j=1}^m B_j$. Choose a common disjoint refinement of A_1, \dots, A_l and B_1, \dots, B_m , i.e. disjoint sets $C_1, \dots, C_n \in \mathcal{B}[0, 1]$ such that

$$A_i = \bigcup_{k:C_k \subset A_i} C_k \quad \text{and} \quad B_j = \bigcup_{k:C_k \subset B_j} C_k.$$

For

$$h_k := \sum_{i:C_k \subset A_i} f_i = \sum_{j:C_k \subset B_j} g_j$$

we have

$$f(t) = \sum_{i=1}^l f_i \mathbb{1}_{A_i}(t) = \sum_{j=1}^m g_j \mathbb{1}_{B_j}(t) = \sum_{k=1}^n h_k \mathbb{1}_{C_k}(t).$$

Since (all limits in the next calculation are $L^2(\mathbb{P})$ -limits)

$$\begin{aligned} W(A \cup B) &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} G_n \langle \mathbb{1}_A + \mathbb{1}_B, \phi_n \rangle_{L^2} \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} G_n \langle \mathbb{1}_A, \phi_n \rangle_{L^2} + \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} G_n \langle \mathbb{1}_B, \phi_n \rangle_{L^2} \\ &= W(A) + W(B) \end{aligned}$$

for any two disjoint sets $A, B \in \mathcal{B}[0, 1]$, we conclude

$$\begin{aligned} \sum_{i=1}^l f_i W(A_i) &= \sum_{i=1}^l f_i W\left(\bigcup_{k:C_k \subset A_i} C_k\right) \\ &= \sum_{i=1}^l \sum_{k:C_k \subset A_i} f_i W(C_k) \\ &= \sum_{k=1}^n \sum_{i:C_k \subset A_i} f_i W(C_k) \\ &= \sum_{k=1}^n d_k W(C_k) \\ &= \dots = \sum_{j=1}^m g_j W(B_j). \end{aligned}$$

e) Let $f \in \mathcal{S}$ be given by

$$f(t) = \sum_{j=1}^m c_j \mathbb{1}_{B_j}(t)$$

where $c_j \in \mathbb{R}$, $B_j \in \mathcal{B}[0, 1]$ for $j = 1, \dots, m$. Using the definition of I , we get

$$\mathbb{E}(|I(f)|^2) = \sum_{j=1}^m \sum_{k=1}^m c_j c_k \mathbb{E}(W(B_j)W(B_k))$$

$$\begin{aligned} &\stackrel{b)}{=} \sum_{j=1}^m \sum_{k=1}^m c_j c_k \lambda(B_j \cap B_k) \\ &= \int_0^1 \left(\sum_{j=1}^m c_j \mathbb{1}_{B_j}(t) \right)^2 \lambda(dt) = \int_0^1 |f|^2 d\lambda. \end{aligned}$$

Since the family of step functions \mathcal{S} is dense in $L^2([0, 1], \lambda)$, the isometry allows us to extend the operator I to $L^2([0, 1], \lambda)$: Let $f \in L^2([0, 1], \lambda)$, then there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $f_n \rightarrow f$ in $L^2([0, 1], \lambda)$. From

$$\mathbb{E}(|I(f_n) - I(f_m)|^2) = \mathbb{E}(|I(f_n - f_m)|^2) = \int_0^1 |f_n - f_m|^2 d\lambda$$

we see that the sequence $(I(f_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{P})$. Therefore, the limit

$$I(f) := L^2(\mathbb{P})\text{-}\lim_{n \rightarrow \infty} I(f_n)$$

exists. Note that the isometry implies that $I(f)$ does not depend on the approximating sequence $(f_n)_{n \in \mathbb{N}}$. Consequently, I is well-defined.

Remark: Using the results from Section 3.1 it is clear that $W_t := W([0, t])$, $t \in [0, 1]$, has all properties of a Brownian motion. As usual, the continuity of the paths $t \mapsto W_t$ is not obvious and needs arguments along the lines of, say, the Lévy–Ciesielski construction in Section 3.2.

Problem 3.3. Solution:

a) From the definition of the Schauder functions $S_n(t)$, $n \geq 0$, $t \in [0, 1]$, we find

$$\begin{aligned} 0 &\leq S_n(t) \quad \forall n, t \\ S_{2^j+k}(t) &\leq S_{2^j+k}((2k+1)/2^{j+1}) = 2^{-j/2}/2^{j+1} = \frac{1}{2} 2^{-j/2} \quad \forall j, k, t \\ \sum_{k=0}^{2^j-1} S_{2^j+k}(t) &\leq \frac{1}{2} 2^{-j/2} \quad (\text{disjoint supports!}) \end{aligned}$$

By assumption,

$$\exists C > 0, \quad \exists \epsilon \in (0, \frac{1}{2}), \quad \forall n : |a_n| \leq C \cdot n^\epsilon.$$

Thus, we find

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| S_n(t) &\leq |a_0| + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} |a_{2^j+k}| S_{2^j+k}(t) \\ &\leq |a_0| + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} C \cdot (2^{j+1})^\epsilon S_{2^j+k}(t) \\ &\leq |a_0| + \sum_{j=0}^{\infty} C \cdot 2^{(j+1)\epsilon} \frac{1}{2} 2^{-j} < \infty. \end{aligned}$$

(The series is convergent since $\epsilon < 1/2$).

This shows that $\sum_{n=0}^{\infty} a_n S_n(t)$ converges absolutely and uniformly for $t \in [0, 1]$.

b) For $C > \sqrt{2}$ we find from

$$\mathbb{P}\left(|G_n| > \sqrt{\log n}\right) \leq \sqrt{\frac{2}{\pi}} \frac{1}{C \sqrt{\log n}} e^{-\frac{1}{2} C^2 \log n} \leq \sqrt{\frac{2}{\pi}} \frac{1}{C} n^{-C^2/2} \quad \forall n \geq 3$$

that the following series converges:

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|G_n| > \sqrt{\log n}\right) < \infty.$$

By the Borel–Cantelli Lemma we find that $G_n(\omega) = O(\sqrt{\log n})$ for almost all ω , thus $G_n(\omega) = O(n^\epsilon)$ for any $\epsilon \in (0, 1/2)$.

From part a) we know that the series $\sum_{n=0}^{\infty} G_n(\omega) S_n(t)$ converges a.s. uniformly for $t \in [0, 1]$.

■ ■

Problem 3.4. Solution: Set $\|f\|_p := (\mathbb{E}|f|^p)^{1/p}$

Solution 1: We observe that the space $L^p(\Omega, \mathcal{A}, \mathbb{P}; S) = \{X : X \in S, \|d(X, 0)\|_p < \infty\}$ is complete and that the condition stated in the problem just says that $(X_n)_n$ is a Cauchy sequence in the space $L^p(\Omega, \mathcal{A}, \mathbb{P}; S)$. A good reference for this is, for example, the monograph by F. Trèves [23, Chapter 46]. You will find the ‘pedestrian’ approach as Solution 2 below.

Solution 2: By assumption

$$\forall k \geq 0 \quad \exists N_k \geq 1 : \sup_{m \geq N_k} \|d(X_{N_k}, X_m)\|_p \leq 2^{-k}.$$

Without loss of generality we can assume that $N_k \leq N_{k+1}$. In particular

$$\|d(X_{N_k}, X_{N_{k+1}})\|_p \leq 2^{-k} \xrightarrow{\forall l > k} \|d(X_{N_k}, X_{N_l})\|_p \leq \sum_{j=k}^{l-1} 2^{-j} \leq \frac{2}{2^k}.$$

Fix $m \geq 1$. Then we see that

$$\|d(X_{N_k}, X_m) - d(X_{N_l}, X_m)\|_p \leq \|d(X_{N_k}, X_{N_l})\|_p \xrightarrow{k, l \rightarrow \infty} 0.$$

This means that that $(d(X_{N_k}, X_m))_{k \geq 0}$ is a Cauchy sequence in $L^p(\mathbb{P}; \mathbb{R})$. By the completeness of the space $L^p(\mathbb{P}; \mathbb{R})$ there is some $f_m \in L^p(\mathbb{P}; \mathbb{R})$ such that

$$d(X_{N_k}, X_m) \xrightarrow[k \rightarrow \infty]{\text{in } L^p} f_m$$

and, for a subsequence $(n_k) \subset (N_k)_k$ we find

$$d(X_{n_k}, X_m) \xrightarrow[k \rightarrow \infty]{\text{almost surely}} f_m.$$

The subsequence n_k may also depend on m . Since $(n_k(m))_k$ is still a subsequence of (N_k) , we still have $d(X_{n_k(m)}, X_{m+1}) \rightarrow f_{m+1}$ in L^p , hence we can find a subsequence

$(n_k(m+1))_k \subset (n_k(m))_k$ such that $d(X_{n_k(m+1)}, X_{m+1}) \rightarrow f_{m+1}$ a.s. Iterating this we see that we can assume that $(n_k)_k$ does not depend on m .

In particular, we have almost surely

$$\forall \epsilon > 0 \quad \exists L = L(\epsilon) \geq 1 \quad \forall k \geq L : |d(X_{n_k}, X_m) - f_m| \leq \epsilon.$$

Moreover,

$$\begin{aligned} \lim_{m \rightarrow \infty} \|f_m\|_p &= \lim_{m \rightarrow \infty} \left\| \lim_{k \rightarrow \infty} d(X_{n_k}, X_m) \right\|_p \leq \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \|d(X_{n_k}, X_m)\|_p \\ &\leq \lim_{k \rightarrow \infty} \sup_{m \geq n_k} \|d(X_{n_k}, X_m)\|_p = 0. \end{aligned}$$

Thus, $f_m \rightarrow 0$ in L^p and, for a subsequence m_k we get

$$\forall \epsilon > 0 \quad \exists K = K(\epsilon) \geq 1 \quad \forall r \geq K : |f_{m_r}| \leq \epsilon.$$

Therefore,

$$\begin{aligned} d(X_{n_k}, X_{n_l}) &\leq d(X_{n_k}, X_{m_r}) + d(X_{n_k}, X_{m_r}) \\ &\leq |d(X_{n_k}, X_{m_r}) - f_{m_r}| + |d(X_{n_k}, X_{m_r}) - f_{m_r}| + 2|f_{m_r}|. \end{aligned}$$

Fix $\epsilon > 0$ and pick $r > K$. Then let $k, l \rightarrow \infty$. This gives

$$d(X_{n_k}, X_{n_l}) \leq |d(X_{n_k}, X_{m_r}) - f_{m_r}| + |d(X_{n_k}, X_{m_r}) - f_{m_r}| + 2\epsilon \leq 4\epsilon \quad \forall k, l \geq L(\epsilon).$$

Since S is complete, this proves that $(X_{n_k})_{k \geq 0}$ converges to some $X \in S$ almost surely.

Remark: If we replace the condition of the Problem by

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{m \geq n} d^p(X_n, X_m) \right) = 0$$

things become MUCH simpler:

This condition says that the sequence $d_n := \sup_{m \geq n} d^p(X_n, X_m)$ converges in $L^p(\mathbb{P}; \mathbb{R})$ to zero. Hence there is a subsequence $(n_k)_k$ such that

$$\lim_{k \rightarrow \infty} \sup_{m \geq n_k} d(X_{n_k}, X_m) = 0$$

almost surely. This shows that $d(X_{n_k}, X_{n_l}) \rightarrow 0$ as $k, l \rightarrow \infty$, i.e. we find by the completeness of the space S that $X_{n_k} \rightarrow X$.

■ ■

Problem 3.5. Solution: Fix $n \geq 1$, $0 \leq t_1 \leq \dots \leq t_n$ and Borel sets A_1, \dots, A_n . By assumption, we know that

$$\mathbb{P}(X_t = Y_t) = 1 \quad \forall t \geq 0 \implies \mathbb{P}(X_{t_j} = Y_{t_j}, j = 1, \dots, n) = \mathbb{P} \left(\bigcap_{j=1}^n \{X_{t_j} = Y_{t_j}\} \right) = 1.$$

Thus,

$$\begin{aligned}
 \mathbb{P}\left(\bigcap_{j=1}^n \{X_{t_j} \in A_j\}\right) &= \mathbb{P}\left(\bigcap_{j=1}^n \{X_{t_j} \in A_j\} \cap \bigcap_{j=1}^n \{X_{t_j} = Y_{t_j}\}\right) \\
 &= \mathbb{P}\left(\bigcap_{j=1}^n \{X_{t_j} \in A_j\} \cap \{X_{t_j} = Y_{t_j}\}\right) \\
 &= \mathbb{P}\left(\bigcap_{j=1}^n \{Y_{t_j} \in A_j\} \cap \{X_{t_j} = Y_{t_j}\}\right) \\
 &= \mathbb{P}\left(\bigcap_{j=1}^n \{Y_{t_j} \in A_j\}\right).
 \end{aligned}$$

■ ■

Problem 3.6. Solution: indistinguishable \implies modification:

$$\mathbb{P}(X_t = Y_t \forall t \geq 0) = 1 \implies \forall t \geq 0 : \mathbb{P}(X_t = Y_t) = 1.$$

modification \implies equivalent: see the previous Problem 3.5

Now assume that I is countable or $t \mapsto X_t, t \mapsto Y_t$ are (left or right) continuous.

modification \implies indistinguishable: By assumption, $\mathbb{P}(X_t \neq Y_t) = 0$ for any $t \in I$. Let $D \subset I$ be any countable dense subset. Then

$$\mathbb{P}\left(\bigcup_{q \in D} \{X_q \neq Y_q\}\right) \leq \sum_{q \in D} \mathbb{P}(X_q \neq Y_q) = 0$$

which means that $\mathbb{P}(X_q = Y_q \forall q \in D) = 1$. If I is countable, we are done. In the other case we have, by the density of D ,

$$\mathbb{P}(X_t = Y_t \forall t \in I) = \mathbb{P}\left(\lim_{D \ni q} X_q = \lim_{D \ni q} Y_q \forall t \in I\right) \geq \mathbb{P}(X_q = Y_q \forall q \in D) = 1.$$

equivalent $\not\implies$ modification: To see this let $(B_t)_{t \geq 0}$ and $(W_t)_{t \geq 0}$ be two **independent** one-dimensional Brownian motions defined on the **same probability space**. Clearly, these processes have the same finite-dimensional distributions, i.e. they are equivalent. On the other hand, for any $t > 0$

$$\mathbb{P}(B_t = W_t) = \int_{-\infty}^{\infty} \mathbb{P}(B_t = y) \mathbb{P}(W_t \in dy) = \int_{-\infty}^{\infty} 0 \mathbb{P}(W_t \in dy) = 0.$$

■ ■

Problem 3.7. Solution: We use the characterization from Lemma 2.8. Its proof shows that we can derive (2.15)

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^n \langle \xi_j, X_{q_j} - X_{q_{j-1}} \rangle + i \langle \xi_0, X_{q_0} \rangle\right)\right] = \exp\left(-\frac{1}{2} \sum_{j=1}^n |\xi_j|^2 (q_j - q_{j-1})\right)$$

on the basis of (B0)–(B3) for $(B_q)_{q \in \mathbb{Q} \cap [0, \infty)}$ and $q_0 = 0, q_1, \dots, q_n \in \mathbb{Q} \cap [0, \infty)$.

Now set $t_0 = q_0 = 0$ and pick $t_1, \dots, t_n \in \mathbb{R}$ and approximate each t_j by a rational sequence $q_j^{(k)}, k \geq 1$. Since (2.15) holds for $q_j^{(k)}, j = 0, \dots, n$ and every $k \geq 0$, we can easily perform the limit $k \rightarrow \infty$ on both sides (on the left we use dominated convergence!) since B_t is continuous.

This proves (2.15) for $(B_t)_{t \geq 0}$, and since $(B_t)_{t \geq 0}$ has continuous paths, Lemma 2.8 proves that $(B_t)_{t \geq 0}$ is a BM¹.

Problem 3.8. Solution: Without loss of generality, we assume that all random variables have values in \mathbb{R}^d . We use Kac's characterization of independent random variables (Kac's theorem). Fix $\xi, \eta, \zeta \in \mathbb{R}^d$. Then

$$\begin{aligned} \mathbb{E} e^{i\langle \xi, X' \rangle + i\langle \eta, X'' \rangle + i\langle \zeta, Z \rangle} & \stackrel{(X', X'') \perp Z}{=} \mathbb{E} e^{i\langle \xi, X' \rangle + i\langle \eta, X'' \rangle} \mathbb{E} e^{i\langle \zeta, Z \rangle} \\ & \stackrel{X' \perp X''}{=} \mathbb{E} e^{i\langle \xi, X' \rangle} \mathbb{E} e^{i\langle \eta, X'' \rangle} \mathbb{E} e^{i\langle \zeta, Z \rangle}. \end{aligned}$$

Problem 3.9. Solution: The joint density of $(W(t_0), W(t), W(t_1))$ is

$$f_{t_0, t, t_1}(x_0, x, x_1) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{(t_1 - t)(t - t_0)t_0}} \exp\left(-\frac{1}{2} \left[\frac{(x_1 - x)^2}{t_1 - t} + \frac{(x - x_0)^2}{t - t_0} + \frac{x_0^2}{t_0} \right]\right)$$

while the joint density of $(W(t_0), W(t_1))$ is

$$f_{t_0, t_1}(x_0, x_1) = \frac{1}{(2\pi)} \frac{1}{\sqrt{(t_1 - t_0)t_0}} \exp\left(-\frac{1}{2} \left[\frac{(x_1 - x_0)^2}{t_1 - t_0} + \frac{x_0^2}{t_0} \right]\right).$$

The conditional density of $W(t)$ given $(W(t_0), W(t_1))$ is

$$\begin{aligned} & f_{t|t_0, t_1}(x | x_1, x_2) \\ &= \frac{f_{t_0, t, t_1}(x_0, x, x_1)}{f_{t_0, t_1}(x_0, x_1)} \\ &= \frac{\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{(t_1 - t)(t - t_0)t_0}} \exp\left(-\frac{1}{2} \left[\frac{(x_1 - x)^2}{t_1 - t} + \frac{(x - x_0)^2}{t - t_0} + \frac{x_0^2}{t_0} \right]\right)}{\frac{1}{(2\pi)} \frac{1}{\sqrt{(t_1 - t_0)t_0}} \exp\left(-\frac{1}{2} \left[\frac{(x_1 - x_0)^2}{t_1 - t_0} + \frac{x_0^2}{t_0} \right]\right)} \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)}} \exp\left(-\frac{1}{2} \left[\frac{(x_1 - x)^2}{t_1 - t} + \frac{(x - x_0)^2}{t - t_0} - \frac{(x_1 - x_0)^2}{t_1 - t_0} \right]\right) \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)}} \exp\left(-\frac{1}{2} \left[\frac{(t - t_0)(x_1 - x)^2 + (t_1 - t)(x - x_0)^2 - (x_1 - x_0)^2}{(t_1 - t)(t - t_0)} \right]\right) \end{aligned}$$

Now consider the argument in the square brackets [...] of the exp-function

$$\left[\frac{(t - t_0)(x_1 - x)^2 + (t_1 - t)(x - x_0)^2 - (x_1 - x_0)^2}{(t_1 - t)(t - t_0)} \right]$$

$$\begin{aligned}
&= \frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)} \left[\frac{t - t_0}{t_1 - t_0} (x_1 - x)^2 + \frac{t_1 - t}{t_1 - t_0} (x - x_0)^2 - \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)^2} (x_1 - x_0)^2 \right] \\
&= \frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)} \left[\left(\frac{t - t_0}{t_1 - t_0} + \frac{t_1 - t}{t_1 - t_0} \right) x^2 + \left(\frac{t - t_0}{t_1 - t_0} - \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)^2} \right) x_1^2 \right. \\
&\quad \left. + \left(\frac{t_1 - t}{t_1 - t_0} - \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)^2} \right) x_0^2 \right. \\
&\quad \left. - 2 \frac{t - t_0}{t_1 - t_0} x_1 x - 2 \frac{t_1 - t}{t_1 - t_0} x x_0 + 2 \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)^2} x_1 x_0 \right] \\
&= \frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)} \left[x^2 + \frac{(t - t_0)^2}{(t_1 - t_0)^2} x_1^2 + \frac{(t_1 - t)^2}{(t_1 - t_0)^2} x_0^2 \right. \\
&\quad \left. - 2 \frac{t - t_0}{t_1 - t_0} x_1 x - 2 \frac{t_1 - t}{t_1 - t_0} x x_0 + 2 \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)^2} x_1 x_0 \right] \\
&= \frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)} \left[x - \frac{t - t_0}{t_1 - t_0} x_1 - \frac{t_1 - t}{t_1 - t_0} x_0 \right]^2 \\
&= \frac{(t_1 - t_0)}{(t_1 - t)(t - t_0)} \left[x - \left(\frac{t - t_0}{t_1 - t_0} x_1 + \frac{t_1 - t}{t_1 - t_0} x_0 \right) \right]^2.
\end{aligned}$$

Set

$$\sigma^2 = \frac{(t_1 - t)(t - t_0)}{(t_1 - t_0)} \quad \text{and} \quad m = \frac{t - t_0}{t_1 - t_0} x_1 + \frac{t_1 - t}{t_1 - t_0} x_0$$

then our calculation shows that

$$f_{t|t_0, t_1}(x | x_1, x_2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right).$$

■ ■

Problem 3.10. Solution: Following the hint we first construct two independent Brownian motions (wlog on the same probability space), say $(B'_t)_{t \geq 0}$ and $(B''_t)_{t \geq 0}$ and define a new process for any $t \in \mathbb{R}$ by

$$W_t := \begin{cases} B'_t, & t \geq 0, \\ B''_{-t}, & t < 0. \end{cases}$$

Obviously, $W_0 = 0$ a.s., $t \mapsto W_t$ is continuous a.s. If $s, t \geq 0$ or $s, t \leq 0$, the properties (B2), (B3) are clear. Thus, assume that $s < 0 < t$. Then, by the independence of B' and B''

$$W_t - W_s = (B'_t - B'_0) + (B''_0 - B''_{-s}) \sim \mathbf{N}(0, t)^{\otimes d} * \mathbf{N}(0, -s)^{\otimes d} = \mathbf{N}(0, t - s)^{\otimes d}.$$

This proves (B2), (B3).

Finally, assume that $-\infty < s_m < s_{m-1} < \dots < s_1 \leq 0 \leq t_1 < \dots < t_n < \infty$. We introduce, artificially, $s_0 = 0 = t_0$. Then the increments of the W -process are given by

$$\begin{aligned}
&W_{s_{m-1}} - W_{s_m}, \dots, W_{s_1} - W_{s_2}, \\
&W_{t_1} - W_{s_1} = (W_{t_1} - W_{t_0}) + (W_{s_0} - W_{s_1}), \\
&W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}
\end{aligned}$$

and, by definition, these are

$$\begin{aligned} & B''_{-s_{m-1}} - B''_{-s_m}, \dots, B''_{-s_1} - B''_{-s_2}, \\ B'_{t_1} - B''_{-s_1} &= (B'_{t_1} - B'_{t_0}) + (B''_{-s_0} - B''_{-s_1}), \\ & B'_{t_2} - B'_{t_1}, \dots, B'_{t_n} - B'_{t_{n-1}} \end{aligned}$$

and it is clear from this representation that the increments of W are independent, since $B' \perp\!\!\!\perp B''$ and since the increments of B' and B'' are independent.

■ ■

4 The canonical model

Problem 4.1. Solution: Let $F : \mathbb{R} \rightarrow [0, 1]$ be a distribution function. We begin with a general lemma: F has a unique generalized monotone increasing right continuous inverse:

$$F^{-1}(u) = G(u) = \inf\{x : F(x) > u\} \quad (4.1)$$

$$\left[= \sup\{x : F(x) \leq u\} \right].$$

We have $F(G(u)) = u$ if $F(t)$ is continuous in $t = G(u)$, otherwise, $F(G(u)) \geq u$.

Indeed: For those t where F is strictly increasing and continuous, there is nothing to show.

Let us look at the two problem cases: F jumps and F is flat.

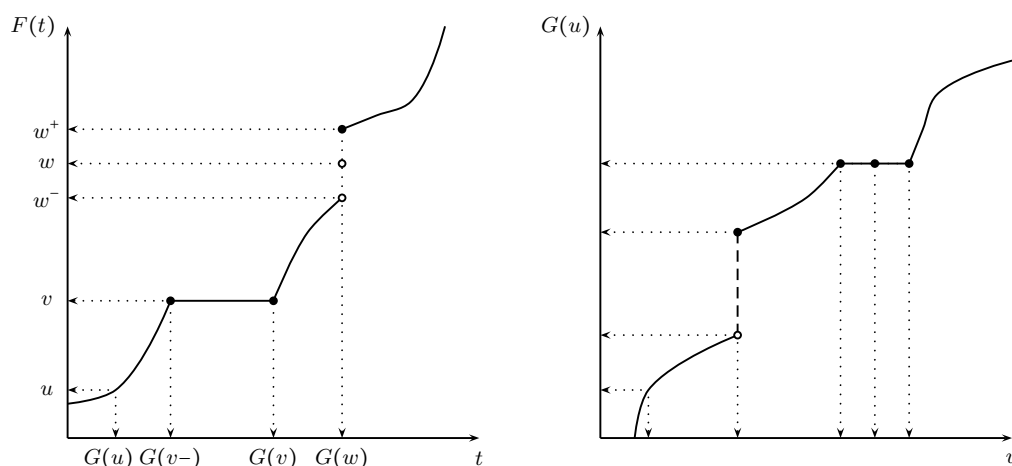


Figure 4.1: An illustration of the problem cases

If $F(t)$ jumps, we have $G(w) = G(w^+) = G(w^-)$ and if $F(t)$ is flat, we take the right endpoint of the ‘flatness interval’ $[G(v-), G(v)]$ to define G (this leads to right continuity of G)

- a) Let $(\Omega, \mathcal{A}, \mathbb{P}) = ([0, 1], \mathcal{B}[0, 1], du)$ (du stands for Lebesgue measure) and define $X = G$ ($G = F^{-1}$ as before). Then

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\})$$

$$= \lambda(\{u \in [0, 1] : G(u) \leq x\})$$

(the discontinuities of F are countable, i.e. a Lebesgue null set)

$$\begin{aligned} &= \lambda(\{t \in [0, 1] : t \leq F(x)\}) \\ &= \lambda([0, F(x)]) = F(x). \end{aligned}$$

Measurability is clear because of monotonicity.

- b) Use the product construction and part a). To be precise, we do the construction for two random variables. Let $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega' \rightarrow \mathbb{R}$ be two iid copies. We define on the product space

$$(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', \mathbb{P} \times \mathbb{P}')$$

the new random variables $\xi(\omega, \omega') := X(\omega)$ and $\eta(\omega, \omega') := Y(\omega')$. Then we have

- ξ, η live on the same probability space
- $\xi \sim X, \eta \sim Y$

$$\begin{aligned} \mathbb{P} \times \mathbb{P}'(\xi \in A) &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : \xi(\omega, \omega') \in A\}) \\ &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : X(\omega) \in A\}) \\ &= \mathbb{P} \times \mathbb{P}'(\{\omega \in \Omega : X(\omega) \in A\} \times \Omega') \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \\ &= \mathbb{P}(X \in A). \end{aligned}$$

and a similar argument works for η .

- $\xi \perp \eta$

$$\begin{aligned} \mathbb{P} \times \mathbb{P}'(\xi \in A, \eta \in B) &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : \xi(\omega, \omega') \in A, \eta(\omega, \omega') \in B\}) \\ &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : X(\omega) \in A, Y(\omega') \in B\}) \\ &= \mathbb{P} \times \mathbb{P}'(\{\omega \in \Omega : X(\omega) \in A\} \times \{\omega' \in \Omega' : Y(\omega') \in B\}) \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \mathbb{P}'(\{\omega' \in \Omega' : Y(\omega') \in B\}) \\ &= \mathbb{P}(X \in A) \mathbb{P}(Y \in B) \\ &= \mathbb{P} \times \mathbb{P}'(\xi \in A) \mathbb{P} \times \mathbb{P}'(\eta \in B) \end{aligned}$$

The same type of argument works for arbitrary products, since independence is always defined for any finite-dimensional subfamily. In the infinite case, we have to invoke the theorem on the existence of infinite product measures (which are constructed via their finite marginals) and which can be seen as a particular case of Kolmogorov's theorem, cf. Theorem 4.8 and Theorem A.2 in the appendix.

- c) The statements are the same if one uses the same construction as above. A difficulty is to identify a multidimensional distribution function $F(x)$. Roughly speaking, these are functions of the form

$$F(x) = \mathbb{P}\left(X \in (-\infty, x_1] \times \cdots \times (-\infty, x_n]\right)$$

where $X = (X_1, \dots, X_n)$ and $x = (x_1, \dots, x_n)$, i.e. x is the 'upper right' endpoint of an infinite rectangle. An abstract characterisation is the following

- $F : \mathbb{R}^n \rightarrow [0, 1]$
- $x_j \mapsto F(x)$ is monotone increasing
- $x_j \mapsto F(x)$ is right continuous
- $F(x) = 0$ if at least one entry $x_j = -\infty$
- $F(x) = 1$ if all entries $x_j = +\infty$
- $\sum (-1)^{\sum_{k=1}^n \epsilon_k} F(\epsilon_1 a_1 + (1 - \epsilon_1) b_1, \dots, \epsilon_n a_n + (1 - \epsilon_n) b_n) \geq 0$ where $-\infty < a_j < b_j < \infty$ and where the outer sum runs over all tuples $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$

The last property is equivalent to

- $\Delta_{h_1}^{(1)} \dots \Delta_{h_n}^{(n)} F(x) \geq 0 \quad \forall h_1, \dots, h_n \geq 0$ where $\Delta_h^{(k)} F(x) = F(x + h e_k) - F(x)$ and e_k is the k^{th} standard unit vector of \mathbb{R}^n .

In principle we can construct such a multidimensional F from its marginals using the theory of copulas, in particular, Sklar's theorem etc. etc. etc.

Another way would be to take $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$ where μ is the probability measure induced by $F(x)$. Then the random variables X_n are just the identity maps! The independent copies are then obtained by the usual product construction.

■ ■

Problem 4.2. Solution: *Step 1:* Let us first show that $\mathbb{P}(\lim_{s \rightarrow t} X_s \text{ exists}) < 1$.

Since $X_r \perp\!\!\!\perp X_s$ and $X_s \sim -X_s$ we get

$$X_r - X_s \sim X_r + X_s \sim \mathbf{N}(0, s + r) \sim \sqrt{s + r} \mathbf{N}(0, 1).$$

Thus,

$$\mathbb{P}(|X_r - X_s| > \epsilon) = \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{s + r}}\right) \xrightarrow{r, s \rightarrow t} \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{2t}}\right) \neq 0.$$

This proves that X_s is not a Cauchy sequence in probability, i.e. it does not even converge **in probability** towards a limit, so a.e. convergence is impossible.

In fact we have

$$\left\{ \omega : \lim_{s \rightarrow t} X_s(\omega) \text{ does not exist} \right\} \supset \bigcap_{k=1}^{\infty} \left\{ \sup_{s, r \in [t-1/k, t+1/k]} |X_s - X_r| > 0 \right\}$$

and so we find with the above calculation

$$\mathbb{P}\left(\lim_{s \rightarrow t} X_s \text{ does not exist}\right) \geq \lim_k \mathbb{P}\left(\sup_{s, r \in [t-1/k, t+1/k]} |X_s - X_r| > 0\right) \geq \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{2t}}\right)$$

This shows, in particular that for any sequence $t_n \rightarrow t$ we have

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} X_{t_n} \text{ exists}\right) < q < 1.$$

where $q = q(t)$ (but independent of the sequence).

Step 2: Fix $t > 0$, fix a sequence $(t_n)_n$ with $t_n \rightarrow t$, and set

$$A = \{\omega \in \Omega : \lim_{s \rightarrow t} X_s(\omega) \text{ exists}\} \quad \text{and} \quad A(t_n) = \{\omega \in \Omega : \lim_{n \rightarrow \infty} X_{t_n}(\omega) \text{ exists}\}.$$

Clearly, $A \subset A(t_n)$ for any such sequence. Moreover, take two sequences $(s_n)_n, (t_n)_n$ such that $s_n \rightarrow t$ and $t_n \rightarrow t$ and which have no points in common; then we get by independence and step 1

$$(X_{s_1}, X_{s_2}, X_{s_3} \dots) \perp\!\!\!\perp (X_{t_1}, X_{t_2}, X_{t_3} \dots) \implies A(t_n) \perp\!\!\!\perp A(s_n)$$

and so, $\mathbb{P}(A) \leq \mathbb{P}(A(s_n) \cap A(t_n)) = \mathbb{P}(A(s_n)) \mathbb{P}(A(t_n)) = q^2$.

By Step 1, $q < 1$. Since there are infinitely many sequences having all no points in common, we get $0 \leq \mathbb{P}(A) \leq \lim_{k \rightarrow \infty} q^k = 0$.

■

Problem 4.3. Solution: Write $\Sigma := \cup\{\sigma(\mathcal{C}) : \mathcal{C} \subset \mathcal{E}, \mathcal{C} \text{ is countable}\}$.

If $\mathcal{C} \subset \mathcal{E}$ we get $\sigma(\mathcal{C}) \subset \sigma(\mathcal{E})$, and so $\Sigma \subset \sigma(\mathcal{E})$.

Conversely, it is clear that $\mathcal{E} \subset \Sigma$, just take $\mathcal{C} := \mathcal{C}_E := \{E\}$ for each $E \in \mathcal{E}$. If we can show that Σ is a σ -algebra we get $\sigma(\mathcal{E}) \subset \sigma(\Sigma) = \Sigma$ and equality follows.

- Clearly, $\emptyset \in \Sigma$.
- If $S \in \Sigma$, then $S \in \sigma(\mathcal{C}_S)$ for some countable $\mathcal{C}_S \subset \mathcal{E}$. Moreover, $S^c \in \sigma(\mathcal{C}_S)$, i.e. $S^c \in \Sigma$.
- If $(S_n)_{n \geq 0} \subset \Sigma$ are countably many sets, then $S_n \in \sigma(\mathcal{C}_n)$ for some countable $\mathcal{C}_n \subset \mathcal{E}$ and each $n \geq 0$. Set $\mathcal{C} := \cup_n \mathcal{C}_n$. This is again countable and we get $S_n \in \sigma(\mathcal{C})$ for all n , hence $\cup_n S_n \in \sigma(\mathcal{C})$ and so $\cup_n S_n \in \Sigma$.

■

Problem 4.4. Solution:

- a) Following the hint, we use $\mathcal{E} = \cup\{\pi_K^{-1}(A) : A \in \mathcal{B}^K(E), K \subset I, \#K < \infty\}$, i.e. the cylinder sets. By definition, $\sigma(\mathcal{E}) = \mathcal{B}^I(E)$. If $\mathcal{C} \subset \mathcal{E}$ is a countable set, then there is a countable index set $J \subset I$ such that $\mathcal{C} \subset \pi_J^{-1}(\mathcal{B}^J(E))$, and since the right-hand side is a σ -algebra, $\sigma(\mathcal{C}) \subset \pi_J^{-1}(\mathcal{B}^J(E))$. This proves the claim.
- b) The previous part shows that every $B \in \mathcal{B}^I(E)$ is of the form $B = \pi_J^{-1}(A)$ with $A \in \mathcal{B}^J(E)$, i.e. it is a cylinder with a countable base. All other indices (and since I is not countable, there are uncountably many left) are E . Since E contains at least two points, we see that $B \neq \emptyset$ is uncountable.
- c) Use the fact that the open cylinders $\pi_K^{-1}(U)$, where $U \subset E^K$ is open and $\#K < \infty$, are open sets in E^I and (a).
- d) Let $e \in E$; then $F := \{e\}$ is a non-void compact set. By Tychonov's theorem $F^I \subset E^I$ is compact and as such it is contained in the Borel σ -algebra $\mathcal{B}(E^I)$. But F^I contains exactly one point, i.e. it is countable. By part (b) it cannot be in $\mathcal{B}^I(E)$.

- e) Assume that I is countable. Then $\mathcal{B}(E^I)$ is generated by countably many sets, namely $\pi_K^{-1}(\mathbb{B}(q, r) \cap E)$ where $K \subset I$ is finite and $\mathbb{B}(q, r)$ are open balls with rational radii $r > 0$ and rational centres $q \in E$. (Note: There are only countably many finite sets contained in a countable set I ! Here the argument would break down for uncountable index sets.) These are but the cylinder sets, i.e. they also generate $\mathcal{B}^I(E)$, and this proves $\mathcal{B}(E^I) \subset \mathcal{B}^I(E)$.

Problem 4.5. Solution: $X_t(\omega)$ is a ‘random’ path starting at the randomly chosen point ω and moving uniformly with constant speed $|v|$ in the direction $v/|v|$. Note that only the starting point is random and it is ‘drawn’ using the law μ or δ_x , i.e. in the latter case we start a.s. at x .

The finite-dimensional distributions are for $0 \leq t_1 < t_2 < \dots < t_n$ given by

$$\begin{aligned} \mathbb{P}^\mu(X_{t_1} \in dy_1, X_{t_2} \in dy_2, \dots, X_{t_n} \in dy_n) \\ = \int_{\mathbb{R}^d} \mu(dx) \delta_{x+t_1v}(dy_1) \otimes \delta_{x+t_2v}(dy_2) \otimes \dots \otimes \delta_{x+t_nv}(dy_n). \end{aligned}$$

Problem 4.6. Solution:

- a) We have to show that any linear combination $\sum_{j=1}^n \xi_j W_{t_j}$, $\xi_1, \dots, \xi_n \in \mathbb{R}$, is a normal distribution. But since we can re-write this as a linear combination $\sum_{j=1}^n \xi_j B_{t_j} - \sum_{j=1}^n t_j \xi_j B_1$, so the claim follows from the fact that $B = (B_t)_{t \in [0,1]}$ is a Gaussian process. Mean-zero is trivial as $\mathbb{E} W_t = \mathbb{E} B_t - t \mathbb{E} B_1 = 0$. The covariance function is given by

$$\begin{aligned} \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] &= \mathbb{E}[B_s B_t + stB_1^2 - sB_1 B_t - tB_1 B_t s] \\ &= s \wedge t + st - s(t \wedge 1) - t(s \wedge 1) \\ &= s \wedge t + st - 2st = s \wedge t - st. \end{aligned}$$

- b) Let $0 = t_0 < \dots < t_n < 1$ and set $t'_k := 1 - t_k$ and $X_t := W_t/(1-t)$. As linear combination of Gaussian random variables the vector $\Delta = (X_{t_k} - X_{t_{k-1}})_{k=1, \dots, n}$ is Gaussian. Clearly, the mean is zero, and the covariance is given by

$$\mathbb{E}(X_{t_k} - X_{t_{k-1}})(X_{t_j} - X_{t_{j-1}}) = \mathbb{E}\left(\frac{W_{t_k}}{1-t_k} - \frac{W_{t_{k-1}}}{1-t_{k-1}}\right)\left(\frac{W_{t_j}}{1-t_j} - \frac{W_{t_{j-1}}}{1-t_{j-1}}\right)$$

where we assume, without loss of generality, that $t_{j-1} < t_j \leq t_{k-1} < t_k$

$$\begin{aligned} &= \mathbb{E} \frac{W_{t_k} W_{t_j}}{(1-t_k)(1-t_j)} - \mathbb{E} \frac{W_{t_k} W_{t_{j-1}}}{(1-t_k)(1-t_{j-1})} - \mathbb{E} \frac{W_{t_{k-1}} W_{t_j}}{(1-t_{k-1})(1-t_j)} + \mathbb{E} \frac{W_{t_{k-1}} W_{t_{j-1}}}{(1-t_{k-1})(1-t_{j-1})} \\ &= \frac{t_j - t_k t_j}{(1-t_k)(1-t_j)} - \frac{t_{j-1} - t_k t_{j-1}}{(1-t_k)(1-t_{j-1})} - \frac{t_j - t_{k-1} t_j}{(1-t_{k-1})(1-t_j)} + \frac{t_{j-1} - t_{k-1} t_{j-1}}{(1-t_{k-1})(1-t_{j-1})} \end{aligned}$$

$$= \frac{t_j}{1-t_j} - \frac{t_{j-1}}{1-t_{j-1}} - \frac{t_j}{1-t_j} + \frac{t_{j-1}}{1-t_{j-1}} = 0.$$

Therefore, the entries of the Gaussian vector Δ are uncorrelated, hence independent. Formally taking $t_k = t_j = t$ and $t_{k-1} = t_{j-1} = s$, a very similar calculation shows that $X_t - X_s \sim \mathcal{N}(0, (t-s)/(t's'))$ with $s < t$ and $t' = 1-t$, $s' = 1-s$.

Now let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be any bounded measurable function. We need to calculate

$$\mathbb{E} f(W_{t_1}, \dots, W_{t_n}) = \mathbb{E} f(t'_1 X_{t_1}, \dots, t'_n X_{t_n}) = \mathbb{E} f(D(X_{t_1}, \dots, X_{t_n})^\top)$$

with the diagonal matrix D with diagonal entries (t'_1, \dots, t'_n) . Thus,

$$\mathbb{E} f(W_{t_1}, \dots, W_{t_n}) = \int_{\mathbb{R}^n} f(Dx)g(x) dx = \frac{1}{\det D} \int_{\mathbb{R}^n} f(y)g(D^{-1}y) dy.$$

The density $g(y)$ is the density of the vector $(X_{t_1}, \dots, X_{t_n})$. Using the fact that $X_t - X_s \sim \mathcal{N}(0, (t-s)/(t's'))$, the argument used for Brownian motion for the proof of (2.10) reveals that

$$g(y) = \frac{1}{(2\pi)^{n/2} \sqrt{\prod_{k=1}^n \frac{(t_k - t_{k-1})}{t'_k t'_{k-1}}}} \exp\left(-\frac{1}{2} \sum_{k=1}^n \frac{t'_k t'_{k-1} (y_k - y_{k-1})^2}{(t_k - t_{k-1})}\right).$$

Therefore, we get

$$\begin{aligned} \frac{g(D^{-1}y)}{\det D} &= \frac{1}{(2\pi)^{n/2} \prod_{k=1}^n t'_k \sqrt{\prod_{k=1}^n \frac{(t_k - t_{k-1})}{t'_k t'_{k-1}}}} \exp\left(-\frac{1}{2} \sum_{k=1}^n \frac{t'_k t'_{k-1} (\frac{1}{t'_k} y_k - \frac{1}{t'_{k-1}} y_{k-1})^2}{(t_k - t_{k-1})}\right) \\ &= \frac{1}{(2\pi)^{n/2} \sqrt{t'_n} \sqrt{\prod_{k=1}^n (t_k - t_{k-1})}} \exp\left(-\frac{1}{2} \sum_{k=1}^n \frac{t'_k t'_{k-1} (\frac{1}{t'_k} y_k - \frac{1}{t'_{k-1}} y_{k-1})^2}{(t_k - t_{k-1})}\right). \end{aligned}$$

Let us now simplify the sum in the exponential function. We have

$$\begin{aligned} \frac{t'_k t'_{k-1} (\frac{1}{t'_k} y_k - \frac{1}{t'_{k-1}} y_{k-1})^2}{(t_k - t_{k-1})} &= \frac{(y_k - y_{k-1})^2}{t_k - t_{k-1}} + \frac{t'_{k-1} - 1}{t_k - t_{k-1}} y_k^2 + \frac{t'_k - 1}{t_k - t_{k-1}} y_{k-1}^2 \\ &= \frac{(y_k - y_{k-1})^2}{t_k - t_{k-1}} + \frac{y_k^2}{1-t_k} - \frac{y_{k-1}^2}{1-t_{k-1}} \end{aligned}$$

If we sum this expression over $k = 1, \dots, n$ we get $\sum_{k=1}^n \frac{(y_k - y_{k-1})^2}{t_k - t_{k-1}} + \frac{y_n^2}{1-t_n}$, and so

$$\begin{aligned} \frac{g(D^{-1}y)}{\det D} &= \frac{1}{\sqrt{t'_n}} \frac{1}{(2\pi)^{n/2} \sqrt{\prod_{k=1}^n (t_k - t_{k-1})}} \exp\left(-\frac{1}{2} \left[\sum_{k=1}^n \frac{(y_k - y_{k-1})^2}{t_k - t_{k-1}} + \frac{y_n^2}{1-t_n} \right]\right) \\ &= \frac{2\pi}{\sqrt{2\pi t'_n}} \exp\left(-\frac{y_n^2}{2(1-t_n)}\right) \frac{1}{(2\pi)^{n/2} \sqrt{\prod_{k=1}^n (t_k - t_{k-1})}} \exp\left(-\frac{1}{2} \sum_{k=1}^n \frac{(y_k - y_{k-1})^2}{t_k - t_{k-1}}\right) \end{aligned}$$

which is the formula claimed in the statement.

- c) We have $f_{X|Y}(x, y) = f_{(X,Y)}(x, y)/f_Y(y)$, $X = (B_{t_1}, \dots, B_{t_n})$ and $Y = B_1$. Now observe that

$$\sqrt{2\pi} g_{t_1}(x_1) \cdot g_{t_2-t_1}(x_2 - x_1) \cdot \dots \cdot g_{t_n-t_{n-1}}(x_n - x_{n-1}) \cdot g_{1-t_n}(x_n - x_{n+1})$$

$$\begin{aligned}
 &= \frac{g_{t_1}(x_1) \cdot g_{t_2-t_1}(x_2-x_1) \cdot \dots \cdot g_{t_n-t_{n-1}}(x_n-x_{n-1}) \cdot g_{1-t_n}(x_n)}{g_1(x_{n+1})} \Bigg|_{x_{n+1}=0} \\
 &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \Bigg|_{y=0} = f_{X|Y=0}(x,0).
 \end{aligned}$$

- d) Note that W' is again Gaussian – the time reversal does not change the argument which we have used in the first part of this problem – with mean zero. $W \sim W'$ since they have the same covariance function. Indeed, let $s < t$, so $1-t < 1-s$

$$\begin{aligned}
 \mathbb{E} W_{1-t} W_{1-s} &= (1-t) \wedge (1-s) - (1-s)(1-t) = (1-t) - (1-s)(1-t) \\
 &= s(1-t) = s \wedge t - st = \mathbb{E} W_s W_t.
 \end{aligned}$$

■ ■

Problem 4.7. Solution:

- a) By the definition of B_t and θ_h , we have for all $w \in \mathcal{C}$

$$B_t(\theta_h w) \stackrel{\text{def. } B}{=} (\theta_h w)(t) \stackrel{\text{def. } \theta}{=} w(t+h) \stackrel{\text{def. } B}{=} B_{t+h}(w).$$

Thus,

$$\begin{aligned}
 \theta_h^{-1}(\{B_t \in C\}) &= \{\theta_h^{-1}(w) : B_t(w) \in C\} \\
 &= \{v : B_t(\theta_h v) \in C\} \\
 &= \{v : B_{t+h}(v) \in C\} \\
 &= \{B_{t+h} \in C\}.
 \end{aligned}$$

- b) We have to show that for all $F \in \mathcal{F}_t$ we have $\theta_h^{-1}(F) \in \mathcal{F}_{t+h}$. It is enough to do this for a generator of \mathcal{F}_t , and we choose the generator whose sets are of the form

$$\begin{aligned}
 \{B_{t_1} \in C_1, \dots, B_{t_n} \in C_n\} &= \bigcap_{k=1}^n \{B_{t_k} \in C_k\}, \\
 n \in \mathbb{N}, 0 \leq t_1 < \dots < t_n \leq t, C_1, \dots, C_n &\in \mathcal{B}(\mathbb{R}).
 \end{aligned}$$

The calculation of the first part shows

$$\theta_h^{-1} \{B_{t_1} \in C_1, \dots, B_{t_n} \in C_n\} = \{B_{t_1+h} \in C_1, \dots, B_{t_n+h} \in C_n\} \in \mathcal{F}_{[h, t+h]} \subset \mathcal{F}_{t+h}.$$

The $\mathcal{F}_{[t+h, \infty)} / \mathcal{F}_{[t, \infty)}$ measurability is shown in a similar way, considering now instances $t \leq t_1 < t_2 < \dots < t_n < \infty$.

- c) We set $\tau = \tau_C$. We have, by definition

$$\begin{aligned}
 \theta_h \tau(w) &\stackrel{\text{def.}}{=} \tau(\theta_h w) \\
 &\stackrel{\text{def.}}{=} \inf \{s \geq 0 : B_s(\theta_h w) \in C\} \\
 &= \inf \{s \geq 0 : B_{s+h}(w) \in C\}
 \end{aligned}$$

$$\begin{aligned} &= \inf \{s + h \geq h : B_{s+h}(w) \in C\} - h \\ &= \inf \{t \geq h : B_t(w) \in C\} - h. \end{aligned}$$

If we happen to know that $\tau \geq h$, then $\inf \{t \geq h : B_t(w) \in C\} = \tau$.

We will see later, in Chapter 5.2, that for certain sets C the random times τ_C are measurable, i.e. random variables.

Problem 4.8. Solution:

- a) We should first talk about the spaces and σ -algebras involved here: $w \in \mathcal{C}$ and the σ -algebra is $\mathcal{B}(\mathcal{C})$. Write $\Phi : w \mapsto B(\cdot, w)$. We want to work with a generator of $\mathcal{B}(\mathcal{C})$, i.e. sets of the built $\Gamma = \{w : w(t_1) \in C_1, \dots, w(t_n) \in C_n\}$ where the C_i are Borel sets in \mathbb{R} and $0 \leq t_1 < t_2 < \dots < t_n$, $n \in \mathbb{N}$. Since we have

$$\begin{aligned} \Phi^{-1}(\Gamma) &= \{w \in \mathcal{C} : (B_t(w))_{t \geq 0} \in C\} \\ &= \{w \in \mathcal{C} : w(t_1) \in C_1, \dots, w(t_n) \in C_n\} \\ &= \bigcap_{k=1}^n \{B_{t_k} \in C_k\} \in \mathcal{B}(\mathcal{C}) \end{aligned}$$

we see that Φ is measurable.

- b) Let us write $\mathbb{P}^x := \mathbb{P}^{\delta_x}$. Then the process described by \mathbb{P}^x is a $(W_t)_{t \geq 0}$ Brownian motion which does not visit the sets C_1, \dots, C_n at the times t_1, \dots, t_n but the shifted sets $C_1 - x, \dots, C_n - x$. This means that $B_t = x + W_t$. From this insight it is clear that B behaves again like a Brownian motion.
- c) π is the “initial distribution” of B , i.e. the law of B_0 . This is immediately seen from the definition of \mathbb{P}^π upon taking $n = 0$, $t_0 = 0$. Another way to think about this is to take a standard BM¹ $W = (W_t)_{t \geq 0}$ (starting at $W_0 = 0$), a random variable $X \sim \pi$ being independent of W and setting $B_t := X + W_t$.

Problem 4.9. Solution: (I owe this solution to Dr. Franziska Kühn) Since we are on the canonical path space $\Omega = \mathcal{C}_{(0)}$, the following “stopping operators” exist and satisfy

$$a_t \omega(s) := (a_t \omega)(s) := \omega(t \wedge s) \implies B_s(a_t \omega) = B_{t \wedge s}(\omega).$$

We **claim** that $\mathcal{F}_t = a_t^{-1}(\mathcal{F}_\infty)$.

Indeed: We show first that $a_t^{-1}(\mathcal{F}_\infty) \subset \mathcal{F}_t$, i.e. a_t is $\mathcal{F}_t/\mathcal{F}_\infty$ measurable. Since $\mathcal{F}_\infty = \sigma(B_s, s \geq 0)$, it is enough to show that $B_s \circ a_t : \Omega \rightarrow \mathbb{R}$ is $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$ measurable for each $s \geq 0$. This follows from

$$\{B_s \circ a_t \in A\} = \{B_{s \wedge t} \in A\} \in \mathcal{F}_{s \wedge t} \subset \mathcal{F}_t.$$

For the converse, it is enough to show that B_s is $\mathcal{G}_t := a_t^{-1}(\mathcal{F}_\infty)$ measurable for any $s \leq t$. But this follows from

$$\{B_s \in A\} = \{B_{t \wedge s} \in A\} = \{B_s \circ a_t \in A\} = \{a_t \in B_s^{-1}(A)\} \in a_t^{-1}(\mathcal{F}_\infty),$$

which holds for any Borel set $A \in \mathcal{B}(\mathbb{R})$.

Now we can turn to the question at hand.

a) \Rightarrow b): Fix $t > 0$ and take $F \in \mathcal{F}_t$. Using the claim, we find some $\Gamma \in \mathcal{F}_\infty$ such that $F = a_t^{-1}(\Gamma)$. Assume that $\omega \in F$ and $\omega' \in \Omega$ are such that $B_s(\omega) = B_s(\omega')$ for all $s \leq t$. Then we have $\omega|_{[0,t]} = \omega'|_{[0,t]}$, i.e. $a_t\omega = a_t\omega'$. This shows that

$$\mathbb{1}_F(\omega') = \mathbb{1}_{a_t^{-1}(\Gamma)}(\omega') = \mathbb{1}_\Gamma(a_t\omega') = \mathbb{1}_\Gamma(a_t\omega) = \mathbb{1}_F(\omega) = 1,$$

i.e. we have $\omega' \in F$.

b) \Rightarrow a): The assumption says that $\omega \in F \iff a_t\omega \in F$. Thus,

$$\mathbb{1}_F(\omega) = \mathbb{1}_F(a_t\omega) = \mathbb{1}_{a_t^{-1}(F)}(\omega) \implies F = a_t^{-1}(F) \implies F \in \mathcal{F}_t.$$

■■

5 Brownian motion as a martingale

Problem 5.1. Solution:

a) We have

$$\mathcal{F}_t^B \subset \sigma(\sigma(X), \mathcal{F}_t^B) = \sigma(X, B_s : s \leq t) = \widetilde{\mathcal{F}}_t.$$

Let $s \leq t$. Then $\sigma(B_t - B_s)$, \mathcal{F}_s^B and $\sigma(X)$ are independent, thus $\sigma(B_t - B_s)$ is independent of $\sigma(\sigma(X), \mathcal{F}_s^B) = \widetilde{\mathcal{F}}_s$. This shows that $(\widetilde{\mathcal{F}}_t)_{t \geq 0}$ is an admissible filtration for $(B_t)_{t \geq 0}$.

b) Set $\mathcal{N} := \{N : \exists M \in \mathcal{A} \text{ such that } N \subset M, \mathbb{P}(M) = 0\}$. Then we have

$$\mathcal{F}_t^B \subset \sigma(\mathcal{F}_t^B, \mathcal{N}) = \overline{\mathcal{F}}_t^B.$$

From measure theory we know that $(\Omega, \mathcal{A}, \mathbb{P})$ can be completed to $(\Omega, \mathcal{A}^*, \mathbb{P}^*)$ where

$$\begin{aligned} \mathcal{A}^* &:= \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}, \\ \mathbb{P}^*(A^*) &:= \mathbb{P}(A) \text{ for } A^* = A \cup N \in \mathcal{A}^*. \end{aligned}$$

We find for all $A \in \mathcal{B}(\mathbb{R}^d)$, $F \in \mathcal{F}_s$, $N \in \mathcal{N}$

$$\begin{aligned} \mathbb{P}^*(\{B_t - B_s \in A\} \cap (F \cup N)) &= \mathbb{P}^*(\underbrace{(\{B_t - B_s \in A\} \cap F)}_{\in \mathcal{A}} \cup \underbrace{(\{B_t - B_s \in A\} \cap N)}_{\in \mathcal{N}}) \\ &= \mathbb{P}(\{B_t - B_s \in A\} \cap F) \\ &= \mathbb{P}(B_t - B_s \in A) \mathbb{P}(F) \\ &= \mathbb{P}^*(B_t - B_s \in A) \mathbb{P}^*(F \cup N). \end{aligned}$$

Therefore $\overline{\mathcal{F}}_t^B$ is admissible. ■ ■

Problem 5.2. Solution: Let $t = t_0 < \dots < t_n$, and consider the random variables

$$B(t_1) - B(t_0), \dots, B(t_n) - B(t_{n-1}).$$

Using the argument of Problem 21 we see for any $F \in \mathcal{F}_t$

$$\begin{aligned} \mathbb{E}\left(e^{i \sum_{k=1}^n \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle} \mathbf{1}_F\right) &= \mathbb{E}\left(e^{i \langle \xi_n, B(t_n) - B(t_{n-1}) \rangle} \cdot \underbrace{e^{i \sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle}}_{\substack{\mathcal{F}_{t_{n-1}} \text{ mble.}, \\ \text{hence } \mathbb{1}_{B(t_n) - B(t_{n-1})}}} \mathbf{1}_F\right) \\ &= \mathbb{E}\left(e^{i \langle \xi_n, B(t_n) - B(t_{n-1}) \rangle}\right) \cdot \mathbb{E}\left(e^{i \sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1}) \rangle} \mathbf{1}_F\right) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & = \prod_{k=1}^n \mathbb{E} \left(e^{i(\xi_k, B(t_k) - B(t_{k-1}))} \right) \mathbb{E} \mathbb{1}_F. \end{aligned}$$

This shows that the increments are independent among themselves (use $F = \Omega$) and that they are all together independent of \mathcal{F}_t (use the above calculation and the fact that the increments are among themselves independent to combine again the \prod_1^n under the expected value)

Thus,

$$\mathcal{F}_t \perp \sigma(B(t_k) - B(t_{k-1}) : k = 1, \dots, n)$$

Therefore the statement is implied by

$$\mathcal{F}_t \perp \bigcup_{\substack{t < t_1 < \dots < t_n \\ n \geq 1}} \sigma(B(t_k) - B(t) : k = 1, \dots, n).$$

■ ■

Problem 5.3. Solution:

- a) i) $\mathbb{E}|X_t| < \infty$, since the expectation does not depend on the filtration.
- ii) X_t is \mathcal{F}_t measurable and $\mathcal{F}_t \subset \mathcal{F}_t^*$. Thus X_t is \mathcal{F}_t^* measurable.
- iii) Let \mathcal{N} denote the set of all sets which are subsets of \mathbb{P} -null sets. Denote by \mathbb{P}^* the measure of the completion of $(\Omega, \mathcal{A}, \mathbb{P})$ (compare with the solution to Exercise 1.b)).

Let $t \geq s$. For all $F^* \in \mathcal{F}_s^*$ there exist $F \in \mathcal{F}_s$, $N \in \mathcal{N}$ such that $F^* = F \cup N$ and

$$\int_{F^*} X_s d\mathbb{P}^* = \int_F X_s d\mathbb{P} = \int_F X_t d\mathbb{P} = \int_{F^*} X_t d\mathbb{P}^*.$$

Since F^* is arbitrary this implies that $\mathbb{E}(X_t | F_s^*) = X_s$.

- b) i) $\mathbb{E}|Y_t| = \mathbb{E}|X_t| < \infty$.
- ii) Note that $\{X_t \neq Y_t\}$, its complement and any of its subsets is in \mathcal{F}_t^* . Let $B \in \mathcal{B}(\mathbb{R}^d)$. Then we get

$$\{Y_t \in B\} = \underbrace{(\{X_t \in B\})}_{\in \mathcal{F}_t} \cap \underbrace{(\{X_t \neq Y_t\}^c)}_{\in \mathcal{F}_t^*} \cup \underbrace{\{Y_t \in B, X_t \neq Y_t\}}_{\in \mathcal{F}_t^*}.$$

- iii) Similar to part a-iii). For each $F^* \in \mathcal{F}_s^*$ we get

$$\int_{F^*} Y_s d\mathbb{P}^* = \int_{F^*} X_s d\mathbb{P}^* \stackrel{\text{a)}}{=} \int_{F^*} X_t d\mathbb{P}^* = \int_{F^*} Y_t d\mathbb{P}^*,$$

i.e. $\mathbb{E}(Y_t | \mathcal{F}_s^*) = Y_s$.

■ ■

Problem 5.4. Solution: Let $s < t$ and pick $s_n \downarrow s$ such that $s < s_n < t$. Then

$$\mathbb{E}(X_t | \mathcal{F}_{s+}) \stackrel{\text{sub-MG}}{\leftarrow}_{s_n \downarrow s} \mathbb{E}(X(t) | \mathcal{F}_{s_n}) \geq X(s_n) \xrightarrow[n \rightarrow \infty]{\text{a.e.}} X(s+) \stackrel{\text{continuous}}{\equiv}_{\text{paths}} X(s).$$

The convergence on the left side follows from the (sub-)martingale convergence theorem (Lévy's downward theorem).

Problem 5.5. Solution: Here is a direct proof without using the hint.

We start with calculating the conditional expectations

$$\begin{aligned} \mathbb{E}(B_t^4 | \mathcal{F}_s) &= \mathbb{E}((B_t - B_s + B_s)^4 | \mathcal{F}_s) \\ &= B_s^4 + 4B_s^3 \mathbb{E}(B_t - B_s) + 6B_s^2 \mathbb{E}((B_t - B_s)^2) + 4B_s \mathbb{E}((B_t - B_s)^3) + \mathbb{E}((B_t - B_s)^4) \\ &= B_s^4 + 6B_s^2(t - s) + 3(t - s)^2 \\ &= B_s^4 - 6B_s^2s + 6B_s^2t + 3(t - s)^2, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(B_t^2 | \mathcal{F}_s) &= \mathbb{E}((B_t - B_s + B_s)^2 | \mathcal{F}_s) \\ &= t - s + 2B_s \mathbb{E}(B_t - B_s) + B_s^2 \\ &= B_s^2 + t - s. \end{aligned}$$

Combining these calculations, such that the term $6B_s^2t$ vanishes from the first formula, we get

$$\begin{aligned} \mathbb{E}(B_t^4 - 6tB_t^2 | \mathcal{F}_s) &= B_s^4 - 6sB_s^2 - 6t^2 + 6st + 3t^2 - 6st + 3s^2 \\ &= B_s^4 - 6sB_s^2 + 3s^2 - 3t^2. \end{aligned}$$

Therefore $\pi(t, B_t) := B_t^4 - 6tB_t^2 + 3t^2$ is a martingale.

Problem 5.6. Solution:

- a) Since Brownian motion has exponential moments of any order, we can use the differentiation lemma for parameter-dependent integrals. Following the instructions we get

$$\begin{aligned} \frac{d}{d\xi} e^{\xi B_t - \frac{t}{2}\xi^2} &= (B_t - t\xi) M_t^\xi \\ \frac{d^2}{d\xi^2} e^{\xi B_t - \frac{t}{2}\xi^2} &= ((B_t - t\xi)^2 - t) M_t^\xi \\ \frac{d^3}{d\xi^3} e^{\xi B_t - \frac{t}{2}\xi^2} &= ((B_t - t\xi)^2 - 3t) (B_t - t\xi) M_t^\xi \end{aligned}$$

$$\frac{d^4}{d\xi^4} e^{\xi B_t - \frac{t}{2}\xi^2} = \{((B_t - t\xi)^2 - 3t)((B_t - t\xi)^2 - t) - 2t(B_t - t\xi)^2\} M_t^\xi$$

and so on. The recursion $n \rightarrow n + 1$ is pretty obvious

$$\frac{d}{d\xi} P_n(B, \xi) M_t^\xi = \underbrace{\left[\frac{d}{d\xi} P_n(B, \xi) + P_n(B, \xi)(B - t\xi) \right]}_{P_{n+1}(B, \xi)} M_t^\xi.$$

If we set $\xi = 0$ we find that $P_n(b, 0)|_{b=B_t}$ is a martingale. In particular,

$$\begin{aligned} B_t \\ B_t^2 - t \\ B_t^3 - 3tB_t \\ B_t^4 - 6tB_t^2 + 3t^2 \end{aligned}$$

are martingales.

b) Part (a) shows the general recursion scheme

$$P_1(b, \xi) = b - t\xi, \quad P_{n+1}(b, \xi) = \frac{d}{d\xi} P_n(b, \xi) + (b - t\xi)P_n(b, \xi).$$

c) Using the fact that $M_t := B_t^4 - 6tB_t^2 + 3t^2$ is a martingale with $M_0 = 0$ we get for the bounded stopping times $\tau \wedge n$ by optional stopping

$$0 = \mathbb{E}[M_{\tau \wedge n}] = \mathbb{E}[B_{\tau \wedge n}^4] - 6\mathbb{E}[(\tau \wedge n)B_{\tau \wedge n}^2] + 3\mathbb{E}[(\tau \wedge n)^2]$$

and, by rearranging this equality, and with the Cauchy-Schwarz inequality

$$\begin{aligned} 3\mathbb{E}[(\tau \wedge n)^2] &= 6\mathbb{E}[(\tau \wedge n)B_{\tau \wedge n}^2] - \mathbb{E}[B_{\tau \wedge n}^4] \\ &\leq 6\mathbb{E}[(\tau \wedge n)B_{\tau \wedge n}^2] \\ &\leq 6\sqrt{\mathbb{E}[(\tau \wedge n)^2]}\sqrt{\mathbb{E}[B_{\tau \wedge n}^4]}. \end{aligned}$$

Thus,

$$\sqrt{\mathbb{E}[(\tau \wedge n)^2]} \leq 2\sqrt{\mathbb{E}[B_{\tau \wedge n}^4]}.$$

Since $|B_{\tau \wedge n}| \leq \max\{a, b\}$, we can use monotone convergence (on the left side) and dominated convergence (on the right), and the first inequality follows.

The second inequality follows in a similar way: By optional stopping we get

$$\begin{aligned} \mathbb{E}[B_{\tau \wedge n}^4] &= 6\mathbb{E}[(\tau \wedge n)B_{\tau \wedge n}^2] - 3\mathbb{E}[(\tau \wedge n)^2] \\ &\leq 6\mathbb{E}[(\tau \wedge n)B_{\tau \wedge n}^2] \\ &\leq 6\sqrt{\mathbb{E}[(\tau \wedge n)^2]}\sqrt{\mathbb{E}[B_{\tau \wedge n}^4]}. \end{aligned}$$

Thus,

$$\sqrt{\mathbb{E}[B_{\tau \wedge n}^4]} \leq 6\sqrt{\mathbb{E}[(\tau \wedge n)^2]} \leq 6\sqrt{\mathbb{E}[\tau^2]}$$

and the estimate follows from dominated convergence (on the left).

Problem 5.7. Solution: For $t = 0$ and all c we have

$$\mathbb{E} e^{c|B_0|} = \mathbb{E} e^{c|B_0|^2} = 1.$$

and for $c \leq 0$

$$\mathbb{E} e^{c|B_0|} \leq 1 \quad \text{and} \quad \mathbb{E} e^{c|B_0|^2} \leq 1.$$

Now let $t > 0$ and $c > 0$. There exists some $R > 0$ such that $c|x| < \frac{1}{4t}|x|^2$ for all $|x| > R$. Thus

$$\begin{aligned} \mathbb{E} e^{c|B_t|} &= \tilde{c} \int e^{c|x|} e^{-\frac{1}{2t}|x|^2} dx \\ &\leq \tilde{c} \int_{|x| \leq R} e^{c|x|} e^{-\frac{1}{2t}|x|^2} dx + \tilde{c} \int_{|x| > R} e^{\frac{1}{4t}|x|^2} e^{-\frac{1}{2t}|x|^2} dx \\ &\leq e^{cR} + \tilde{c} \int_{|x| > R} e^{-\frac{1}{4t}|x|^2} dx < \infty, \end{aligned}$$

i.e., $\mathbb{E} e^{c|B_t|} < \infty$ for all c, t . Furthermore

$$\mathbb{E} e^{c|B_t|^2} = \tilde{c} \int e^{c|x|^2 - \frac{1}{2t}|x|^2} dx = \tilde{c} \int e^{|x|^2(c - \frac{1}{2t})} dx$$

and this integral is finite if, and only if, $c - \frac{1}{2t} < 0$ or equivalently $c < \frac{1}{2t}$.

Problem 5.8. Solution:

a) We have $p(t, x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$. By the chain rule we get

$$\frac{\partial}{\partial t} p(t, x) = -\frac{d}{2} t^{-\frac{d}{2}-1} (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}} + (2\pi t)^{-\frac{d}{2}} (-1) t^{-2} (-1) \frac{|x|^2}{2} e^{-\frac{|x|^2}{2t}}$$

and for all $j = 1, \dots, d$

$$\begin{aligned} \frac{\partial}{\partial x_j} p(t, x) &= (2\pi t)^{-\frac{d}{2}} \left(-\frac{2x_j}{2t} \right) e^{-\frac{|x|^2}{2t}}, \\ \frac{\partial^2}{\partial x_j^2} p(t, x) &= (2\pi t)^{-\frac{d}{2}} \left(-\frac{1}{t} \right) e^{-\frac{|x|^2}{2t}} + (2\pi t)^{-\frac{d}{2}} \frac{x_j^2}{t^2} e^{-\frac{|x|^2}{2t}}. \end{aligned}$$

Adding these terms and noting that $|x|^2 = \sum_{j=1}^d x_j^2$ we get

$$\frac{1}{2} \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} p(t, x) = -\frac{d}{2} (2\pi t)^{-\frac{d}{2}} t^{-1} e^{-\frac{|x|^2}{2t}} + \frac{(2\pi t)^{-\frac{d}{2}}}{2} \frac{|x|^2}{t^2} e^{-\frac{|x|^2}{2t}} = \frac{\partial}{\partial t} p(t, x).$$

b) A formal calculation yields

$$\begin{aligned} &\int p(t, x) \frac{1}{2} \frac{\partial^2}{\partial x_j^2} f(t, x) dx \\ &= p(t, x) \frac{1}{2} \frac{\partial}{\partial x_j} f(t, x) \Big|_{-\infty}^{\infty} - \int \frac{\partial}{\partial x_j} p(t, x) \cdot \frac{1}{2} \frac{\partial}{\partial x_j} f(t, x) dx \end{aligned}$$

$$\begin{aligned}
 &= 0 - \frac{\partial}{\partial x_j} p(t, x) \cdot \frac{1}{2} f(t, x) \Big|_{-\infty}^{\infty} + \int \frac{\partial^2}{\partial x_j^2} p(t, x) \cdot \frac{1}{2} f(t, x) dx \\
 &= \int \frac{\partial^2}{\partial x_j^2} p(t, x) \cdot \frac{1}{2} f(t, x) dx.
 \end{aligned}$$

By the same arguments as in Exercise 7 we find that all terms are integrable and vanish as $|x| \rightarrow \infty$. This justifies the above calculation. Furthermore summing over $j = 1, \dots, d$ we obtain the statement. ■ ■

Problem 5.9. Solution: Note that $\mathbb{E}|X_t| < \infty$ for all a, b , cf. Problem 5.7. We have

$$\begin{aligned}
 \mathbb{E} \left(e^{aB_t + bt} \mid \mathcal{F}_s \right) &= \mathbb{E} \left(e^{a(B_t - B_s)} e^{aB_s + bt} \mid \mathcal{F}_s \right) \\
 &= e^{aB_s + bt} \mathbb{E} e^{aB_{t-s}} \\
 &= e^{aB_s + bt + (t-s)a^2/2}.
 \end{aligned}$$

Thus, X_t is a martingale if, and only if, $bs = bt + (t-s)\frac{a^2}{2}$, i.e., $b = -\frac{1}{2}a^2$. ■ ■

Problem 5.10. Solution: Measurability (i.e. adaptedness to the Filtration \mathcal{F}_t) and integrability is no issue, see also Problem 5.7.

a) U_t is only a martingale for $c = 0$.

Solution 1: see Exercise 9.

Solution 2: if $c \neq 0$, $\mathbb{E}U_t$ is not constant, i.e. cannot be a martingale. If $c = 0$, U_t is trivially a martingale.

b) V_t is a martingale since

$$\begin{aligned}
 \mathbb{E} \left(V_t \mid \mathcal{F}_s \right) &= t \mathbb{E}(B_t - B_s) + tB_s - \mathbb{E} \left(\int_0^s B_r dr \mid \mathcal{F}_s \right) - \mathbb{E} \left(\int_s^t B_r dr \mid \mathcal{F}_s \right) \\
 &= tB_s - \int_0^s B_r dr - \mathbb{E} \left(\int_s^t (B_r - B_s) + B_s dr \mid \mathcal{F}_s \right) \\
 &= tB_s - \int_0^s B_r dr - (t-s)B_s \\
 &= V_s.
 \end{aligned}$$

(c) and (e) Let $a \in \mathbb{R}$. Then we get

$$\begin{aligned}
 \mathbb{E} \left(aB_t^3 - tB_t \mid \mathcal{F}_s \right) &= \mathbb{E} \left(a(B_t - B_s + B_s)^3 - t(B_t - B_s) - tB_s \mid \mathcal{F}_s \right) \\
 &= aB_s^3 + 3aB_s^2 \mathbb{E} B_{t-s} + 3aB_s \mathbb{E} B_{t-s}^2 + a \mathbb{E} B_{t-s}^3 - 0 - tB_s \\
 &= aB_s^3 + (3a(t-s) - t)B_s.
 \end{aligned}$$

This is a martingale if, and only if, $-s = 3a(t-s) - t$, i.e., $a = \frac{1}{3}$. Thus Y_t is a martingale and W_t is not a martingale.

d) We have seen in part c) and b) that

$$\mathbb{E}(B_t^3 | \mathcal{F}_s) = B_s^3 + 3(t-s)B_s$$

and

$$3\mathbb{E}\left(\int_0^t B_r dr \middle| \mathcal{F}_s\right) = 3\int_0^s B_r dr + 3(t-s)B_s.$$

Thus, X_t is a martingale.

(f) Z_t is only a martingale for $c = \frac{1}{2}$, see Exercise 9.

Problem 5.11. Solution: Let $s < t$. We have

$$\begin{aligned} M_t &= f(t)B_t - \int_0^t f'(r)B_r dr \\ &= f(t)B_t - \int_0^s f'(r)B_r dr - \int_s^t f'(r)(B_r - B_s) dr - \int_s^t f'(r) dr B_s \\ &= f(s)B_s - \int_0^s f'(r)B_r dr - \int_s^t f'(r)(B_r - B_s) dr \\ &= M_s - \int_s^t f'(r)(B_r - B_s) dr. \end{aligned}$$

The integral term depends only on the increments $B_r - B_s$ with $r \geq s$, thus it is independent of \mathcal{F}_s . Taking conditional expectation yields, using pull-out and independence

$$\begin{aligned} \mathbb{E}[M_t | \mathcal{F}_s] &= M_s - \mathbb{E}\left[\int_s^t f'(r)(B_r - B_s) dr \middle| \mathcal{F}_s\right] \\ &= M_s - \mathbb{E}\left[\int_s^t f'(r)(B_r - B_s) dr\right] \\ &= M_s - \int_s^t f'(r)\mathbb{E}[(B_r - B_s)] dr = 0. \end{aligned}$$

Remark: A formal integration by parts yields that $M_t = \int_0^t f(s) dB_s$ (this is a Wiener–Itô integral). We will see in Remark 13.5 and later in Chapter 15 that this operation can be justified. Note that $t \mapsto B_t$ has unbounded variation (Corollary 9.2) and, therefore, the integral $\int_0^t f(s) dB_s$ cannot be defined for all continuous $f \in \mathcal{C}(\mathbb{R})$ using the classical approach, see Corollary A.36 and Remark A.37.

Problem 5.12. Solution: We have

$$\mathbb{E}\left(\frac{1}{d}|B_t|^2 - t \middle| \mathcal{F}_s\right) = -t + \frac{1}{d} \sum_{j=1}^d \mathbb{E}\left(\left(B_t^{(j)}\right)^2 \middle| \mathcal{F}_s\right) \stackrel{\text{Pr. 5}}{=} -t + \frac{1}{d} \sum_{j=1}^d \left(\left(B_s^{(j)}\right)^2 + t - s\right) = \frac{1}{d}|B_s|^2 - s.$$

Problem 5.13. Solution: For a)–c) we prove only the statements for τ° , the statements for τ are proved analogously.

a) The following implications hold:

$$A \subset C \implies \{t \geq 0 : X_t \in A\} \subset \{t \geq 0 : X_t \in C\} \implies \tau_A^\circ \geq \tau_C^\circ.$$

b) By part a) we have $\tau_{A \cup C}^\circ \leq \tau_A^\circ$ and $\tau_{A \cup C}^\circ \leq \tau_C^\circ$. Thus,

$$\tau_{A \cup C}^\circ \stackrel{a)}{\leq} \min\{\tau_A^\circ, \tau_C^\circ\}.$$

To see the converse, $\min\{\tau_A^\circ, \tau_C^\circ\} \leq \tau_{A \cup C}^\circ$, it is enough to show that

$$X_t(\omega) \in A \cup C \implies t \geq \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}$$

since this implication shows that $\tau_{A \cup C}^\circ(\omega) \geq \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}$ holds.

Now observe that

$$\begin{aligned} X_t(\omega) \in A \cup C &\implies X_t(\omega) \in A \text{ or } X_t(\omega) \in C \\ &\implies t \geq \tau_A^\circ(\omega) \text{ or } t \geq \tau_C^\circ(\omega) \\ &\implies t \geq \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}. \end{aligned}$$

c) Part a) implies $\max\{\tau_A^\circ, \tau_C^\circ\} \leq \tau_{A \cap C}^\circ$.

Remark: we cannot expect “=” . To see this consider a BM¹ starting at $B_0 = 0$ and the set

$$A = [4, 6] \quad \text{and} \quad C = [1, 2] \cup [5, 7].$$

Then B_t has to reach first C and A before it hits $A \cap C$.

d) as in b) it is clear that $\tau_A^\circ \leq \tau_{A_n}^\circ$ for all $n \geq 1$, hence

$$\tau_A^\circ \leq \inf_{n \geq 1} \tau_{A_n}^\circ.$$

In order to show the converse, $\tau_A^\circ \geq \inf_{n \geq 1} \tau_{A_n}^\circ$, it is enough to check that

$$X_t(\omega) \in A \implies t \geq \inf_{n \geq 1} \tau_{A_n}^\circ(\omega)$$

since, if this is true, this implies that $\tau_A^\circ(\omega) \geq \inf_{n \geq 0} \tau_{A_n}^\circ(\omega)$.

Now observe that

$$\begin{aligned} X_t(\omega) \in A = \cup_n A_n &\implies X_t(\omega) \in A_n \text{ for some } n \in \mathbb{N} \\ &\implies t \geq \tau_{A_n}^\circ(\omega) \text{ for some } n \in \mathbb{N} \\ &\implies t \geq \inf_{n \geq 0} \tau_{A_n}^\circ(\omega). \end{aligned}$$

e) Note that $\inf\{s \geq 0 : X_{s+\frac{1}{n}} \in A\} = \inf\{s \geq \frac{1}{n} : X_s \in A\}$ is monotone decreasing as $n \rightarrow \infty$. Thus we get

$$\begin{aligned} \inf_n \left(\frac{1}{n} + \inf\{s \geq \frac{1}{n} : X_s \in A\} \right) &= 0 + \inf_n \inf\{s \geq \frac{1}{n} : X_s \in A\} \\ &= \inf\{s > 0 : X_s \in A\} \\ &= \tau_A. \end{aligned}$$

f) Let $X_t = x_0 + t$. Then $\tau_{\{x_0\}}^\circ = 0$ and $\tau_{\{x_0\}} = \infty$.

More generally, a similar situation may happen if we consider a process with continuous paths, a closed set F , and if we let the process start on the boundary ∂F . Then $\tau_F^\circ = 0$ a.s. (since the process is in the set) while $\tau_F > 0$ is possible with positive probability.

Problem 5.14. Solution: We have $\tau_U^\circ \leq \tau_U$.

Let $x_0 \in U$. Then $\tau_U^\circ = 0$ and, since U is open and X_t is continuous, there exists an $N > 0$ such that

$$X_{\frac{1}{n}} \in U \text{ for all } n \geq N.$$

Thus $\tau_U = 0$.

If $x_0 \notin U$, then $X_t(\omega) \in U$ can only happen if $t > 0$. Thus, $\tau_U^\circ = \tau_U$.

Problem 5.15. Solution: Suppose $d(x, A) \geq d(z, A)$. Then

$$\begin{aligned} d(x, A) - d(z, A) &= \inf_{y \in A} |x - y| - \inf_{y \in A} |z - y| \\ &\leq \inf_{y \in A} (|x - z| + |z - y|) - \inf_{y \in A} |z - y| \\ &= |x - z| \end{aligned}$$

and, with an analogous argument for $d(x, A) \leq d(z, A)$, we conclude

$$|d(x, A) - d(z, A)| \leq |x - z|.$$

Thus $x \mapsto d(x, A)$ is globally Lipschitz continuous, hence uniformly continuous.

Problem 5.16. Solution: It is enough to show that τ_F° is a stopping time for \mathcal{F}_{t+} , the assertion for τ_F to be a stopping time for \mathcal{F}_t now follows as in the proof of Lemma 5.8.

In order not to run into measurability problems, we assume that we work with a filtration \mathcal{F}_t which contains all subsets of measurable null sets.

Let F be a closed set and approximate F by a sequence of open sets $U_n = F + \mathbb{B}(0, 1/n) = \{x \in \mathbb{R}^d : \inf_{y \in F} |x - y| < \frac{1}{n}\}$. Clearly, $U_n \downarrow F$ i.e. $F = \bigcap_n U_n = \bigcap_n \overline{U}_n$.

Define a further stopping time $\sigma := \sup_n \tau_{U_n}^\circ$ and note that $\sigma \leq \tau_F^\circ$. The last inequality is due to the fact that $F \subset U_n$.

Let us show the converse inequality.

Claim: On $\{\tau_A^\circ < \infty\}$ we have $X(\tau_A^\circ) \in \overline{A}$ for any set $A \subset \mathbb{R}^d$. Indeed, if $\omega \in \{\tau_A^\circ < \infty\}$ we know that

$$\forall \delta > 0 \quad \exists t \in [\tau_A^\circ(\omega), \tau_A^\circ(\omega) + \delta) : X_t(\omega) \in A$$

and, by right continuity, $X(\tau_A^\circ) \in \overline{A}$.

Now we apply this to $A = U_n$ and see that $X(\tau_{U_n}^\circ) \in \overline{U_n}$ for all $n \in \mathbb{N}$. Using quasi left continuity (qlc), we get, a.s. on $\{0 < \sigma < \infty\}$

$$X(\sigma) \stackrel{\text{qlc}}{=} \lim_n X(\tau_{U_n}^\circ) \in \bigcap_n \overline{U_n} = F.$$

Consequently, $\sigma \geq \tau_F^\circ$ on $\{0 < \sigma < \infty\}$.

If $\sigma = 0$, then $\tau_{U_n}^\circ = 0$ for all n and $X_0 \in F$, so $\sigma \geq \tau_F^\circ$ on $\{\sigma = 0\}$.

If $\sigma = \infty$, then $\tau_F^\circ = \infty$ on $\{\sigma = \infty\}$.

Together, we see that $\tau_F^\circ = \lim_n \tau_{U_n}^\circ$ and that $\{\tau_F^\circ \leq t\} = \bigcap_n \{\tau_{U_n}^\circ \leq t\} \in \mathcal{F}_t$ and $\{\tau_F^\circ = 0\} = \{X_0 \in F\}$.

■ ■

Problem 5.17. Solution: Denote by $\Omega_0 = \{\omega \in \Omega : t \mapsto X_t(\omega) \text{ is right continuous}\}$. Then the proof of Lemma 18 shows only that $\{\tau_U < t\} \cap \Omega_0 \in \Omega_0 \cap \mathcal{F}_t^X$. If we know that $\Omega_0 \in \mathcal{F}_t$ along with all subsets of measurable null sets, then we get $\{\tau_U < t\} \in \mathcal{F}_t$.

The same holds for the proof of Lemma 5.8.

■ ■

Problem 5.18. Solution: We treat the two cases simultaneously and check the three properties of a sigma algebra:

i) We have $\Omega \in \mathcal{F}_\infty$ and

$$\Omega \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathcal{F}_t \subset \mathcal{F}_{t+}.$$

ii) Let $A \in \mathcal{F}_{\tau(+)}$. Thus $A \in \mathcal{F}_\infty$, $A^c \in \mathcal{F}_\infty$ and

$$A^c \cap \{\tau \leq t\} = \Omega \setminus A \cap \{\tau \leq t\} = \underbrace{(\Omega \cap \{\tau \leq t\})}_{\in \mathcal{F}_t \subset \mathcal{F}_{t+}} \setminus \underbrace{(A \cap \{\tau \leq t\})}_{\in \mathcal{F}_{t(+) \text{ since } A \in \mathcal{F}_{\tau(+)}} \in \mathcal{F}_{t(+)}.$$

iii) Let $A_n \in \mathcal{F}_{\tau(+)}$. Then $A_n, \bigcup_n A_n \in \mathcal{F}_\infty$ and

$$\bigcup_n A_n \cap \{\tau \leq t\} = \bigcup_n \underbrace{(A_n \cap \{\tau \leq t\})}_{\in \mathcal{F}_{t(+)}} \in \mathcal{F}_{t(+)}.$$

Therefore \mathcal{F}_τ and $\mathcal{F}_{\tau+}$ are σ -algebras.

■ ■

Problem 5.19. Solution:

a) Let $F \in \mathcal{F}_{\tau+}$, i.e., $F \in \mathcal{F}_\infty$ and for all s we have $F \cap \{\tau \leq s\} \in \mathcal{F}_{s+}$.

Let $t > 0$. Then

$$F \cap \{\tau < t\} = \bigcup_{s < t} (F \cap \{\tau \leq s\}) \in \bigcup_{s < t} \mathcal{F}_{s+} \subset \mathcal{F}_t.$$

For the converse: Note that $\{\tau < \infty\} = \bigcup_{n \in \mathbb{N}} \{\tau \leq n\} \in \mathcal{F}_\infty$. If $\tau < \infty$ a.s. then $F = \bigcup_{t>0} (F \cap \{\tau \leq t\}) \in \mathcal{F}_\infty$ and

$$F \cap \{\tau \leq s\} = \bigcap_{t>s} (F \cap \{\tau < t\}) \in \bigcap_{t>s} \mathcal{F}_t = \mathcal{F}_{s+}.$$

If $\tau = \infty$ occurs with strictly positive probability, then we have to assume that $F \in \mathcal{F}_\infty$.

b) For all $s \geq 0$ we have

$$\{\tau \wedge t \leq s\} = \begin{cases} \{\tau \leq s\} \in \mathcal{F}_s, & \text{if } t > s; \\ \Omega \in \mathcal{F}_s, & \text{if } t \leq s; \end{cases}$$

so $\tau \wedge t$ is indeed a stopping time.

c) We have $\{\tau \leq t\} \in \mathcal{F}_t \subset \mathcal{F}_\infty$ and

$$\{\tau \leq t\} \cap \{\tau \wedge t \leq r\} = \begin{cases} \{\tau \leq t\} \in \mathcal{F}_t & \text{if } r \geq t; \\ \{\tau \leq r\} \in \mathcal{F}_r \subset \mathcal{F}_t & \text{if } r < t. \end{cases}$$

■

Problem 5.20. Solution:

a) $e^{i\xi B_t + \frac{1}{2} t|\xi|^2}$ is a martingale for all $\xi \in \mathbb{R}$ by Example 5.2 d). By optional stopping

$$1 = \mathbb{E} e^{\frac{1}{2} (\tau \wedge t)c^2 + icB_{\tau \wedge t}}.$$

Since the left-hand side is real, we get

$$1 = \mathbb{E} \left(e^{\frac{1}{2} (\tau \wedge t)c^2} \cos(cB_{\tau \wedge t}) \right).$$

Set $m := a \vee b$. Since $|B_{\tau \wedge t}| \leq m$, we see that for $mc < \frac{1}{2} \pi$ the cosine is positive. By Fatou's lemma we get for all $mc < \frac{1}{2} \pi$

$$\begin{aligned} 1 &= \underline{\lim}_{t \rightarrow \infty} \mathbb{E} \left(e^{\frac{1}{2} (\tau \wedge t)c^2} \cos(cB_{\tau \wedge t}) \right) \\ &\geq \mathbb{E} \left(\underline{\lim}_{t \rightarrow \infty} e^{\frac{1}{2} (\tau \wedge t)c^2} \cos(cB_{\tau \wedge t}) \right) \\ &\geq \mathbb{E} \left(e^{\frac{1}{2} \tau c^2} \cos(cB_\tau) \right) \\ &\geq \cos(mc) \mathbb{E} e^{\frac{1}{2} \tau c^2}. \end{aligned}$$

Thus, $\mathbb{E} e^{\gamma \tau} < \infty$ for any $\gamma < \frac{1}{2} c^2$ and all $c < \pi/(2m)$. Since

$$e^t = \sum_{j=0}^{\infty} \frac{t^j}{j!} \implies \forall t \geq 0, j \geq 0 : e^t \geq \frac{t^j}{j!}$$

we see that $\mathbb{E} \tau^j \leq j! \gamma^{-j} \mathbb{E} e^{\gamma \tau} < \infty$ for any $j \geq 0$.

b) By Exercise 10 d) the process $B_t^3 - 3 \int_0^t B_s ds$ is a martingale. By optional stopping we get

$$\mathbb{E}\left(B_{\tau \wedge t}^3 - 3 \int_0^{\tau \wedge t} B_s ds\right) = 0 \quad \text{for all } t \geq 0. \quad (*)$$

Set $m = \max\{a, b\}$. By the definition of τ we see that $|B_{\tau \wedge t}| \leq m$; since τ is integrable we get

$$|B_{\tau \wedge t}^3| \leq m^3 \quad \text{and} \quad \left| \int_0^{\tau \wedge t} B_s ds \right| \leq \tau \cdot m.$$

Therefore, we can use in (*) the dominated convergence theorem and let $t \rightarrow \infty$:

$$\begin{aligned} \mathbb{E}\left(\int_0^\tau B_s ds\right) &= \frac{1}{3} \mathbb{E}(B_\tau^3) \\ &= \frac{1}{3}(-a)^3 \mathbb{P}(B_\tau = -a) + \frac{1}{3} b^3 \mathbb{P}(B_\tau = b) \\ &\stackrel{(5.13)}{=} \frac{1}{3} \frac{-a^3 b + b^3 a}{a + b} \\ &= \frac{1}{3} ab(b - a). \end{aligned}$$

Problem 5.21. Solution: By Example 5.2 c) $|B_t|^2 - d \cdot t$ is a martingale. Thus we get by optional stopping

$$\mathbb{E}(t \wedge \tau_R) = \frac{1}{d} \mathbb{E}|B_{t \wedge \tau_R}|^2 \quad \text{for all } t \geq 0.$$

Since $|B_{t \wedge \tau_R}| \leq R$, we can use monotone convergence on the left and dominated convergence on the right-hand side to get

$$\mathbb{E} \tau_R = \sup_{t \geq 0} \mathbb{E}(t \wedge \tau_R) = \lim_{t \rightarrow \infty} \frac{1}{d} \mathbb{E}|B_{t \wedge \tau_R}|^2 = \frac{1}{d} \mathbb{E}|B_{\tau_R}|^2 = \frac{1}{d} R^2.$$

Problem 5.22. Solution:

a) For all t we have

$$\{\sigma \wedge \tau \leq t\} = \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_t} \cup \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t.$$

b) For all t we have

$$\begin{aligned} \{\sigma < \tau\} \cap \{\sigma \wedge \tau \leq t\} &= \bigcup_{0 \leq r \in \mathbb{Q}} (\{\sigma \leq r < \tau\} \cap \{\sigma \wedge \tau \leq t\}) \\ &= \bigcup_{r \in \mathbb{Q} \cap [0, t]} ((\{\sigma \leq r\} \cap \{\tau \leq r\}^c) \cap \{\sigma \wedge \tau \leq t\}) \in \mathcal{F}_t. \end{aligned}$$

This shows that $\{\sigma < \tau\}, \{\sigma \geq \tau\} = \{\sigma < \tau\}^c \in \mathcal{F}_{\sigma \wedge \tau}$. Since σ and τ play symmetric roles, we get with a similar argument that $\{\sigma > \tau\}, \{\sigma \leq \tau\} = \{\sigma > \tau\}^c \in \mathcal{F}_{\sigma \wedge \tau}$, and the claim follows.

- c) Since $\tau \wedge \sigma$ is an integrable stopping time, we get from Wald's identities, Theorem 5.10, that

$$\mathbb{E} B_{\tau \wedge \sigma}^2 = \mathbb{E}(\tau \wedge \sigma) < \infty.$$

Following the hint we get

$$\begin{aligned} \mathbb{E}(B_\sigma B_\tau \mathbf{1}_{\{\sigma \leq \tau\}}) &= \mathbb{E}(B_{\sigma \wedge \tau} B_\tau \mathbf{1}_{\{\sigma \leq \tau\}}) \\ &= \mathbb{E}\left(\mathbb{E}(B_{\sigma \wedge \tau} B_\tau \mathbf{1}_{\{\sigma \leq \tau\}} \mid \mathcal{F}_{\tau \wedge \sigma})\right) \\ &\stackrel{\text{b)}}{=} \mathbb{E}(B_{\sigma \wedge \tau} \mathbf{1}_{\{\sigma \leq \tau\}} \mathbb{E}(B_\tau \mid \mathcal{F}_{\tau \wedge \sigma})) \\ &\stackrel{(*)}{=} \mathbb{E}(B_{\sigma \wedge \tau}^2 \mathbf{1}_{\{\sigma \leq \tau\}}). \end{aligned}$$

(We will discuss the step marked by (*) below.)

With an analogous calculation for $\tau \leq \sigma$ we conclude

$$\mathbb{E}(B_\sigma B_\tau) = \mathbb{E}(B_{\sigma \wedge \tau} B_\tau \mathbf{1}_{\{\sigma < \tau\}}) + \mathbb{E}(B_{\sigma \wedge \tau} B_\tau \mathbf{1}_{\{\tau \leq \sigma\}}) = \mathbb{E}(B_{\sigma \wedge \tau}^2) = \mathbb{E} \sigma \wedge \tau.$$

In the step marked with (*) we used that for *integrable* stopping times σ, τ we have

$$\mathbb{E}(B_\tau \mid \mathcal{F}_{\sigma \wedge \tau}) = B_{\sigma \wedge \tau}.$$

To see this we use optional stopping which gives

$$\mathbb{E}(B_{\tau \wedge k} \mid \mathcal{F}_{\sigma \wedge \tau \wedge k}) = B_{\sigma \wedge \tau \wedge k} \quad \text{for all } k \geq 1.$$

This is the same as to say that

$$\int_F B_{\tau \wedge k} d\mathbb{P} = \int_F B_{\sigma \wedge \tau \wedge k} d\mathbb{P} \quad \text{for all } k \geq 1, F \in \mathcal{F}_{\sigma \wedge \tau \wedge k}.$$

Since $B_{\tau \wedge k} \xrightarrow[k \rightarrow \infty]{} B_\tau$ in $L^2(\mathbb{P})$, see the proof of Theorem 5.10, we get for some fixed $i < k$ because of $\mathcal{F}_{\sigma \wedge \tau \wedge i} \subset \mathcal{F}_{\sigma \wedge \tau \wedge k}$ that

$$\int_F B_\tau d\mathbb{P} = \lim_{k \rightarrow \infty} \int_F B_{\tau \wedge k} d\mathbb{P} = \lim_{k \rightarrow \infty} \int_F B_{\sigma \wedge \tau \wedge k} d\mathbb{P} = \int_F B_{\sigma \wedge \tau} d\mathbb{P} \quad \text{for all } F \in \mathcal{F}_{\sigma \wedge \tau \wedge i}.$$

Let $\rho = \sigma \wedge \tau$ (or any other stopping time). Since $\mathcal{F}_{\rho \wedge k} = \mathcal{F}_\rho \cap \mathcal{F}_k$ we see that \mathcal{F}_ρ is generated by the \cap -stable generator $\cup_i \mathcal{F}_{\rho \wedge i}$, and (*) follows.

- d) From the above and Wald's identity we get

$$\begin{aligned} \mathbb{E}(|B_\tau - B_\sigma|^2) &= \mathbb{E}(B_\tau^2 - 2B_\tau B_\sigma + B_\sigma^2) \\ &= \mathbb{E} \tau - 2 \mathbb{E} \tau \wedge \sigma + \mathbb{E} \sigma \\ &= \mathbb{E}(\tau - 2(\tau \wedge \sigma) + \sigma) \\ &= \mathbb{E} |\tau - \sigma|. \end{aligned}$$

In the last step we used the elementary relation

$$(a + b) - 2(a \wedge b) = a \wedge b + a \vee b - 2(a \wedge b) = a \vee b - a \wedge b = |a - b|.$$



Problem 5.23. Solution: (I owe this solution to Dr. Franziska Kühn) a) \Rightarrow b): Since we have

$$\{\tau = t\} = \{\tau \leq t\} \setminus \bigcup_{n \in \mathbb{N}} \{\tau \leq t - 1/n\} \in \mathcal{F}_t$$

we can use Problem 4.9 to see that for all $\omega \in \{\tau = t\}$ and $\omega' \in \Omega$

$$\forall s \leq t = \tau(\omega) : B_s(\omega) = B_s(\omega') \implies \omega' \in \{\tau = t\}.$$

This is but what is claimed in b).

b) \Rightarrow a): Set $F = \{\tau \leq t\}$. If $\omega \in F$ and $\omega' \in \Omega$ are such that b) holds, then we get $\tau(\omega') = t$, hence $\omega' \in F$. Using again Problem 4.9 we infer that $\{\tau \leq t\} = F \in \mathcal{F}_t$.

Remark. We may change, in the statement of the test, $\tau(\omega) = t$ and $\tau(\omega') = t$ for $\tau(\omega) \leq t$ and $\tau(\omega') \leq t$.



6 Brownian motion as a Markov process

Problem 6.1. Solution: We write $g_t(x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$ for the one-dimensional normal density.

- a) This follows immediately from our proof of b).
 b) Let $u \in \mathcal{B}_b(\mathbb{R})$ and $s, t \geq 0$. Then, by the independent and stationary increments property of a Brownian motion

$$\begin{aligned} \mathbb{E} [u(|B_{t+s}|) | \mathcal{F}_s] &= \mathbb{E} [u(|(B_{t+s} - B_s) + B_s|) | \mathcal{F}_s] \\ &= \mathbb{E} [u(|(B_{t+s} - B_s) + y|)] \Big|_{y=B_s} \\ &= \mathbb{E} [u(|B_t + y|)] \Big|_{y=B_s}. \end{aligned}$$

Since $B \sim -B$ we also get

$$\mathbb{E} [u(|B_{t+s}|) | \mathcal{F}_s] = \mathbb{E} [u(|B_t + y|)] \Big|_{y=-B_s} = \mathbb{E} [u(|B_t - y|)] \Big|_{y=B_s}$$

and, therefore,

$$\begin{aligned} \mathbb{E} [u(|B_{t+s}|) | \mathcal{F}_s] &= \frac{1}{2} [\mathbb{E} u(|B_t + y|) + \mathbb{E} u(|B_t - y|)] \Big|_{y=B_s} \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} (u(|z + y|) + u(|z - y|)) g_t(z) dz \right] \Big|_{y=B_s} \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} u(|z|) (g_t(z + y) + g_t(z - y)) dz \right] \Big|_{y=B_s} \\ &= \int_0^{\infty} u(|z|) (g_t(z + y) + g_t(z - y)) dz \Big|_{y=B_s} \end{aligned}$$

here we use that the integrand is even in z

$$= \underbrace{\int_0^{\infty} u(|z|) (g_t(z + |y|) + g_t(z - |y|)) dz}_{=: g_{u,s,t+s}(y)} \Big|_{y=B_s}$$

=: $g_{u,s,t+s}(y)$ —it is independent of s !

since the integrand is also even in y ! This shows that

- $\mathbb{E} [u(|B_{t+s}|) | \mathcal{F}_s]$ is a function of $|B_s|$, i.e. we get Markovianity.
- $\mathbb{P}^y(|B_t| \in dz) = g_t(z - y) + g_t(z + y)$ for $z, y \geq 0$, i.e. the form of the transition function.

Remark: $|B_t|$ is called *reflecting* (also: *reflected*) *Brownian motion*.

- c) Set $M_t := \sup_{s \leq t} B_s$ for the running maximum, i.e. $Y_t = M_t - B_t$. From the reflection principle, Theorem 6.10 we know that $Y_t \sim |B_t|$. So the guess is that Y and $|B|$ are two Markov processes with the same transition function!

Let $s, t \geq 0$ and $u \in \mathcal{B}_b(\mathbb{R})$. We have by the independent and stationary increments property of Brownian motion

$$\begin{aligned} \mathbb{E}(u(Y_{t+s}) | \mathcal{F}_s) &= \mathbb{E}(u(M_{t+s} - B_{t+s}) | \mathcal{F}_s) \\ &= \mathbb{E}\left(u\left(\max\left\{\sup_{u \leq s} B_r, \sup_{0 \leq u \leq t} B_{s+u}\right\} - B_{t+s}\right) \middle| \mathcal{F}_s\right) \\ &= \mathbb{E}\left(u\left(\max\left\{\sup_{u \leq s} (B_r - B_s) + (B_s - B_{t+s}), \sup_{0 \leq u \leq t} (B_{s+u} - B_{s+t})\right\}\right) \middle| \mathcal{F}_s\right) \end{aligned}$$

and, as $\sup_{u \leq s} (B_r - B_s)$ is \mathcal{F}_s measurable and $(B_s - B_{t+s}), \sup_{0 \leq u \leq t} (B_{s+u} - B_{s+t}) \perp \mathcal{F}_s$, we get

$$\begin{aligned} &= \mathbb{E}\left(u\left(\max\left\{y + (B_s - B_{t+s}), \sup_{0 \leq u \leq t} (B_{s+u} - B_{s+t})\right\}\right) \middle|_{y = \sup_{u \leq s} (B_r - B_s)}\right) \\ &= \mathbb{E}\left(u\left(\max\left\{y - B_t, \sup_{0 \leq u \leq t} (B_u - B_t)\right\}\right) \middle|_{y = Y_s}\right) \end{aligned}$$

Using time inversion (cf. 2.15) we see that $W = (W_u)_{u \in [0, t]} = (B_{t-u} - B_t)_{u \in [0, t]}$ is again a BM^1 , and we get $(B_t, \sup_{0 \leq u \leq t} (B_u - B_t)) \sim (W_t, \sup_{0 \leq u \leq t} (W_u - W_t)) = (-B_t, \sup_{0 \leq u \leq t} B_u)$ (we understand the vector as a function of the whole process B resp. W and use $B \sim W$)

$$= \mathbb{E}\left(u\left(\max\left\{y + B_t, \sup_{0 \leq u \leq t} B_u\right\}\right) \middle|_{y = Y_s}\right).$$

Using Solution 2 of Problem 6.12 we know the joint distribution of $(B_t, \sup_{u \leq t} B_u)$:

$$\begin{aligned} &\mathbb{E}\left(u\left(\max\left\{y + B_t, \sup_{0 \leq u \leq t} B_u\right\}\right)\right) \\ &= \int_{z=0}^{\infty} \int_{x=-\infty}^z u(\max\{y+x, z\}) \frac{2}{\sqrt{2\pi t}} \frac{2z-x}{t} e^{-(2z-x)^2/2t} dx dz. \end{aligned}$$

Splitting the integral $\int_{x=-\infty}^z$ into two parts $\int_{x=-\infty, y+x \leq z} + \int_{x=-\infty, y+x > z}$ we get

$$I = \int_{z=0}^{\infty} u(z) \frac{2}{\sqrt{2\pi t}} \underbrace{\int_{x=-\infty}^{z-y} \frac{2z-x}{t} e^{-(2z-x)^2/2t} dx}_{= e^{-(2z-x)^2/2t} \Big|_{-\infty}^{z-y}} dz = \frac{2}{\sqrt{2\pi t}} \int_{z=0}^{\infty} u(z) e^{-(z+y)^2/2t} dz$$

and

$$\begin{aligned} II &= \frac{2}{\sqrt{2\pi t}} \int_{z=0}^{\infty} \int_{x=-z-y}^z u(y+x) \frac{2z-x}{t} e^{-(2z-x)^2/2t} dx dz \\ &= \frac{2}{\sqrt{2\pi t}} \int_{x=-y}^{\infty} u(y+x) \underbrace{\int_{z=x}^{x+y} \frac{2z-x}{t} e^{-(2z-x)^2/2t} dz}_{= -\frac{1}{2} e^{-(2z-x)^2/2t} \Big|_{z=x}^{x+y}} dx \end{aligned}$$

$$\begin{aligned} dx &= \frac{1}{\sqrt{2\pi t}} \int_{x=-y}^{\infty} u(y+x) \left[e^{-x^2/2t} - e^{-(x+2y)^2/2t} \right] dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{x=-y}^{\infty} u(\xi) \left[e^{-(\xi-y)^2/2t} - e^{-(\xi+y)^2/2t} \right] dx. \end{aligned}$$

Finally, adding *I* and *II* we end up with

$$\mathbb{E} \left(u \left(\max \left\{ y + B_t, \sup_{0 \leq u \leq t} B_u \right\} \right) \right) = \int_0^\infty u(z) (g_t(z+y) + g_t(z-y)) dz, \quad y \geq 0$$

which is the same transition function as in part b).

d) See part c).

Problem 6.2. Solution: Replace $\mathbb{1}_F$ by $\mathbb{1}_F \mathbb{1}_G$ where $G \in \sigma(B_{\sigma+r} - B_\sigma, r \leq s)$. The whole calculation remains valid for this and yields, in the end,

$$\mathbb{E} \left[e^{i(\xi, B_{\sigma+t} - B_{\sigma+s})} \mathbb{1}_F \mathbb{1}_G \right] = e^{-\frac{1}{2}(t-s)|\xi|^2} \mathbb{P}(F \cap G).$$

The main change is in the third equality of the long, multi-line calculation at the end of the proof where we now have

$$\begin{aligned} &\mathbb{E} \left[e^{i(\xi, B(\sigma+t) - B(\sigma+s))} \mathbb{1}_F \mathbb{1}_G \right] \\ &= \dots \\ &= \lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} \mathbb{E} \left[\underbrace{e^{i(\xi, B(k2^{-j}+t) - B(k2^{-j}+s))}}_{\substack{\mathbb{1}_{\{(k-1)2^{-j} \leq \sigma < k2^{-j}\}} \\ \in \mathcal{F}_{k2^{-j}} \text{ as } F \in \mathcal{F}_{\sigma+} \\ \in \mathcal{F}_{k2^{-j}+s}}} \right] \\ &= \dots \end{aligned}$$

(pay attention to the independence w.r.t. $\mathcal{F}_{k2^{-j}+s}$, that's the main point). Now take $F = \Omega$. Then we're in the situation of Lemma 5.4 and we see that $B_{t+\sigma} - B_\sigma$ is a Brownian motion.

Problem 6.3. Solution: Let us assume, for simplicity, that $|X_t| \leq 1$. The essence is that we have a uniform majorant which is integrable, constants (as in the statement of the theorem) will do nicely.

- For every $s \geq 0$ we have $\{\sigma \wedge n + t \leq s\} = \{\sigma \wedge n \leq s - t\}$ and this is \emptyset if $s < t$, this is Ω if $s \geq t$ & $n \leq s - t$ and this is $\{\sigma \leq s - t\} \in \mathcal{F}_{s-t} \subset \mathcal{F}_s$ if $s \geq t$ & $n > s - t$. In any case it is an element of \mathcal{F}_s , hence $\sigma \wedge n + t$ is a stopping time.
- Since $\sigma \wedge n \leq \sigma$ we have $\mathcal{F}_{\sigma \wedge n} \subset \mathcal{F}_\sigma$, and so $\bigcup_n \mathcal{F}_{\sigma \wedge n} \subset \mathcal{F}_\sigma$ and $\sigma(\bigcup_n \mathcal{F}_{\sigma \wedge n}) \subset \mathcal{F}_\sigma$. To see the first assertion, take $F \in \mathcal{F}_{\sigma \wedge n}$ and observe that for any $t \geq 0$

$$F \cap \{\sigma \leq t\} = \underbrace{F \cap \{\sigma \wedge n \leq t\}}_{\in \mathcal{F}_t \text{ as } F \in \mathcal{F}_{\sigma \wedge n}} \cap \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t.$$

For the reverse inclusion, take $F \in \mathcal{F}_\sigma$ and observe that $\Omega = \{\sigma < \infty\}$. Then

$$F \in \mathcal{F}_\sigma \implies \forall n : F \cap \{\sigma \leq n\} \in \mathcal{F}_{\sigma \wedge n} \implies \forall n : F \cap \{\sigma \leq n\} \in \sigma \left(\bigcup_m \mathcal{F}_{\sigma \wedge m} \right)$$

and taking the union over all n we get $F = F \cap \{\sigma < \infty\} \in \sigma \left(\bigcup_n \mathcal{F}_{\sigma \wedge n} \right)$. If we have only $\sigma < \infty$ a.s., then $F = F \cap \{\sigma < \infty\}$ up to a measurable null set, but as all measurable null sets are in \mathcal{F}_0 , we still have $F \in \sigma \left(\bigcup_n \mathcal{F}_{\sigma \wedge n} \right)$.

- c) Recall that Lévy's upwards theorem says that for an integrable random variable $Z \in L^1(\mathbb{P})$ we have

$$\mathbb{E}(Z | \mathcal{G}_n) \xrightarrow[n \rightarrow \infty]{L^1 \& \text{ a.s.}} \mathbb{E}(Z | \mathcal{G}_\infty), \quad \mathcal{G}_\infty = \sigma \left(\bigcup_n \mathcal{G}_n \right).$$

We would like to take $Z = X_{\sigma \wedge n}$ and $\mathcal{G}_n = \mathcal{F}_{\sigma \wedge n}$, but the problem is the n -dependence of Z . Therefore, we argue like this: Define for $m \in \mathbb{N}$

$$W_m = \sup_{n \geq m} |X_{\sigma \wedge n} - X_\sigma|.$$

Clearly, $W_m \leq 2$ and $W_m \rightarrow 0$ a.s. since X_t has continuous paths. Now apply Lévy's upwards theorem to $Z = W_m$ to get

$$\overline{\lim}_n \mathbb{E}(|X_{\sigma \wedge n} - X_\sigma| | \mathcal{F}_{\sigma \wedge n}) \leq \lim_n \mathbb{E}(W_m | \mathcal{F}_{\sigma \wedge n}) = \mathbb{E}(W_m | \mathcal{F}_\sigma) \xrightarrow[m \rightarrow \infty]{} 0$$

by (conditional) monotone convergence or (conditional) dominated convergence. This proves a.e. convergence. Now the classical dominated convergence theorem kicks in and gives

$$\mathbb{E}|\mathbb{E}(X_{\sigma \wedge n} | \mathcal{F}_{\sigma \wedge n}) - \mathbb{E}(X_\sigma | \mathcal{F}_\sigma)| \xrightarrow[n \rightarrow \infty]{} 0.$$

■ ■

Problem 6.4. Solution:

- a) By the definition of B_t and θ_σ , we have for all $w \in \mathcal{C}$

$$B_t(\theta_\sigma w) \stackrel{\text{def. } B}{=} \theta_\sigma w(t) \stackrel{\text{def. } \theta}{=} w(t + \sigma(w)) \stackrel{\text{def. } B}{=} B_{t+\sigma(w)}(w).$$

Thus,

$$\begin{aligned} \theta_\sigma^{-1}(\{B_t \in C\}) &= \{\theta_\sigma^{-1}(w) : B_t(w) \in C\} \\ &= \{v : B_t(\theta_\sigma v) \in C\} \\ &= \{v : B_{t+\sigma}(v) \in C\} \\ &= \{B_{t+\sigma} \in C\}. \end{aligned}$$

- b) We have to show that for all $F \in \mathcal{F}_t$ we have $\theta_\sigma^{-1}(F) \in \mathcal{F}_{t+\sigma}$. It is enough to do this for a generator of \mathcal{F}_t , and we choose the generator whose sets are of the form

$$\{B_{t_1} \in C_1, \dots, B_{t_n} \in C_n\} = \bigcap_{k=1}^n \{B_{t_k} \in C_k\},$$

$$n \in \mathbb{N}, 0 \leq t_1 < \dots < t_n \leq t, C_1, \dots, C_n \in \mathcal{B}(\mathbb{R}).$$

The calculation of the first part and Corollary 6.26 (used for $\tau = \sigma + t_j$) show

$$\theta_\sigma^{-1} \{B_{t_1} \in C_1, \dots, B_{t_n} \in C_n\} = \{B_{t_1+\sigma} \in C_1, \dots, B_{t_n+\sigma} \in C_n\} \in \mathcal{F}_{[\sigma, t+\sigma]} \subset \mathcal{F}_{t+\sigma}.$$

The $\mathcal{F}_{[t+\sigma, \infty)}/\mathcal{F}_{[t, \infty)}$ measurability is shown in a similar way, considering now instances $t \leq t_1 < t_2 < \dots < t_n < \infty$.

- c) $\sigma + \tau \circ \theta_\sigma$ means that we first have observed σ and after that, we observe τ . Note that $\tau \circ \theta_\sigma$ refers to the shifted paths $w(\cdot + \sigma(\cdot))$. For example, if σ means that “ B_t reaches the level 1” and if τ means “ B reaches the level 3”, then we have $\tau = \sigma + \tau \circ \theta_\sigma$ since we must go to 1 before we can reach 3 (b/o continuity of the paths). But if we are at 1, then we still have to go to 3, i.e. we have the same τ , but for a different set of paths (those starting at 1).

Now for the assertion. For every $t \geq 0$ we have

$$\{\sigma + \tau \circ \theta_\sigma < t\} = \bigcup_{r \in \mathbb{Q}} \{\sigma < t - r\} \cap \{\tau \circ \theta_\sigma < r\} = \bigcup_{r \in \mathbb{Q}} \{\sigma + r < t\} \cap \underbrace{\theta_\sigma^{-1}\{\tau < r\}}_{\in \mathcal{F}_{\sigma+r}}$$

and, by the definition of $\mathcal{F}_{\sigma+r}$ ((5.12) and Problem 5.mar-p37) we see that the set under the countable union is in \mathcal{F}_t , so $\{\sigma + \tau \circ \theta_\sigma < t\} \in \mathcal{F}_t$ i.e. $\{\sigma + \tau \circ \theta_\sigma \leq t\} \in \mathcal{F}_{t+}$

It is interesting to see what happens if we consider $\{\sigma + \tau \circ \theta_\sigma \leq t\}$. If, say, $\tau \equiv 0$ and $t \notin \mathbb{Q}$, then the identity is obviously wrong, so the strict inequality is essential. ■■

Problem 6.5. Solution: Let $s, t \geq 0$. We use the following abbreviations:

$$I_s = \int_0^s B_r dr \quad \text{and} \quad M_s = \sup_{u \leq s} B_u \quad \text{and} \quad \mathcal{F}_s = \mathcal{F}_s^B.$$

- a) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ measurable and bounded. Then

$$\begin{aligned} & \mathbb{E}(f(M_{s+t}, B_{s+t}) \mid \mathcal{F}_s) \\ &= \mathbb{E}\left(f\left(\sup_{s \leq u \leq s+t} B_u \vee M_s, (B_{s+t} - B_s) + B_s\right) \mid \mathcal{F}_s\right) \\ &= \mathbb{E}\left(f\left(\left[B_s + \sup_{s \leq u \leq s+t} (B_u - B_s)\right] \vee M_s, (B_{s+t} - B_s) + B_s\right) \mid \mathcal{F}_s\right). \end{aligned}$$

By the independent increments property of BM we get that the random variables $\sup_{s \leq u \leq s+t} (B_u - B_s)$, $B_{s+t} - B_s \perp \mathcal{F}_s$ while M_s and B_s are \mathcal{F}_s measurable. Thus, we can treat these groups of random variables separately (see, e.g., Lemma A.3:

$$\begin{aligned} & \mathbb{E}(f(M_{s+t}, B_{s+t}) \mid \mathcal{F}_s) \\ &= \mathbb{E}\left(f\left(\left[z + \sup_{s \leq u \leq s+t} (B_u - B_s)\right] \vee y, (B_{s+t} - B_s) + z\right) \mid \mathcal{F}_s\right) \Bigg|_{y=M_s, z=B_s} \end{aligned}$$

$$= \phi(M_s, B_s)$$

where

$$\phi(y, z) = \mathbb{E} \left(f \left(\left[z + \sup_{s \leq u \leq s+t} (B_u - B_s) \right] \vee y, (B_{s+t} - B_s) + z \right) \right).$$

b) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ measurable and bounded. Then

$$\begin{aligned} & \mathbb{E} (f(I_{s+t}, B_{s+t}) \mid \mathcal{F}_s) \\ &= \mathbb{E} \left(f \left(\int_s^{s+t} B_u \, du + I_s, (B_{s+t} - B_s) + B_s \right) \mid \mathcal{F}_s \right) \\ &= \mathbb{E} \left(f \left(\int_s^{s+t} (B_u - B_s) \, du + I_s + tB_s, (B_{s+t} - B_s) + B_s \right) \mid \mathcal{F}_s \right). \end{aligned}$$

By the independent increments property of BM we get that the random variables $\int_s^{s+t} (B_u - B_s) \, du$, $B_{s+t} - B_s \perp \mathcal{F}_s$ while $I_s + tB_s$ and B_s are \mathcal{F}_s measurable. Thus, we can treat these groups of random variables separately (see, e.g., Lemma A.3:

$$\begin{aligned} & \mathbb{E} (f(I_{s+t}, B_{s+t}) \mid \mathcal{F}_s) \\ &= \mathbb{E} \left(f \left(\int_s^{s+t} (B_u - B_s) \, du + y + tz, (B_{s+t} - B_s) + z \right) \right) \Big|_{y=I_s, z=B_s} \\ &= \phi(I_s, B_s) \end{aligned}$$

for the function

$$\phi(y, z) = \mathbb{E} \left(f \left(\int_s^{s+t} (B_u - B_s) \, du + y + tz, (B_{s+t} - B_s) + z \right) \right).$$

c) No! If we use the calculation of a) and b) for the function $f(y, z) = g(y)$, i.e. only depending on M or I , respectively, we see that we still get

$$\mathbb{E} (g(I_{t+s}) \mid \mathcal{F}_s) = \psi(B_s, I_s),$$

i.e. $(I_t, \mathcal{F}_t)_t$ cannot be a Markov process. The same argument applies to $(M_t, \mathcal{F}_t)_t$. ■ ■

Problem 6.6. Solution: We follow the hint.

First, if $f : \mathbb{R}^{d \times n} \rightarrow \mathbb{R}$, $f = f(x_1, \dots, x_n)$, $x_1, \dots, x_n \in \mathbb{R}^d$, we see that

$$\begin{aligned} & \mathbb{E}^x f(B(t_1), \dots, B(t_n)) \\ &= \mathbb{E} f(B(t_1) + x, \dots, B(t_n) + x) \\ &= \underbrace{\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d}}_{n \text{ times}} f(y_1 + x, \dots, y_n + x) \mathbb{P}(B(t_1) \in dy_1, \dots, B(t_n) \in dy_n) \end{aligned}$$

and the last expression is clearly measurable. This applies, in particular, to $f = \prod_{j=1}^n \mathbb{1}_{A_j}$ where $G := \bigcap_{j=1}^n \{B(t_j) \in A_j\}$, i.e. $\mathbb{E}^x \mathbb{1}_G$ is Borel measurable.

Set

$$\Gamma := \left\{ \bigcap_{j=1}^n \{B(t_j) \in A_j\} : n \geq 0, 0 \leq t_1 < \dots < t_n, A_1, \dots, A_n \in \mathcal{B}_b(\mathbb{R}^d) \right\}.$$

Let us see that Σ is a Dynkin system. Clearly, $\emptyset \in \Sigma$. If $A \in \Sigma$, then

$$x \mapsto \mathbb{E}^x \mathbb{1}_{A^c} = \mathbb{E}^x(1 - \mathbb{1}_A) = 1 - \mathbb{E}^x \mathbb{1}_A \in \mathcal{B}_b(\mathbb{R}^d) \implies A^c \in \Sigma.$$

Finally, if $(A_j)_{j \geq 1} \subset \Sigma$ are disjoint and $A := \cup_j A_j$ we get $\mathbb{1}_A = \sum_j \mathbb{1}_{A_j}$. Thus,

$$x \mapsto \mathbb{E}^x \mathbb{1}_A = \sum_j \mathbb{E}^x \mathbb{1}_{A_j} \in \mathcal{B}_b(\mathbb{R}^d).$$

This shows that Σ is a Dynkin System. Denote by $\delta(\cdot)$ the Dynkin system generated by the argument. Then

$$\Gamma \subset \Sigma \subset \mathcal{F}_\infty^B \implies \delta(\Gamma) \subset \delta(\Sigma) = \Sigma \subset \mathcal{F}_\infty^B.$$

But $\delta(\Gamma) = \sigma(\Gamma)$ since Γ is stable under finite intersections and $\sigma(\Gamma) = \mathcal{F}_\infty^B$. This proves, in particular, that $\Sigma = \mathcal{F}_\infty^B$.

Since we can approximate every bounded \mathcal{F}_∞^B measurable function Z by step functions with steps from \mathcal{F}_∞^B , the claim follows. ■ ■

Problem 6.7. Solution: *Solution 1:* Without the assumption that \mathcal{F}_0 contains all null sets, use Corollary 6.26 and the fact that $\mathcal{F}_\tau \subset \mathcal{F}_{\tau+}$.

Solution 2: We follow the hint, but we need to assume that \mathcal{F}_0 contains all null sets from \mathcal{A} . Since $\mathcal{F}_0 \subset \mathcal{F}_t$ and $\mathcal{F}_0 \subset \mathcal{F}_\tau \subset \mathcal{F}_{\tau+}$, all \mathcal{F}_* contain the measurable null sets.

Set $u_n(x) := (-n) \vee x \wedge n$. Then $u_n(x) \rightarrow u(x) := x$. Using (6.7) we see

$$\mathbb{E}[u_n(B_{t+\tau}) | \mathcal{F}_{\tau+}](\omega) \stackrel{\text{a.s.}}{=} \mathbb{E}^{B_\tau(\omega)} u_n(B_t).$$

Now take $t = 0$ to get

$$\mathbb{E}[u_n(B_\tau) | \mathcal{F}_{\tau+}](\omega) \stackrel{\text{a.s.}}{=} u_n(B_\tau)(\omega)$$

and we get

$$\lim_{n \rightarrow \infty} \mathbb{E}[u_n(B_\tau) | \mathcal{F}_{\tau+}](\omega) = \lim_{n \rightarrow \infty} u_n(B_\tau)(\omega) \stackrel{\text{a.s.}}{=} B_\tau(\omega).$$

Since the l.h.s. is $\mathcal{F}_{\tau+}$ measurable (as limit of such measurable functions!) and since $B_\tau(\omega)$ differs from this by at most a measurable null set, the claim follows. At this point we have to use that \mathcal{F}_0 or all \mathcal{F}_* contain the measurable null sets. ■ ■

Problem 6.8. Solution: By the reflection principle, Theorem 6.10,

$$\mathbb{P}\left(\sup_{s \leq t} |B_s| \geq x\right) \leq \mathbb{P}\left(\sup_{s \leq t} B_s \geq x\right) + \mathbb{P}\left(\inf_{s \leq t} B_s \leq -x\right) = \mathbb{P}(|B_t| \geq x) + \mathbb{P}(|B_t| \geq x).$$

Problem 6.9. Solution:

a) Since $B(\cdot) \sim -B(\cdot)$, we get

$$\tau_b = \inf\{s \geq 0 : B_s = b\} \sim \inf\{s \geq 0 : -B_s = b\} = \inf\{s \geq 0 : B_s = -b\} = \tau_{-b}.$$

b) Since $B(c^{-2}\cdot) \sim c^{-1}B(\cdot)$, we get

$$\begin{aligned} \tau_{cb} &= \inf\{s \geq 0 : B_s = cb\} = \inf\{s \geq 0 : c^{-1}B_s = b\} \\ &\sim \inf\{s \geq 0 : B_{s/c^2} = b\} \\ &= \inf\{rc^2 \geq 0 : B_r = b\} \\ &= c^2 \inf\{r \geq 0 : B_r = b\} = c^2\tau_b. \end{aligned}$$

c) We have

$$\tau_b - \tau_a = \inf\{s \geq 0 : B_{s+\tau_a} = b\} = \inf\{s \geq 0 : B_{s+\tau_a} - B_{\tau_a} = b - a\}$$

which shows that $\tau_b - \tau_a$ is independent of \mathcal{F}_{τ_a} by the strong Markov property of Brownian motion.

Now we find for all $s, t \geq 0$ and $c \in [0, a]$

$$\{\tau_c \leq s\} \cap \{\tau_a \leq t\} \stackrel{\tau_c \leq \tau_a}{=} \{\tau_c \leq s \wedge t\} \cap \{\tau_a \leq t\} \in \mathcal{F}_{t \wedge s} \cap \mathcal{F}_t \subset \mathcal{F}_t.$$

This shows that $\{\tau_c \leq s\} \in \mathcal{F}_{\tau_a}$, i.e. τ_c is \mathcal{F}_{τ_a} measurable. Since c is arbitrary, $\{\tau_c\}_{c \in [0, a]}$ is \mathcal{F}_{τ_a} measurable, and the claim follows. ■ ■

Problem 6.10. Solution: We begin with a simpler situation. As usual, we write τ_b for the first passage time of the level b : $\tau_b = \inf\{t \geq 0 : \sup_{s \leq t} B_s = b\}$ where $b > 0$. From Example 5.2

d) we know that $(M_t^\xi := \exp(\xi B_t - \frac{1}{2}t\xi^2))_{t \geq 0}$ is a martingale. By optional stopping we get that $(M_{t \wedge \tau_b}^\xi)_{t \geq 0}$ is also a martingale and has, therefore, constant expectation. Thus, for $\xi > 0$ (and with $\mathbb{E} = \mathbb{E}^0$)

$$1 = \mathbb{E} M_0^\xi = \mathbb{E} \left(\exp(\xi B_{t \wedge \tau_b} - \frac{1}{2}(t \wedge \tau_b)\xi^2) \right)$$

Since the RV $\exp(\xi B_{t \wedge \tau_b})$ is bounded (mind: $\xi \geq 0$ and $B_{t \wedge \tau_b} \leq b$), we can let $t \rightarrow \infty$ and get

$$1 = \mathbb{E} \left(\exp(\xi B_{\tau_b} - \frac{1}{2}\tau_b\xi^2) \right) = \mathbb{E} \left(\exp(\xi b - \frac{1}{2}\tau_b\xi^2) \right)$$

or, if we take $\xi = \sqrt{2\lambda}$,

$$\mathbb{E} e^{-\lambda\tau_b} = e^{-\sqrt{2\lambda}b}.$$

As $B \sim -B$, $\tau_b \sim \tau_{-b}$, and the above calculation yields

$$\mathbb{E} e^{-\lambda\tau_b} = e^{-\sqrt{2\lambda}|b|} \quad \forall b \in \mathbb{R}.$$

Now let us turn to the situation of the problem. Set $\tau = \tau_{(a,b)^c}^\circ$. Here, $B_{t \wedge \tau}$ is bounded (it is in the interval (a, b) , and this makes things easier when it comes to optional stopping. As before, we get by stopping the martingale $(M_t^\xi)_{t \geq 0}$ that

$$e^{\xi x} = \lim_{t \rightarrow \infty} \mathbb{E}^x \left(\exp(\xi B_{t \wedge \tau} - \frac{1}{2}(t \wedge \tau)\xi^2) \right) = \mathbb{E}^x \left(\exp(\xi B_\tau - \frac{1}{2}\tau\xi^2) \right) \quad \forall \xi$$

(and not, as before, for positive ξ ! Mind also the starting point $x \neq 0$, but this does not change things dramatically.) by, e.g., dominated convergence. The problem is now that B_τ does not attain a particular value as it may be a or b . We get, therefore, for all $\xi \in \mathbb{R}$

$$\begin{aligned} e^{\xi x} &= \mathbb{E}^x \left(\exp(\xi B_\tau - \frac{1}{2}\tau\xi^2) \mathbb{1}_{\{B_\tau=a\}} \right) + \mathbb{E}^x \left(\exp(\xi B_\tau - \frac{1}{2}\tau\xi^2) \mathbb{1}_{\{B_\tau=b\}} \right) \\ &= \mathbb{E}^x \left(\exp(\xi a - \frac{1}{2}\tau\xi^2) \mathbb{1}_{\{B_\tau=a\}} \right) + \mathbb{E}^x \left(\exp(\xi b - \frac{1}{2}\tau\xi^2) \mathbb{1}_{\{B_\tau=b\}} \right) \end{aligned}$$

Now pick $\xi = \pm\sqrt{2\lambda}$. This yields 2 equations in two unknowns:

$$\begin{aligned} e^{\sqrt{2\lambda}x} &= e^{\sqrt{2\lambda}a} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) + e^{\sqrt{2\lambda}b} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) \\ e^{-\sqrt{2\lambda}x} &= e^{-\sqrt{2\lambda}a} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) + e^{-\sqrt{2\lambda}b} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) \end{aligned}$$

Solving this system of equations gives

$$\begin{aligned} e^{\sqrt{2\lambda}(x-a)} &= \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) + e^{\sqrt{2\lambda}(b-a)} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) \\ e^{-\sqrt{2\lambda}(x-a)} &= \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) + e^{-\sqrt{2\lambda}(b-a)} \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) \end{aligned}$$

and so

$$\mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) = \frac{\sinh(\sqrt{2\lambda}(x-a))}{\sinh(\sqrt{2\lambda}(b-a))} \quad \text{and} \quad \mathbb{E}^x \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) = \frac{\sinh(\sqrt{2\lambda}(b-x))}{\sinh(\sqrt{2\lambda}(b-a))}.$$

This answers Problem b).

For the solution of Problem a) we only have to add these two expressions:

$$\mathbb{E} e^{-\lambda\tau} = \mathbb{E} \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=a\}} \right) + \mathbb{E} \left(e^{-\lambda\tau} \mathbb{1}_{\{B_\tau=b\}} \right) = \frac{\sinh(\sqrt{2\lambda}(b-x)) + \sinh(\sqrt{2\lambda}(x-a))}{\sinh(\sqrt{2\lambda}(b-a))}.$$

Problem 6.11. Solution: (I owe this solution to Dr. Franziska Kühn) We write $W_t = (a, b)^\top + (B_t^1, B_t^2)^\top$ where $(B_t^i)_{t \geq 0}$, $i = 1, 2$, are independent standard BM¹. WLOG we assume that $b > 0$. The problem can be re-phrased in terms of stopping times as follows:

$$\tau := \inf\{t \geq 0 : W_t \in \mathbb{R} \times (-\infty, 0]\} = \inf\{t \geq 0 : b + B_t^2 \leq 0\}.$$

(Draw a picture. In order to hit the x -axis, we have go into the negative with the second coordinate!). The second equality shows that τ depends only on the canonical filtration

\mathcal{F}_t^2 of the second coordinate B^2 . Thus, τ is independent of $(B_t^1)_{t \geq 0}$, and we get for the probability which we are looking for (now comes the condition which half is hit first!)

$$p = \mathbb{P}(a + B_\tau^1 \geq 0) = \int \mathbb{P}(a + B_t^1 \geq 0) \mathbb{P}(\tau \in dt).$$

From Theorem 6.10 we know the law of τ : $\mathbb{P}(\tau \in dt) = |b|(2\pi t^3)^{-1/2} e^{-b^2/(2t)} \mathbf{1}_{(0, \infty)} dt$. Thus,

$$\begin{aligned} p &= \int_0^\infty \int_{-a}^\infty \frac{|b|}{\sqrt{2\pi t^3}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{b^2 + y^2}{2t}\right) dy dt \\ &= \frac{|b|}{2\pi} \int_{-a}^\infty \int_0^\infty \exp\left(-\frac{b^2 + y^2}{2t}\right) \frac{dt}{t^2} dy \\ &= \frac{|b|}{2\pi} \int_{-a}^\infty \int_0^\infty \exp\left(-\frac{s(b^2 + y^2)}{2}\right) ds dy \\ &= \frac{1}{\pi} \int_{-a}^\infty \frac{|b|}{b^2 + y^2} dy \\ &= \frac{1}{\pi} \arctan \frac{x}{|b|} \Big|_{-a}^\infty = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{-a}{|b|}\right). \end{aligned}$$

Note: we have actually shown that B_τ^1 is Cauchy distributed. ■ ■

Problem 6.12. Solution: Solution 1 (direct calculation): Denote by $\tau = \tau_y = \inf\{s > 0 : B_s = y\}$ the first passage time of the level y . Then $B_\tau = y$ and we get for $y \geq x$

$$\begin{aligned} \mathbb{P}(B_t \leq x, M_t \geq y) &= \mathbb{P}(B_t \leq x, \tau \leq t) \\ &= \mathbb{P}(B_{t \vee \tau} \leq x, \tau \leq t) \\ &= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{B_{t \vee \tau} \leq x\}} \mid \mathcal{F}_{\tau+}\right) \cdot \mathbf{1}_{\{\tau \leq t\}}\right) \end{aligned}$$

by the tower property and pull-out. Now we can use Theorem 6.12

$$\begin{aligned} &= \int \mathbb{P}^{B_\tau(\omega)}(B_{t-\tau(\omega)} \leq x) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &= \int \mathbb{P}^y(B_{t-\tau(\omega)} \leq x) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &= \int \mathbb{P}(B_{t-\tau(\omega)} \leq x - y) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &\stackrel{B \sim B}{=} \int \mathbb{P}(B_{t-\tau(\omega)} \geq y - x) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &= \int \mathbb{P}^y(B_{t-\tau(\omega)} \geq 2y - x) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &= \int \mathbb{P}^{B_\tau(\omega)}(B_{t-\tau(\omega)} \geq 2y - x) \cdot \mathbf{1}_{\{\tau \leq t\}}(\omega) \mathbb{P}(d\omega) \\ &= \dots = \mathbb{P}(B_t \geq 2y - x, M_t \geq y) \stackrel{y \geq x}{=} \mathbb{P}(B_t \geq 2y - x). \end{aligned}$$

This means that

$$\mathbb{P}(B_t \leq x, M_t \geq y) = \mathbb{P}(B_t \geq 2y - x) = \int_{2y-x}^\infty (2\pi t)^{-1/2} e^{-z^2/(2t)} dz$$

and differentiating in x and y yields

$$\mathbb{P}(B_t \in dx, M_t \in dy) = \frac{2(2y-x)}{\sqrt{2\pi t^3}} e^{-(2y-x)^2/(2t)} dx dy.$$

Solution 2 (using Theorem 6.19): We have (with the notation of Theorem 6.19)

$$\mathbb{P}(M_t < y, B_t \in dx) = \lim_{a \rightarrow -\infty} \mathbb{P}(m_t > a, M_t < y, B_t \in dx) \stackrel{(6.20)}{=} \frac{dx}{\sqrt{2\pi t}} \left[e^{-\frac{x^2}{2t}} - e^{-\frac{(x-2y)^2}{2t}} \right]$$

and if we differentiate this expression in y we get

$$\mathbb{P}(B_t \in dx, M_t \in dy) = \frac{2(2y-x)}{\sqrt{2\pi t^3}} e^{-(2y-x)^2/(2t)} dx dy.$$

Problem 6.13. Solution: This is the so-called *absorbed* or *killed Brownian motion*. The result is

$$\mathbb{P}^x(B_t \in dz, \tau_0 > t) = (g_t(x-z) - g_t(x+z)) dz = \frac{1}{\sqrt{2\pi t}} \left(e^{-(x-z)^2/(2t)} - e^{-(x+z)^2/(2t)} \right) dz,$$

for $x, z > 0$ or $x, z < 0$.

To see this result we assume that $x > 0$. Write $M_t = \sup_{s \leq t} B_s$ and $m_t = \inf_{s \leq t} B_s$ for the running maximum and minimum, respectively. Then we have for $A \subset [0, \infty)$

$$\begin{aligned} \mathbb{P}^x(B_t \in A, \tau_0 > t) &= \mathbb{P}^x(B_t \in A, m_t > 0) \\ &= \mathbb{P}^x(B_t \in A, x \geq m_t > 0) \end{aligned}$$

(we start in $x > 0$, so the minimum is smaller!)

$$\begin{aligned} &= \mathbb{P}^0(B_t \in A-x, 0 \geq m_t > -x) \\ &\stackrel{B \rightsquigarrow -B}{=} \mathbb{P}^0(-B_t \in A-x, 0 \geq -M_t > -x) \\ &= \mathbb{P}^0(B_t \in x-A, 0 \leq M_t < x) \\ &= \iint \mathbf{1}_A(x-a) \mathbf{1}_{[0,x)}(b) \mathbb{P}^0(B_t \in da, M_t \in db) \end{aligned}$$

Now we use the result of Problem 6.12:

$$\mathbb{P}^0(B_t \in da, M_t \in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) da db$$

and we get

$$\begin{aligned} \mathbb{P}^x(B_t \in A, \tau_0 > t) &= \int \mathbf{1}_A(x-a) \left[\int_0^x \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) db \right] da \\ &= \int \mathbf{1}_A(x-a) \frac{t}{\sqrt{2\pi t^3}} \left[\int_0^x \frac{2 \cdot 2 \cdot (2b-a)}{2t} \exp\left(-\frac{(2b-a)^2}{2t}\right) db \right] da \\ &= \int \mathbf{1}_A(x-a) \frac{1}{\sqrt{2\pi t}} \left[\int_0^x \frac{2 \cdot (2b-a)}{t} \exp\left(-\frac{(2b-a)^2}{2t}\right) db \right] da \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(x-a) \left[-\exp\left(-\frac{(2b-a)^2}{2t}\right) \right]_{b=0}^x da \\
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(x-a) \left\{ \exp\left(-\frac{a^2}{2t}\right) - \exp\left(-\frac{(2x-a)^2}{2t}\right) \right\} da \\
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(z) \left\{ \exp\left(-\frac{(x-z)^2}{2t}\right) - \exp\left(-\frac{(x+z)^2}{2t}\right) \right\} da.
 \end{aligned}$$

The calculation for $x < 0$ is similar (actually easier): Let $A \subset (-\infty, 0]$

$$\begin{aligned}
 \mathbb{P}^x(B_t \in A, \tau_0 > 0) &= \mathbb{P}^x(B_t \in A, -x \leq M_t < 0) \\
 &= \mathbb{P}^0(B_t \in A - x, 0 \leq M_t < -x) \\
 &= \iint \mathbf{1}_A(a+x) \mathbf{1}_{[0, -x)}(b) \frac{2(2b-a)}{\sqrt{2\pi t^2}} \exp\left(-\frac{(2b-a)^2}{2t}\right) db da \\
 &= \int \mathbf{1}_A(a+x) \frac{t}{\sqrt{2\pi t^3}} \int_0^{-x} \frac{2 \cdot (2b-a)}{t} \exp\left(-\frac{(2b-a)^2}{2t}\right) db da \\
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(a+x) \left[-\exp\left(-\frac{(2b-a)^2}{2t}\right) \right]_{b=0}^{-x} da \\
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(a+x) \left\{ \exp\left(-\frac{a^2}{2t}\right) - \exp\left(-\frac{(2x+a)^2}{2t}\right) \right\} da \\
 &= \frac{1}{\sqrt{2\pi t}} \int \mathbf{1}_A(y) \left\{ \exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right) \right\} da.
 \end{aligned}$$

■

Problem 6.14. Solution: For a compact set $K \subset \mathbb{R}^d$ the set $U_n := K + \mathbb{B}(0, 1/n) := \{x+y : x \in K, |y| < 1/n\}$ is open.

$$\phi_n(x) := d(x, U_n^c) / (d(x, K) + d(x, U_n^c)).$$

Since for $d(x, z) := |x - z|$ and all $x, z \in \mathbb{R}^d$

$$d(x, A) \leq d(x, z) + d(z, A) \implies |d(x, A) - d(z, A)| \leq d(x, z),$$

we see that $\phi_n(x)$ is continuous. Obviously, $\mathbf{1}_{U_n}(x) \geq \phi_n(x) \geq \phi_{n+1} \geq \mathbf{1}_K$, and $\mathbf{1}_K = \inf_n \phi_n$ follows.

■

Problem 6.15. Solution: Recall that $\mathbb{P} = \mathbb{P}^0$. We have for all $a \geq t \geq 0$

$$\begin{aligned}
 \mathbb{P}(\tilde{\xi}_t > a) &= \mathbb{P}(\inf \{s \geq t : B_s = 0\} > a) \\
 &= \mathbb{P}(\inf \{h \geq 0 : B_{t+h} = 0\} + t > a) \\
 &= \mathbb{E}[\mathbb{P}^{B_t}(\inf \{h \geq 0 : B_h = 0\} > a - t)] \\
 &= \mathbb{E}\left[\mathbb{P}^0(\inf \{h \geq 0 : B_h + x = 0\} > a - t) \Big|_{x=B_t}\right] \\
 &= \mathbb{E}\left[\mathbb{P}(\inf \{h \geq 0 : B_h = -x\} > a - t) \Big|_{x=B_t}\right]
 \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[\mathbb{P} (\tau_{-x} > a - t) \Big|_{x=B_t} \right] \\
 &\stackrel{B \sim B}{=} \mathbb{E} [\mathbb{P} (\tau_{B_t} > a - t)] \\
 &\stackrel{(6.14)}{=} \mathbb{E} \left[\int_{a-t}^{\infty} \frac{|B_t|}{\sqrt{2\pi s^3}} e^{-B_t^2/(2s)} ds \right] \\
 &= \int_{a-t}^{\infty} \mathbb{E} \left[\frac{|B_t|}{\sqrt{2\pi s^3}} e^{-B_t^2/(2s)} \right] ds.
 \end{aligned}$$

Thus, differentiating with respect to a and using Brownian scaling yields

$$\begin{aligned}
 \mathbb{P}(\tilde{\xi}_t \in da) &= \mathbb{E} \left[\frac{|B_t|}{\sqrt{2\pi(a-t)^3}} \exp \left(-\frac{B_t^2}{2(a-t)} \right) \right] \\
 &= \frac{1}{(a-t)\sqrt{\pi}} \mathbb{E} \left[\frac{\sqrt{t}}{\sqrt{a-t}} \frac{|B_1|}{\sqrt{2}} \exp \left(-\frac{1}{2} B_1^2 \frac{t}{a-t} \right) \right] \\
 &= \frac{1}{(a-t)\sqrt{\pi}} \mathbb{E} [|cB_1| \exp(-cB_1^2)] \\
 &= \frac{1}{(a-t)\sqrt{\pi}} \mathbb{E} [|B_{c^2}| \exp(-B_{c^2}^2)]
 \end{aligned}$$

where $c^2 = \frac{1}{2} \frac{t}{a-t}$.

Now let us calculate for $s = c^2$

$$\begin{aligned}
 \mathbb{E} [|B_s| e^{-B_s^2}] &= (2\pi s)^{-1/2} \int_{-\infty}^{\infty} |x| e^{-x^2} e^{-x^2/(2s)} dx \\
 &= (2\pi s)^{-1/2} 2 \int_0^{\infty} x e^{-x^2(1+(2s)^{-1})} dx \\
 &= (2\pi s)^{-1/2} \frac{1}{(1+(2s)^{-1})} \int_0^{\infty} 2(1+(2s)^{-1}) x e^{-x^2(1+(2s)^{-1})} dx \\
 &= \frac{1}{\sqrt{2\pi s}} \frac{2s}{2s+1} \left[e^{-x^2(1+(2s)^{-1})} \right]_{x=0}^{\infty} \\
 &= \frac{1}{\sqrt{2\pi s}} \frac{2s}{2s+1}.
 \end{aligned}$$

Let $(B_t)_{t \geq 0}$ be a BM¹. Find the distribution of $\tilde{\xi}_t := \inf\{s \geq t : B_s = 0\}$. This gives

$$\begin{aligned}
 \mathbb{P}(\tilde{\xi}_t \in da) &= \frac{1}{(a-t)\sqrt{\pi}} \frac{1}{\sqrt{2\pi c}} \frac{2c^2}{2c^2+1} \\
 &= \frac{1}{(a-t)\pi} \frac{\sqrt{a-t}}{\sqrt{t}} \frac{t}{(a-t)a/(a-t)} \\
 &= \frac{1}{a\pi} \sqrt{\frac{t}{a-t}}.
 \end{aligned}$$

Problem 6.16. Solution: We have seen in Problem 6.1 that $M - B$ is a Markov process with the same law as $|B|$. This entails immediately that $\xi \sim \eta$.

Attention: this problem shows that it is not enough to have only $M_t - B_t \sim |B_t|$ for all $t \geq 0$, we do need that the finite-dimensional distributions coincide. The Markov property guarantees just this once the one-dimensional distributions coincide!

Problem 6.17. Solution:

a) We have

$$\mathbb{P}(B_t = 0 \text{ for some } t \in (u, v)) = 1 - \mathbb{P}(B_t \neq 0 \text{ for all } t \in (u, v)).$$

But the complementary probability is known from Theorem 6.20.

$$\mathbb{P}(B_t \neq 0 \text{ for all } t \in (u, v)) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{v}}$$

and so

$$\mathbb{P}(B_t = 0 \text{ for some } t \in (u, v)) = 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{u}{v}}.$$

b) Since $(u, v) \subset (u, w)$ we find with the classical conditional probability that

$$\begin{aligned} & \mathbb{P}(B_t \neq 0 \forall t \in (u, w) \mid B_t \neq 0 \forall t \in (u, v)) \\ &= \frac{\mathbb{P}(\{B_t \neq 0 \forall t \in (u, w)\} \cap \{B_t \neq 0 \forall t \in (u, v)\})}{\mathbb{P}(B_t \neq 0 \forall t \in (u, v))} \\ &= \frac{\mathbb{P}(B_t \neq 0 \forall t \in (u, w))}{\mathbb{P}(B_t \neq 0 \forall t \in (u, v))} \\ &\stackrel{\text{a)}}{=} \frac{\arcsin \sqrt{\frac{u}{w}}}{\arcsin \sqrt{\frac{u}{v}}} \end{aligned}$$

c) We have

$$\begin{aligned} & \mathbb{P}(B_t \neq 0 \forall t \in (0, w) \mid B_t \neq 0 \forall t \in (0, v)) \\ &= \lim_{u \rightarrow 0} \mathbb{P}(B_t \neq 0 \forall t \in (u, w) \mid B_t \neq 0 \forall t \in (u, v)) \\ &\stackrel{\text{b)}}{=} \lim_{u \rightarrow 0} \frac{\arcsin \sqrt{\frac{u}{w}}}{\arcsin \sqrt{\frac{u}{v}}} \\ &\stackrel{\text{a)}}{=} \lim_{\text{L'H\^opital } u \rightarrow 0} \frac{\sqrt{v}\sqrt{v-u}}{\sqrt{w}\sqrt{w-u}} \\ &= \frac{\sqrt{v}}{\sqrt{w}}. \end{aligned}$$

■ ■

7 Brownian motion and transition semigroups

Problem 7.1. Solution: Banach space: It is obvious that $\mathcal{C}_\infty(\mathbb{R}^d)$ is a linear space. Let us show that it is closed. By definition, $u \in \mathcal{C}_\infty(\mathbb{R}^d)$ if

$$\forall \epsilon > 0 \quad \exists R > 0 \quad \forall |x| > R : |u(x)| < \epsilon. \quad (*)$$

Let $(u_n)_n \subset \mathcal{C}_\infty(\mathbb{R}^d)$ be a Cauchy sequence for the uniform convergence. It is clear that the uniform limit $u = \lim_n u_n$ is again continuous. Fix ϵ and pick R as in (*). Then we get

$$|u(x)| \leq |u_n(x) - u(x)| + |u_n(x)| \leq \|u_n - u\|_\infty + |u_n(x)|.$$

By uniform convergence, there is some $n(\epsilon)$ such that

$$|u(x)| \leq \epsilon + |u_{n(\epsilon)}(x)| \quad \text{for all } x \in \mathbb{R}^d.$$

Since $u_{n(\epsilon)} \in \mathcal{C}_\infty$, we find with (*) some $R = R(n(\epsilon), \epsilon) = R(\epsilon)$ such that

$$|u(x)| \leq \epsilon + |u_{n(\epsilon)}(x)| \leq \epsilon + \epsilon \quad \forall |x| > R(\epsilon).$$

Density: Fix an ϵ and pick $R > 0$ as in (*), and pick a cut-off function $\chi = \chi_R \in \mathcal{C}(\mathbb{R}^d)$ such that

$$\mathbf{1}_{\overline{B}(0,R)} \leq \chi_R \leq \mathbf{1}_{B(0,2R)}.$$

Clearly, $\text{supp } \chi_R$ is compact, $\chi_R \uparrow 1$, $\chi_R u \in \mathcal{C}_c(\mathbb{R}^d)$ and

$$\sup_x |u(x) - \chi_R(x)u(x)| = \sup_{|x|>R} |\chi_R(x)u(x)| \leq \sup_{|x|>R} |u(x)| < \epsilon.$$

This shows that $\mathcal{C}_c(\mathbb{R}^d)$ is dense in $\mathcal{C}_\infty(\mathbb{R}^d)$. ■ ■

Problem 7.2. Solution: Fix $(t, y, v) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{C}_\infty(\mathbb{R}^d)$, $\epsilon > 0$, and take any $(s, x, u) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{C}_\infty(\mathbb{R}^d)$. Then we find using the triangle inequality

$$\begin{aligned} |P_s u(x) - P_t v(y)| &\leq |P_s u(x) - P_s v(x)| + |P_s v(x) - P_t v(x)| + |P_t v(x) - P_t v(y)| \\ &\leq \sup_x |P_s u(x) - P_s v(x)| + \sup_x |P_s v(x) - P_s P_{t-s} v(x)| + |P_t v(x) - P_t v(y)| \\ &= \|P_s(u - v)\|_\infty + \|P_s(v - P_{t-s} v)\|_\infty + |P_t v(x) - P_t v(y)| \\ &\leq \|u - v\|_\infty + \|v - P_{t-s} v\|_\infty + |P_t v(x) - P_t v(y)| \end{aligned}$$

where we used the contraction property of P_s .

- Since $y \mapsto P_t v(y)$ is continuous, there is some $\delta_1 = \delta_1(t, y, v, \epsilon)$ such that $|x - y| < \delta \implies |P_t v(x) - P_t v(y)| < \epsilon$.
 - Using the strong continuity of the semigroup (Proposition 7.3 f) there is some $\delta_2 = \delta_2(t, v, \epsilon)$ such that $|t - s| < \delta_2 \implies \|P_{t-s} v - v\|_\infty \leq \epsilon$.
- . This proves that for $\delta := \min\{\epsilon, \delta_1, \delta_2\}$

$$|s - t| + |x - y| + \|u - v\|_\infty \leq \delta \implies |P_s u(x) - P_t v(y)| \leq 3\epsilon.$$

Problem 7.3. Solution: By the tower property we find

$$\begin{aligned} \mathbb{E}^x(f(X_t)g(X_{t+s})) &\stackrel{\substack{\text{tower} \\ \text{property}}}{=} \mathbb{E}^x\left(\mathbb{E}^x(f(X_t)g(X_{t+s}) \mid \mathcal{F}_t)\right) \\ &\stackrel{\substack{\text{pull} \\ \text{out}}}{=} \mathbb{E}^x\left(f(X_t) \mathbb{E}^x(g(X_{t+s}) \mid \mathcal{F}_t)\right) \\ &\stackrel{\substack{\text{Markov} \\ \text{property}}}{=} \mathbb{E}^x\left(f(X_t) \mathbb{E}^{X_t}(g(X_s))\right) \\ &= \mathbb{E}^x(f(X_t)h(X_t)) \end{aligned}$$

where, for every s ,

$$h(y) = \mathbb{E}^y g(X_s) \text{ is again in } \mathcal{C}_\infty.$$

Thus, $\mathbb{E}^x f(X_t)g(X_{t+s}) = \mathbb{E}^x \phi(X_t)$ and $\phi(y) = f(y)h(y)$ is in \mathcal{C}_∞ . This shows that $x \mapsto \mathbb{E}^x(f(X_t)g(X_{t+s}))$ is in \mathcal{C}_∞ .

Using semigroups we can write the above calculation in the following form:

$$\mathbb{E}^x(f(X_t)g(X_{t+s})) = \mathbb{E}^x(f(X_t)P_s g(X_t)) = P_t(fP_s g)(x)$$

i.e. $h = P_s$ and $\phi = f \cdot P_s g$, and since P_t preserves \mathcal{C}_∞ , the claim follows.

Problem 7.4. Solution: Set $u(t, z) := P_t u(z) = p_t \star u(z) = (2\pi t)^{d/2} \int_{\mathbb{R}^d} u(y) e^{-|z-y|^2/2t} dy$.

$u(t, \cdot)$ is in \mathcal{C}^∞ for $t > 0$: Note that the Gauss kernel

$$p_t(z - y) = (2\pi t)^{-d/2} e^{-|z-y|^2/2t}, \quad t > 0$$

can be arbitrarily often differentiated in z and

$$\partial_z^k p_t(z - y) = Q_k(z, y, t) p_t(z - y)$$

where the function $Q_k(z, y, t)$ grows at most polynomially in z and y . Since $p_t(z - y)$ decays exponentially, we see — as in the proof of Proposition 7.3 g) — that for each z

$$\begin{aligned} &|\partial_z^k p_t(z - y)| \\ &\leq \sup_{|y| \leq 2R} |Q_k(z, y, t)| \mathbf{1}_{B(0, 2R)}(y) + \sup_{|y| \geq 2R} |Q_k(z, y, t) e^{-|y|^2/(16t)}| e^{-|y|^2/(16t)} \mathbf{1}_{B^c(0, 2R)}(y). \end{aligned}$$

This inequality holds uniformly in a small neighbourhood of z , i.e. we can use the differentiation lemma from measure and integration to conclude that $\partial^k P_t u \in \mathcal{C}_b$.

$x \mapsto \partial_t u(t, x)$ is in \mathcal{C}^∞ for $t > 0$: This follows from the first part and the fact that

$$\begin{aligned} \partial_t p_t(z-y) &= -\frac{d}{2}(2\pi t)^{-d/2-1} e^{-|z-y|^2/2t} + (2\pi t)^{-d/2} e^{-|z-y|^2/2t} \frac{|z-y|^2}{2t^2} \\ &= \frac{1}{2} \left(\frac{|z-y|^2}{t^2} - \frac{d}{t} \right) p_t(z-y). \end{aligned}$$

Again with the domination argument of the first part we see that $\partial_t \partial_x^k u(t, x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$.

Problem 7.5. Solution:

- a) Note that $|u_n| \leq |u| \in L^p$. Since $|u_n - u|^p \leq (|u_n| + |u|)^p \leq (|u| + |u|)^p = 2^p |u|^p \in L^1$ and since $|u_n(x) - u(x)| \rightarrow 0$ for every x as $n \rightarrow \infty$, the claim follows by dominated convergence.
- b) Let $u \in L^p$ and $m < n$. We have

$$\|P_t u_n - P_t u_m\|_{L^p} = \|p_t \star (u_n - u_m)\|_{L^p} \stackrel{\text{Young}}{\leq} \|p_t\|_{L^1} \|u_n - u_m\|_{L^p} = \|u_n - u_m\|_{L^p}.$$

Since $(u_n)_n$ is an L^p Cauchy sequence (it converges in L^p towards $u \in L^p$), so is $(P_t u_n)_n$, and therefore $\tilde{P}_t u := \lim_n P_t u_n$ exists in L^p .

If v_n is any other sequence in L^p with limit u , the above argument shows that $\lim_n P_t v_n$ also exists. ‘Mixing’ the sequences $(w_n) := (u_1, v_1, u_2, v_2, u_3, v_3, \dots)$ produces yet another convergent sequence with limit u , and we conclude that

$$\lim_n P_t u_n = \lim_n P_t w_n = \lim_n P_t v_n,$$

i.e. \tilde{P}_t is well-defined.

- c) Any $u \in L^p$ with $0 \leq u \leq 1$ has a representative $u \in \mathcal{B}_b$. And then the claim follows since P_t is sub-Markovian.
- d) Recall that $y \mapsto \|u(\cdot + y) - u\|_{L^p}$ is for $u \in L^p(dx)$ a continuous function. By Fubini’s theorem and the Hölder inequality

$$\begin{aligned} \|P_t u - u\|_{L^p}^p &= \int |\mathbb{E} u(x + B_t) - u(x)|^p dx \\ &\leq \mathbb{E} \left(\int |u(x + B_t) - u(x)|^p dx \right) \\ &= \mathbb{E} (\|u(\cdot + B_t) - u\|_{L^p}^p). \end{aligned}$$

The integrand is bounded by $2^p \|u\|_{L^p}^p$, and continuous as a function of t ; therefore we can use the dominated convergence theorem to conclude that $\lim_{t \rightarrow 0} \|P_t u - u\|_{L^p} = 0$.

Problem 7.6. Solution: Let $u \in \mathcal{C}_b$. Then we have, by definition

$$\begin{aligned} T_{t+s}u(x) &= \int_{\mathbb{R}^d} u(z) p_{t+s}(x, dz) \\ T_t(T_s u)(x) &= \int_{\mathbb{R}^d} T_s u(y) p_t(x, dy) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(z) p_s(y, dz) p_t(x, dy) \\ &= \int_{\mathbb{R}^d} u(z) \int_{\mathbb{R}^d} p_s(y, dz) p_t(x, dy) \end{aligned}$$

By the semigroup property, $T_{t+s} = T_t T_s$, and we see that

$$p_{t+s}(x, dz) = \int_{\mathbb{R}^d} p_s(y, dz) p_t(x, dy).$$

If we pick $u = \mathbb{1}_C$, this formal equality becomes

$$p_{t+s}(x, C) = \int_{\mathbb{R}^d} p_s(y, C) p_t(x, dy).$$

Problem 7.7. Solution: Using $T_t \mathbb{1}_C(x) = p_t(x, C) = \int \mathbb{1}_C(y) p_t(x, dy)$ we get

$$\begin{aligned} & p_{t_1, \dots, t_n}^x(C_1 \times \dots \times C_n) \\ &= T_{t_1} \left(\mathbb{1}_{C_1} \left[T_{t_2-t_1} \mathbb{1}_{C_2} \left\{ \dots T_{t_{n-1}-t_{n-2}} \int \mathbb{1}_{C_n}(x_n) p_{t_n-t_{n-1}}(\cdot, dx_n) \dots \right\} \right] \right) (x) \\ &= T_{t_1} \left(\mathbb{1}_{C_1} \left[T_{t_2-t_1} \mathbb{1}_{C_2} \left\{ \dots \int \mathbb{1}_{C_{n-1}}(x_{n-1}) \int \mathbb{1}_{C_n}(x_n) p_{t_n-t_{n-1}}(x_{n-1}, dx_n) \times \right. \right. \right. \\ &\quad \left. \left. \left. \times p_{t_{n-1}-t_{n-2}}(\cdot, dx_{n-1}) \dots \right\} \right] \right) (x) \\ &\vdots \\ &= \underbrace{\int \dots \int}_{n \text{ integrals}} \mathbb{1}_{C_1}(x_1) \mathbb{1}_{C_2}(x_2) \dots \mathbb{1}_{C_n}(x_n) p_{t_n-t_{n-1}}(x_{n-1}, dx_n) p_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1}) \times \\ &\quad \dots \times p_{t_2-t_1}(x_2, dx_2) p_{t_1}(x, dx_1) \\ &= \underbrace{\int \dots \int}_{n \text{ integrals}} \mathbb{1}_{C_1 \times \dots \times C_n}(x_1, \dots, x_n) \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, dx_j) \end{aligned}$$

(we set $t_0 := 0$ and $x_0 := x$).

This shows that $p_{t_1, \dots, t_n}^x(C_1 \times \dots \times C_n)$ is the restriction of

$$p_{t_1, \dots, t_n}^x(\Gamma) = \underbrace{\int \dots \int}_{n \text{ integrals}} \mathbb{1}_{\Gamma}(x_1, \dots, x_n) \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, dx_j), \quad \Gamma \in \mathcal{B}(\mathbb{R}^{d \cdot n})$$

and the right-hand side clearly defines a probability measure. By the uniqueness theorem for measures, each measure is uniquely defined by its values on the rectangles, so we are done.

Problem 7.8. Solution:

a) Let $x, y \in \mathbb{R}^d$ and $a \in A$. Then

$$\inf_{\alpha \in A} |x - \alpha| \leq |x - a| \leq |x - y| + |a - y|$$

Since this holds for all $a \in A$, we get

$$\inf_{\alpha \in A} |x - \alpha| \leq |x - y| + \inf_{a \in A} |a - y|$$

and, since x, y play symmetric roles,

$$|d(x, A) - d(y, A)| = \left| \inf_{\alpha \in A} |x - \alpha| - \inf_{a \in A} |a - y| \right| \leq |x - y|.$$

b) By definition, $U_n = K + \mathbb{B}(0, 1/n)$ and $u_n(x) := \frac{d(x, U_n^c)}{d(x, K) + d(x, U_n^c)}$. Being a combination of continuous functions, see Part (a), u_n is clearly continuous. Moreover,

$$u_n|_K \equiv 1 \quad \text{and} \quad u_n|_{U_n^c} \equiv 0.$$

This shows that $\mathbf{1}_K \leq u_n \leq \mathbf{1}_{U_n^c} \xrightarrow{n \rightarrow \infty} \mathbf{1}_K$.

Picture: u_n is piecewise linear.

c) Assume, without loss of generality, that $\text{supp } \chi_n \subset \mathbb{B}(0, 1/n^2)$. Since $0 \leq u_n \leq 1$, we find

$$\chi_n \star u_n(x) = \int \chi_n(x - y) u_n(y) dy \leq \int \chi_n(x - y) dy = 1 \quad \forall x.$$

Now we observe that for $\gamma \in (0, 1)$

$$u_n(y) = \frac{d(y, U_n^c)}{d(y, K) + d(y, U_n^c)} \geq \frac{(1 - \gamma)/n}{1/n} = 1 - \gamma. \quad \forall y \in K + \mathbb{B}(0, \gamma/n)$$

(Essentially this means that u_n is ‘linear’ for $x \in U_n \setminus K$!). Thus, if $\gamma > 1/n$,

$$\begin{aligned} \chi_n \star u_n(x) &= \int \chi_n(x - y) u_n(y) dy \\ &\geq (1 - \gamma) \int \chi_n(x - y) \mathbf{1}_{K + \mathbb{B}(0, \gamma/n)}(y) dy \\ &= (1 - \gamma) \int \chi_n(x - y) \mathbf{1}_{\mathbb{B}(0, 1/n^2)}(x - y) \mathbf{1}_{K + \mathbb{B}(0, \gamma/n)}(y) dy \\ &= (1 - \gamma) \int \chi_n(x - y) \mathbf{1}_{x + \mathbb{B}(0, 1/n^2)}(y) \mathbf{1}_{K + \mathbb{B}(0, \gamma/n)}(y) dy \\ &\geq (1 - \gamma) \int \chi_n(x - y) \mathbf{1}_{x + \mathbb{B}(0, 1/n^2)}(y) dy \\ &= 1 - \gamma \quad \forall x \in K. \end{aligned}$$

This shows that

$$1 - \gamma \leq \liminf_n \chi_n \star u_n(x) \leq \limsup_n \chi_n \star u_n(x) \leq 1 \quad \forall x \in K,$$

hence,

$$\lim_{n \rightarrow \infty} \chi_n \star u_n(x) = x \quad \text{for all } x \in K.$$

On the other hand, if $x \in K^c$, there is some $n \geq 1$ such that $d(x, K) > \frac{1}{n} + \frac{1}{n^2}$. Since

$$\frac{1}{n} + \frac{1}{n^2} < d(x, K) \leq d(x, y) + d(y, K) \implies d(x, y) > \frac{1}{n^2} \quad \text{or} \quad d(y, K) > \frac{1}{n},$$

and so, using that $\text{supp } \chi_n \subset \mathbb{B}(0, 1/n^2)$ and $\text{supp } u_n \subset K + \mathbb{B}(0, 1/n)$,

$$\chi_n \star u_n(x) = \int \chi_n(x-y)u_n(y) dy = 0 \quad \forall x : d(x, K) > \frac{1}{n} + \frac{1}{n^2}.$$

It follows that $\lim_n \chi_n \star u_n(x) = 0$ for $x \in K^c$.

Remark 1: If we are just interested in a smooth function approximating $\mathbb{1}_K$ we could use $v_n := \chi_n \star \mathbb{1}_{K+\text{supp } u_n}$ where $(\chi_n)_n$ is any sequence of type δ . Indeed, as before,

$$\chi_n \star \mathbb{1}_{K+\text{supp } u_n}(x) = \int \chi_n(x-y)\mathbb{1}_{K+\text{supp } u_n}(y) dy \leq \int \chi_n(x-y) dy = 1 \quad \forall x.$$

For $x \in K$ we find

$$\begin{aligned} \chi_n \star \mathbb{1}_{K+\text{supp } u_n}(x) &= \int \chi_n(x-y)\mathbb{1}_{K+\text{supp } u_n}(y) dy \\ &= \int \chi_n(y)\mathbb{1}_{K+\text{supp } u_n}(x-y) dy \\ &= \int \chi_n(y) dy \\ &= 1 \quad \forall x \in K. \end{aligned}$$

As before we get $\chi_n \star \mathbb{1}_{K+\text{supp } u_n}(x) = 0$ if $d(x, K) > 2/n$.

Thus, $\lim_n \chi_n \star \mathbb{1}_{K+\text{supp } u_n}(x) = 0$ if $x \in K^c$.

Remark 2: The naive approach $\chi_n \star \mathbb{1}_K$ will, in general, not lead to a (pointwise everywhere) approximation of $\mathbb{1}_K$: consider $K = \{0\}$, then $\chi_n \star \mathbb{1}_K \equiv 0$. In fact, since $\mathbb{1}_K \in L^1$ we get $\chi_n \star \mathbb{1}_K \rightarrow \mathbb{1}_K$ in L^1 hence, for a subsequence, a.e. ...

Problem 7.9. Solution:

- a) This follows from part (c).
- b) This follows by approximating $\mathbb{1}_K$ from above by a decreasing sequence of \mathcal{C}_∞ functions. Such a sequence exists, see Problem 7.8 above.

Remark: If we know that the kernel $p_t(x, K) := T_t \mathbb{1}_K(x)$ is inner (compact) regular — or outer (open) regular, which is the same and does always hold in this topologically nice situation with bounded measures, see Schilling [18, Appendix H] — then we get that $(t, x) \mapsto p_t(x, C)$ is measurable for all Borel measures $C \subset \mathbb{R}^d$. Just observe that $p_t(x, C) = \sup\{p_t(x, K) : K \subset C, K \text{ compact}\}$ and use the fact the the supremum is attained by a sequence (which may depend on C , of course).

- c) This is a standard $3\text{-}\epsilon$ -trick. Fix $\epsilon > 0$, fix (s, x, u) and consider another point (t, y, w) . Without loss of generality we assume that $s \leq t$. Then, by the triangle inequality, the semigroup property and the contractivity (in the last step), we get

$$|T_s u(x) - T_t w(y)| \leq |T_s u(x) - T_s u(y)| + |T_s u(y) - T_t u(y)| + |T_t u(y) - T_t w(y)|$$

$$\begin{aligned}
 &\leq |T_s u(x) - T_s u(y)| + \|T_s u - T_t u\|_\infty + \|T_t(u - w)\|_\infty \\
 &\leq |T_s u(x) - T_s u(y)| + \|T_s(u - T_{t-s}u)\|_\infty + \|T_t(u - w)\|_\infty \\
 &\leq |T_s u(x) - T_s u(y)| + \|u - T_{t-s}u\|_\infty + \|u - w\|_\infty.
 \end{aligned}$$

Now we know that for given ϵ there are $\delta_1, \delta_2, \delta_3$ such that

$$\begin{aligned}
 \|u - w\|_\infty < \delta_1 &\implies \|u - w\|_\infty < \epsilon && \text{(pick } \delta_1 = \epsilon) \\
 |t - s| < \delta_2 &\implies \|T_{t-s}u - u\|_\infty < \epsilon && \text{(by strong continuity)} \\
 |x - y| < \delta_3 &\implies |T_s u(x) - T_s u(y)| < \epsilon && \text{(by the Feller property).}
 \end{aligned}$$

This proves continuity with $\delta := \min(\delta_1, \delta_2, \delta_3)$; note that δ may (and will) depend on ϵ as well as on the fixed point (s, x, u) , as we require continuity at this point only. Mind that there are minor, but obvious, changes necessary if $s = 0$.

Remark: A full account on Feller semigroups can be found in Böttcher–Schilling–Wang [1, Chapter 1].

■ ■

Problem 7.10. Solution:

a) Existence, contractivity: Let us, first of all, check that the series converges. Denote by $\|A\|$ any matrix norm in \mathbb{R}^d . Then we see

$$\|P_t\| = \left\| \sum_{j=0}^{\infty} \frac{(tA)^j}{j!} \right\| \leq \sum_{j=0}^{\infty} \frac{t^j \|A^j\|}{j!} \leq \sum_{j=0}^{\infty} \frac{t^j \|A\|^j}{j!} = e^{t\|A\|}.$$

This shows that, in general, P_t is not a contraction. We can make it into a contraction by setting $Q_t := e^{-t\|A\|} P_t$. It is clear that Q_t is again a semigroup, if P_t is a semigroup.

Semigroup property: This is shown using as for the one-dimensional exponential series. Indeed,

$$\begin{aligned}
 e^{(t+s)A} &= \sum_{k=0}^{\infty} \frac{(t+s)^k A^k}{k!} \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{1}{k!} \binom{k}{j} t^j s^{k-j} A^k \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{t^j A^j}{j!} \frac{s^{k-j} A^{k-j}}{(k-j)!} \\
 &= \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} \sum_{k=j}^{\infty} \frac{s^{k-j} A^{k-j}}{(k-j)!} \\
 &= \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} \sum_{l=0}^{\infty} \frac{s^l A^l}{l!} \\
 &= e^{tA} e^{sA}.
 \end{aligned}$$

Strong continuity: We have

$$\|e^{tA} - \text{id}\| = \left\| \sum_{j=1}^{\infty} \frac{t^j A^j}{j!} \right\| = t \left\| \sum_{j=1}^{\infty} \frac{t^{j-1} A^j}{j!} \right\|$$

and, as in the first calculation, we see that the series converges absolutely. Letting $t \rightarrow 0$ shows strong continuity, even continuity in the operator norm.

(Strictly speaking, strong continuity means that for each vector $v \in \mathbb{R}^d$

$$\lim_{t \rightarrow 0} |e^{tA}v - v| = 0.$$

Since

$$|e^{tA}v - v| \leq \|e^{tA} - \text{id}\| \cdot |v|$$

strong continuity is implied by uniform continuity. One can show that the generator of a norm-continuous semigroup is already a bounded operator, see e.g. Pazy.)

b) Let $s, t > 0$. Then

$$e^{tA} - e^{sA} = \sum_{j=0}^{\infty} \left(\frac{t^j A^j}{j!} - \frac{s^j A^j}{j!} \right) = \sum_{j=1}^{\infty} \frac{(t^j - s^j) A^j}{j!}$$

Since the sum converges absolutely, we get

$$\frac{e^{tA} - e^{sA}}{t - s} = \sum_{j=1}^{\infty} \frac{(t^j - s^j) A^j}{t - s} \frac{1}{j!} \xrightarrow{s \rightarrow t} \sum_{j=1}^{\infty} j t^{j-1} \frac{A^j}{j!}.$$

The last expression, however, is

$$\sum_{j=1}^{\infty} j t^{j-1} \frac{A^j}{j!} = A \sum_{j=1}^{\infty} t^{j-1} \frac{A^{j-1}}{(j-1)!} = A e^{tA}.$$

A similar calculation, pulling out A to the back, yields that the sum is also $e^{tA}A$.

c) Assume first that $AB = BA$. Repeated applications of this rule show $A^j B^k = B^k A^j$ for all $j, k \geq 0$. Thus,

$$e^{tA} e^{tB} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j A^j}{j!} \frac{t^k B^k}{k!} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j t^k A^j B^k}{j! k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^k t^j B^k A^j}{k! j!} = e^{tB} e^{tA}.$$

Conversely, if $e^{tA} e^{tB} = e^{tB} e^{tA}$ for all $t > 0$, we get

$$\lim_{t \rightarrow 0} \frac{e^{tA} - \text{id}}{t} \frac{e^{tB} - \text{id}}{t} = \lim_{t \rightarrow 0} \frac{e^{tB} - \text{id}}{t} \frac{e^{tA} - \text{id}}{t}$$

and this proves $AB = BA$.

Alternative solution for the converse: If $s = j/n$ and $t = k/n$ for some common denominator n , we get from $e^{tA} e^{tB} = e^{tB} e^{tA}$ that

$$e^{tA} e^{sB} = \underbrace{e^{\frac{1}{n}A} \dots e^{\frac{1}{n}A}}_k \underbrace{e^{\frac{1}{n}B} \dots e^{\frac{1}{n}B}}_j = \underbrace{e^{\frac{1}{n}B} \dots e^{\frac{1}{n}B}}_j \underbrace{e^{\frac{1}{n}A} \dots e^{\frac{1}{n}A}}_k = e^{sB} e^{tA}.$$

Thus, if $s, t > 0$ are dyadic numbers, we get

$$A e^{sB} = \lim_{t \rightarrow 0} \frac{e^{tA} - \text{id}}{t} e^{sB} = e^{sB} \lim_{t \rightarrow 0} \frac{e^{tA} - \text{id}}{t} = e^{sB} A$$

and,

$$AB = A \lim_{s \rightarrow 0} \frac{e^{sB} - \text{id}}{s} = \lim_{s \rightarrow 0} \frac{e^{sB} - \text{id}}{s} A = BA.$$

d) We have

$$e^{A/k} = \text{id} + \frac{1}{k} A + \rho_k \quad \text{and} \quad k^2 \rho_k = \sum_{j=2}^{\infty} \frac{A^j}{j!} \frac{1}{k^{j-2}}.$$

Note that $k^2 \rho_k$ is bounded. Do the same for B (with the remainder term ρ'_k) and multiply these expansions to get

$$e^{A/k} e^{B/k} = \text{id} + \frac{1}{k} A + \frac{1}{k} B + \sigma_k$$

where $k^2 \sigma_k$ is again bounded. In particular, if $k \gg 1$,

$$\left\| \frac{1}{k} A + \frac{1}{k} B + \sigma_k \right\| < 1.$$

This allows us to (formally) apply the logarithm series

$$\log(e^{A/k} e^{B/k}) = \frac{1}{k} A + \frac{1}{k} B + \sigma_k + \sigma'_k$$

where $k^2 \sigma'_k$ is bounded. Multiply with k to get

$$k \log(e^{A/k} e^{B/k}) = A + B + \tau_k$$

with $k \tau_k$ bounded. Then we get

$$\begin{aligned} e^{A+B} &= \lim_{k \rightarrow \infty} e^{A+B+\tau_k} \\ &= \lim_{k \rightarrow \infty} e^{k \log(e^{A/k} e^{B/k})} \\ &= \lim_{k \rightarrow \infty} \left(e^{\log(e^{A/k} e^{B/k})} \right)^k \\ &= \lim_{k \rightarrow \infty} \left(e^{A/k} e^{B/k} \right)^k \end{aligned}$$

Alternative Solution: Set $S_k = e^{(A+B)/k}$ and $T_k = e^{A/k} e^{B/k}$. Then

$$S_k^k - T_k^k = \sum_{j=0}^{k-1} S_k^j (S_k - T_k) T_k^{k-1-j}.$$

This shows that

$$\begin{aligned} \|S_k^k - T_k^k\| &\leq \sum_{j=0}^{k-1} \|S_k^j (S_k - T_k) T_k^{k-1-j}\| \\ &\leq \sum_{j=0}^{k-1} \|S_k^j\| \cdot \|S_k - T_k\| \cdot \|T_k^{k-1-j}\| \\ &\leq k \|S_k - T_k\| \cdot \max\{\|S_k\|, \|T_k\|\}^{k-1} \\ &\leq k \|S_k - T_k\| \cdot e^{\|A\| + \|B\|}. \end{aligned}$$

Observe that

$$\|S_k - T_k\| = \left\| \sum_{j=0}^{\infty} \frac{(A+B)^j}{k^j j!} - \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{A^j B^l}{k^j j! k^l l!} \right\| \leq \frac{C}{k^2}$$

with a constant C depending only on $\|A\|$ and $\|B\|$. This yields $S_k^k - T_k^k \rightarrow 0$.



Problem 7.11. Solution:

a) Let $0 < s < t$ and assume throughout that $h \in \mathbb{R}$ is such that $t - s - h > 0$. We have

$$\begin{aligned} P_{t-(s+h)}T_{s+h} - P_{t-s}T_s &= P_{t-(s+h)}T_{s+h} - P_{t-(s+h)}T_s + P_{t-(s+h)}T_s - P_{t-s}T_s \\ &= P_{t-(s+h)}(T_{s+h} - T_s) + (P_{t-(s+h)} - P_{t-s})T_s \\ &= (P_{t-(s+h)} - P_{t-s})(T_{s+h} - T_s) + P_{t-s}(T_{s+h} - T_s) + (P_{t-(s+h)} - P_{t-s})T_s. \end{aligned}$$

Divide by $h \neq 0$ to get for all $u \in \mathfrak{D}(B) \subset \mathfrak{D}(A)$

$$\begin{aligned} &\frac{1}{h}(P_{t-(s+h)}T_{s+h}u - P_{t-s}T_su) \\ &= (P_{t-(s+h)} - P_{t-s})\frac{T_{s+h}u - T_su}{h} + P_{t-s}\frac{T_{s+h}u - T_su}{h} + \frac{P_{t-(s+h)} - P_{t-s}}{h}T_su \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

Letting $h \rightarrow 0$ gives for all $u \in \mathfrak{D}(B) \subset \mathfrak{D}(A)$

$$\text{II} \rightarrow P_{t-s}BT_s \quad \text{and} \quad \text{III} \rightarrow -P_{t-s}AT_s$$

(we use for the last asserting that $T_s(\mathfrak{D}(B)) \stackrel{\text{Lemma 7.10.a)}}{\subset} \mathfrak{D}(B) \subset \mathfrak{D}(A)$). Let us show that $\text{I} \rightarrow 0$. We have

$$\text{I} = (P_{t-(s+h)} - P_{t-s})\left(\frac{T_{s+h}u - T_su}{h} - T_sBu\right) + (P_{t-(s+h)} - P_{t-s})T_sBu = \text{I}_1 + \text{I}_2.$$

By the strong continuity of the semigroup $(P_t)_t$, we see that $\text{I}_2 \rightarrow 0$ as $h \rightarrow 0$. Furthermore, by contractivity,

$$\|\text{I}_1\| \leq (\|P_{t-(s+h)}\| + \|P_{t-s}\|) \cdot \left\| \frac{T_{s+h}u - T_su}{h} - T_sBu \right\| \leq 2 \left\| \frac{T_{s+h}u - T_su}{h} - T_sBu \right\| \rightarrow 0$$

since $u \in \mathfrak{D}(B)$.

Remark. Usually this identity is used if $\mathfrak{D}(A) = \mathfrak{D}(B)$, for example we have used it in this way in the proof of Corollary 7.11.c) on page 96. Another, typical application is the situation where $B-A$ is a bounded operator (hence, $\mathfrak{D}(A) = \mathfrak{D}(B)$). Integrating the identity of part a) yields

$$T_tu - P_tu = \int_0^t \frac{d}{ds}(P_{t-s}T_s)u ds = \int_0^t P_{t-s}(B-A)T_su ds$$

which is often referred to as *Duhamel's formula*. This formula holds first for $u \in \mathfrak{D}(B)$ and then, by extension of bounded linear operators defined on a dense set, for all u in the closure $\overline{\mathfrak{D}(B)}$.

- b) In general, no. The problem is the semigroup property (unless T_t and P_s commute for all $s, t \geq 0$):

$$U_t U_s = T_t P_t T_s P_s \neq T_t T_s P_t P_s = T_{t+s} P_{t+s} = U_{t+s}.$$

In (c) we see how this can be ‘remedied’.

It is interesting to note (and helpful for the proof of (c)) that U_t is an operator on \mathcal{C}_∞ :

$$U_t : \mathcal{C}_\infty \xrightarrow{P_t} \mathcal{C}_\infty \xrightarrow{T_t} \mathcal{C}_\infty$$

and that U_t is strongly continuous: for all $s, t \geq 0$ and $f \in \mathcal{C}_\infty$

$$\begin{aligned} \|U_t f - U_s f\| &= \|T_t P_t f - T_s P_t f + T_s P_t f - T_s P_s f\| \\ &\leq \|(T_t - T_s) P_t f\| + \|T_s (P_t - P_s) f\| \\ &\leq \|(T_t - T_s) P_t f\| + \|(P_t - P_s) f\| \end{aligned}$$

and, as $s \rightarrow t$, both expressions tend to 0 since $f, P_t f \in \mathcal{C}_\infty$.

- c) Set $U_{t,n} := (T_{t/n} P_{t/n})^n$.

U_t is a contraction on \mathcal{C}_∞ : By assumption, $P_{t/n}$ and $T_{t/n}$ map \mathcal{C}_∞ into itself and, therefore, $T_{t/n} P_{t/n} : \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty$ as well as $U_{t,n}$.

We have $\|U_{t,n} f\| = \|T_{t/n} P_{t/n} \cdots T_{t/n} P_{t/n} f\| \leq \prod_{j=1}^n \|T_{t/n}\| \|P_{t/n}\| \|f\| \leq \|f\|$. So, by the continuity of the norm

$$\|U_t f\| = \left\| \lim_n U_{t,n} f \right\| = \lim_n \|U_{t,n} f\| \leq \|f\|.$$

Strong continuity: Since the limit defining U_t is locally uniform in t , it is enough to show that $U_{t,n}$ is strongly continuous. Let X, Y be contractions in \mathcal{C}_∞ . Then we get

$$\begin{aligned} X^n - Y^n &= X^{n-1} X - X^{n-1} Y + X^{n-1} Y - Y^{n-1} Y \\ &= X^{n-1} (X - Y) + (X^{n-1} - Y^{n-1}) Y \end{aligned}$$

hence, by the contraction property,

$$\|X^n f - Y^n f\| \leq \|(X - Y) f\| + \|(X^{n-1} - Y^{n-1}) Y f\|.$$

By iteration, we get

$$\|X^n f - Y^n f\| \leq \sum_{k=0}^{n-1} \|(X - Y) Y^k f\|.$$

Take $Y = T_{t/n} P_{t/n}$, $X = T_{s/n} P_{s/n}$ where n is fixed. Then letting $s \rightarrow t$ shows the strong continuity of each $t \mapsto U_{t,n}$.

Semigroup property: Let $s, t \in \mathbb{Q}$ and write $s = j/m$ and $t = k/m$ for the same m . Then we take $n = l(j+k)$ and get

$$\left(T_{\frac{s+t}{n}} P_{\frac{s+t}{n}} \right)^n = \left(T_{\frac{1}{lm}} P_{\frac{1}{lm}} \right)^{l(j+k)}$$

$$\begin{aligned}
 &= \left(T_{\frac{1}{lm}} P_{\frac{1}{lm}}\right)^{lj} \left(T_{\frac{1}{lm}} P_{\frac{1}{lm}}\right)^{lk} \\
 &= \left(T_{\frac{j}{ljm}} P_{\frac{j}{ljm}}\right)^{lj} \left(T_{\frac{k}{lkm}} P_{\frac{k}{lkm}}\right)^{lk} \\
 &= \left(T_{\frac{s}{lj}} P_{\frac{s}{lj}}\right)^{lj} \left(T_{\frac{t}{lk}} P_{\frac{t}{lk}}\right)^{lk}
 \end{aligned}$$

Since $n \rightarrow \infty \iff l \rightarrow \infty \iff lk, lj \rightarrow \infty$, we see that $U_{s+t} = U_s U_t$ for rational s, t .

For arbitrary s, t the semigroup property follows by approximation and the strong continuity of U_t : let $\mathbb{Q} \ni s_n \rightarrow s$ and $\mathbb{Q} \ni t_n \rightarrow t$. Then, by the contraction property,

$$\begin{aligned}
 \|U_s U_t f - U_{s_n} U_{t_n} f\| &\leq \|U_s U_t f - U_s U_{t_n} f\| + \|U_s U_{t_n} f - U_{s_n} U_{t_n} f\| \\
 &\leq \|U_t f - U_{t_n} f\| + \|(U_s - U_{s_n})(U_{t_n} - U_t)f\| + \|(U_s - U_{s_n})U_t f\| \\
 &\leq \|U_t f - U_{t_n} f\| + 2\|(U_{t_n} - U_t)f\| + \|(U_s - U_{s_n})U_t f\|
 \end{aligned}$$

and the last expression tends to 0. The limit $\lim_n U_{s_n+t_n} u = U_{s+t} u$ is obvious.

Generator: Let us begin with a heuristic argument (by ? and ?? indicate the steps which are questionable!). By the chain rule

$$\begin{aligned}
 \left.\frac{d}{dt}\right|_{t=0} U_t g &= \left.\frac{d}{dt}\right|_{t=0} \lim_n (T_{t/n} P_{t/n})^n g \\
 &\stackrel{?}{=} \lim_n \left.\frac{d}{dt}\right|_{t=0} (T_{t/n} P_{t/n})^n g \\
 &\stackrel{??}{=} \lim_n \left[n (T_{t/n} P_{t/n})^{n-1} (T_{t/n} \frac{1}{n} B P_{t/n} + T_{t/n} \frac{1}{n} A P_{t/n}) g \Big|_{t=0} \right] \\
 &= Bg + Ag.
 \end{aligned}$$

So it is sensible to assume that $\mathfrak{D}(A) \cap \mathfrak{D}(B)$ is not empty. For the rigorous argument we have to justify the steps marked by question marks.

Since $\mathfrak{D}(B) \subset \mathfrak{D}(A)$, we can argue as follows: ?? We have to show that $\frac{d}{ds} T_s P_s f$ exists and is $T_s A f + B P_s f$ for $f \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$. This follows similar to (a) since we have for $s, h > 0$

$$\begin{aligned}
 T_{s+h} P_{s+h} f - T_s P_s f &= T_{s+h} (P_{s+h} - P_s) f + (T_{s+h} - T_s) P_s f \\
 &= (T_{s+h} - T_s) (P_{s+h} - P_s) f + T_s (P_{s+h} - P_s) f + (T_{s+h} - T_s) P_s f.
 \end{aligned}$$

Divide by h . Then the first term converges to 0 as $h \rightarrow 0$, while the other two terms tend to $T_s A f$ and $B P_s f$, respectively.

? This is a matter of interchanging limit and differentiation. Recall the following theorem from calculus, e.g. Rudin [15, Theorem 7.17].

Theorem. *Let $(f_n)_n$ be a sequence of differentiable functions on $[0, \infty)$ which converges for some $t_0 > 0$. If $(f'_n)_n$ converges [locally] uniformly, then $(f_n)_n$ converges [locally] uniformly to a differentiable function f and we have $f' = \lim_n f'_n$.*

This theorem holds for functions with values in any Banach space and, therefore, we can apply it to the situation at hand: Fix $g \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$; we know that $f_n(t) := U_{t,n}g$ converges (even locally uniformly) and, because of [??], that $f'_n(t) = (T_{t/n}P_{t/n})^{n-1}(T_{t/n}A + BP_{t/n})g$.

Since $\lim_n (T_{t/n}P_{t/n})^n u$ converges locally uniformly, so does $\lim_n (T_{t/n}P_{t/n})^{n-1}u$; moreover, by the strong continuity, $T_{t/n}A + BP_{t/n} \rightarrow (A + B)g$ locally uniformly for $g \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$. Therefore, the assumptions of the theorem are satisfied and we may interchange the limits in the calculation above.

Remark. It is surprisingly difficult to verify that $A + B$ is the generator of the semigroup U_t – even if one already knows that the Trotter Formula converges. The obvious failure is that the canonical (pre-)domain of $A + B$, the set $\mathfrak{D}(A) \cap \mathfrak{D}(B)$ is too small, i.e. not dense or even empty!. The assumption that $\mathfrak{D}(B) \subset \mathfrak{D}(A)$ is a strong, but still reasonable assumption. Alternatively one can require that A and B commute.

The usual statements of Trotter’s formula, see e.g. the excellent monograph by Engel & Nagel [5, Chapter III.5], is such that one has a condition on $\mathfrak{D}(A) \cap \mathfrak{D}(B)$ which also ensures that the limit defining U_t exists. In an L^2 -context one can find a counterexample on p. 229 of [5].

■ ■

Problem 7.12. Solution: The idea is to show that $A = -\frac{1}{2} \Delta$ is closed when defined on $\mathcal{C}_\infty^2(\mathbb{R})$. Since $\mathcal{C}_\infty^2(\mathbb{R}) \subset \mathfrak{D}(A)$ and since $(A, \mathfrak{D}(A))$ is the smallest closed extension, we are done. So let $(u_n)_n \subset \mathcal{C}_\infty^2(\mathbb{R})$ be a sequence such that $u_n \rightarrow u$ uniformly and $(Au_n)_n$ is a \mathcal{C}_∞ Cauchy sequence. Since $\mathcal{C}_\infty(\mathbb{R})$ is complete, we can assume that $u''_n \rightarrow 2g$ uniformly for some $g \in \mathcal{C}_\infty(\mathbb{R}^d)$. The aim is to show that $u \in \mathcal{C}_\infty^2$.

a) By the fundamental theorem of differential and integral calculus we get

$$u_n(x) - u_n(0) - xu'_n(0) = \int_0^x (u'_n(y) - u'_n(0)) dy = \int_0^x \int_0^y u''_n(z) dz.$$

Since $u''_n \rightarrow 2g$ uniformly, we get

$$u_n(x) - u_n(0) - xu'_n(0) = \int_0^x \int_0^y u''_n(z) dz \rightarrow \int_0^x \int_0^y 2g(z) dz.$$

Since $u_n(x) \rightarrow u(x)$ and $u_n(0) \rightarrow u(0)$, we conclude that $u'_n(0) \rightarrow c$ converges.

b) Recall the following theorem from calculus, e.g. Rudin [15, Theorem 7.17].

Theorem. Let $(f_n)_n$ be a sequence of differentiable functions on $[0, \infty)$ which converges for some $t_0 > 0$. If $(f'_n)_n$ converges uniformly, then $(f_n)_n$ converges uniformly to a differentiable function f and we have $f' = \lim_n f'_n$.

If we apply this with $f'_n = u''_n \rightarrow 2g$ and $f_n(0) = u'_n(0) \rightarrow c$, we get that $u'_n(x) - u'_n(0) \rightarrow \int_0^x 2g(z) dt$.

Let us determine the constant $c' := \lim_n u'_n(0)$. Since u'_n converges uniformly, the limit as $n \rightarrow \infty$ is in \mathcal{C}_∞ , and so we get

$$-\lim_{n \rightarrow \infty} u'_n(0) = \lim_{x \rightarrow -\infty} \lim_{n \rightarrow \infty} (u'_n(x) - u'_n(0)) = \lim_{x \rightarrow -\infty} \int_0^x 2g(z) dz$$

i.e. $c' = \int_{-\infty}^0 g(z) dz$. We conclude that $u'_n(x) \rightarrow \int_{-\infty}^x g(z) dz$ uniformly.

- c) Again by the Theorem quoted in (b) we get $u_n(x) - u_n(0) \rightarrow \int_0^x \int_{-\infty}^y 2g(z) dz$ uniformly, and with the same argument as in (b) we get $u_n(0) = \int_{-\infty}^0 \int_{-\infty}^y 2g(z) dz$.

Problem 7.13. Solution: By definition, (for all $\alpha > 0$ and formally but justifiable via monotone convergence also for $\alpha = 0$)

$$\begin{aligned} U_\alpha \mathbf{1}_C(x) &= \int_0^\infty e^{-\alpha t} P_t \mathbf{1}_C(x) dt \\ &= \int_0^\infty e^{-\alpha t} \mathbb{E} \mathbf{1}_C(B_t + x) dt \\ &= \mathbb{E} \int_0^\infty e^{-\alpha t} \mathbf{1}_{C-x}(B_t) dt. \end{aligned}$$

This is the ‘discounted’ (with ‘interest rate’ α) total amount of time a Brownian motion spends in the set $C - x$.

Problem 7.14. Solution: This is a consequence of the recurrence of a one- or two-dimensional Brownian motion. Here is the intuition: If $u \geq 0$, monotone convergence shows that

$$U_0 u(x) = \sup_{\alpha > 0} U_\alpha u(x) = \sup_{\alpha > 0} \mathbb{E}^x \int_0^\infty e^{-t\alpha} u(B_t) dt = \mathbb{E}^x \int_0^\infty u(B_t) dt$$

and the integral measures “how much time” B_t spends in the region where $\{u > 0\}$. Since it is recurrent, the time should be infinitely long and $U_0 u(x) = \infty$.

Now for the rigorous argument. We can use the above calculation, then use the law of B_t and Tonelli so see (compare Example 7.14) that

$$U_0 u(x) = \sup_{\alpha > 0} \int_{\mathbb{R}^d} G_{\alpha,d}(y) u(x - y) dy.$$

If $d = 1$ we get

$$U_0 u(x) = \sup_{\alpha > 0} \int_{\mathbb{R}} \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|y|} u(x - y) dy \stackrel{\text{BL}}{=} \int_{\mathbb{R}} \sup_{\alpha > 0} \frac{1}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|y|} u(x - y) dy = \infty.$$

If $d = 2$ we get

$$U_0 u(x) = \sup_{\alpha > 0} \int_{\mathbb{R}^2} K_0(\sqrt{2\alpha}|y|) u(x - y) dy \stackrel{\text{BL}}{=} \int_{\mathbb{R}^2} \sup_{\alpha > 0} K_0(\sqrt{2\alpha}|y|) u(x - y) dy = \infty$$

since $K_0(0) = \infty$ (see also the explicit asymptotic relation used in the proof of Lemma 7.26).

Problem 7.15. Solution: First formula: We use induction. The induction start with $n = 0$ is clearly correct. Let us assume that the formula holds for some n and we do the induction step $n \rightsquigarrow n + 1$. We have for $\beta \neq \alpha$

$$\begin{aligned} \frac{d^{n+1}}{d\alpha^{n+1}}U_\alpha f(x) &= \lim_{\beta \rightarrow \alpha} \frac{\frac{d^n}{d\alpha^n}U_\alpha f(x) - \frac{d^n}{d\beta^n}U_\beta f(x)}{\beta - \alpha} \\ &= \lim_{\beta \rightarrow \alpha} \frac{n!(-1)^n U_\alpha^{n+1} f(x) - n!(-1)^n U_\beta^{n+1} f(x)}{\beta - \alpha} \\ &= n!(-1)^n \lim_{\beta \rightarrow \alpha} \frac{U_\alpha^{n+1} f(x) - U_\beta^{n+1} f(x)}{\beta - \alpha} \end{aligned}$$

Using the identity $a^{n+1} - b^{n+1} = (a - b) \sum_{j=0}^n a^{n-j} b^j$ we get, since the resolvents commute,

$$\frac{U_\alpha^{n+1} f(x) - U_\beta^{n+1} f(x)}{\beta - \alpha} = \frac{U_\alpha - U_\beta}{\beta - \alpha} \sum_{j=0}^n U_\alpha^{n-j} U_\beta^j f(x) = -U_\alpha U_\beta \sum_{j=0}^n U_\alpha^{n-j} U_\beta^j f(x)$$

In the last line we used the resolvent identity. Now we can let $\beta \rightarrow \alpha$ to get

$$\xrightarrow{\beta \rightarrow \alpha} -U_\alpha U_\alpha \sum_{j=0}^n U_\alpha^{n-j} U_\alpha^j f(x) = -(n+1)U_\alpha^{n+2} f(x).$$

This finishes the induction step.

Second formula: We use Leibniz' formula for the derivative of a product:

$$\partial^n(fg) = \sum_{j=0}^n \binom{n}{j} \partial^j f \partial^{n-j} g$$

and we get, using the first formula

$$\begin{aligned} \partial^n(\alpha U_\alpha f(x)) &= \binom{n}{0} \alpha \partial^n U_\alpha f(x) + \binom{n}{1} \partial^{n-1} U_\alpha f(x) \\ &= \alpha n!(-1)^n U_\alpha^{n+1} f(x) + n(n-1)!(-1)^{n-1} U_\alpha^n f(x) \\ &= n!(-1)^{n+1} (\text{id} - \alpha U_\alpha) U_\alpha^n f(x). \end{aligned}$$



Problem 7.16. Solution: Using Dini's Theorem (e.g.: Rudin, p. 150) we see that

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |f(x) - f_n(x)| = 0$$

for any compact set $K \subset \mathbb{R}^d$. Fix $\epsilon > 0$ and pick a compact set $K = K_\epsilon \subset \mathbb{R}^d$ such that $0 \leq f_n(x) \leq f(x) \leq \epsilon$ on $\mathbb{R}^d \setminus K$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_x |f_n(x) - f(x)| &\leq \lim_{n \rightarrow \infty} \sup_{x \in K} |f_n(x) - f(x)| + \overline{\lim}_{n \rightarrow \infty} \sup_{x \notin K} |f_n(x)| + \sup_{x \notin K} |f(x)| \\ &\leq 3\epsilon. \end{aligned}$$

Remark: Positivity is, in fact, not needed. Here is the argument: Let $f_1 \leq f_2 \leq \dots \leq f_n \leq \dots$ and $f_n \in \mathcal{C}_\infty(\mathbb{R}^d)$ (any sign is now allowed!) and set $f = \sup_n f_n$. Using Dini's Theorem (e.g.: Rudin, p. 150) we see that

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |f(x) - f_n(x)| = 0$$

for any compact set $K \subset \mathbb{R}^d$. Fix $\epsilon > 0$ and pick a compact set $K = K_\epsilon \subset \mathbb{R}^d$ such that $-\epsilon \leq f_1(x) \leq f_n(x) \leq f(x) \leq \epsilon$ on $\mathbb{R}^d \setminus K$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_x |f_n(x) - f(x)| &\leq \lim_{n \rightarrow \infty} \sup_{x \in K} |f_n(x) - f(x)| + \overline{\lim}_{n \rightarrow \infty} \sup_{x \notin K} (f(x) - f_n(x)) \\ &\leq \lim_{n \rightarrow \infty} \sup_{x \in K} |f_n(x) - f(x)| + 2\epsilon \\ &\leq 3\epsilon. \end{aligned}$$

Alternative Solution (non-positive case): Apply the solution of the positive case to the sequence $g_n := f_n - f_1$. This is possible since $0 \leq g_n \leq g_{n+1}$ and $\sup_n g_n = f - f_1$.

Problem 7.17. Solution: Because of Lemma 7.24 c),d) it is enough to show that there are positive functions $u, v \in \mathcal{C}_\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha > 0} U_\alpha u, \quad \sup_{\alpha > 0} U_\alpha v \in \mathcal{C}_\infty(\mathbb{R}^d) \quad \text{but} \quad \boxed{U_0(u - v)(x) = \sup_{\alpha > 0} U_\alpha u(x) - \sup_{\alpha > 0} U_\alpha v(x) \notin \mathcal{C}^2(\mathbb{R}^d).}$$

As in Example 7.25 we have

$$U_0 u(x) = \alpha_d \int_{\mathbb{R}^d} |y - x|^{2-d} u(y) dy$$

for any $u \in \mathcal{C}_\infty^+(\mathbb{R}^d)$ with $\alpha_d = \pi^{-d/2} \Gamma(\frac{d}{2}) / (d - 2)$. Since $\int_{|y| \leq 1} |y|^{2-d} dy < \infty$, we see with dominated convergence that the function $U_0 w$ is in $\mathcal{C}_\infty(\mathbb{R}^d)$ for all $u \in \mathcal{C}_\infty^+(\mathbb{R}^d) \cap L^1(dy)$.

Pick any $f \in \mathcal{C}_c([0, 1])$ such that $f(0) = 0$. We denote by (x_1, \dots, x_d) points in \mathbb{R}^d and set $r^2 := x_1^2 + \dots + x_d^2$. Then let

$$u(x_1, \dots, x_d) := \gamma \frac{x_d^2}{r^2} f(r), \quad v(x_1, \dots, x_d) := f(r)$$

and

$$w(x_1, \dots, x_d) := u(x_1, \dots, x_d) - v(x_1, \dots, x_d).$$

We will show that there is some f and a constant $\gamma > 0$ such that $w \in \mathfrak{D}(U_0)$ and $U_0 w \notin \mathcal{C}^2(\mathbb{R}^d)$. The first assertion follows directly from Lemma 7.24 d). Introducing polar coordinates

$$\begin{aligned} y_d &= r \cos \theta_{d-2}, \\ y_{d-1} &= r \cos \theta_{d-3} \cdot \sin \theta_{d-2}, \\ y_{d-2} &= r \cos \theta_{d-4} \cdot \sin \theta_{d-3} \cdot \sin \theta_{d-2}, \\ &\vdots \\ y_2 &= r \cos \phi \sin \theta_1 \cdot \dots \cdot \sin \theta_{d-2}, \\ y_1 &= r \sin \phi \sin \theta_1 \cdot \dots \cdot \sin \theta_{d-2}, \end{aligned}$$

and using the integral formula for U_0u and U_0v , we get for $x_d \in (0, 1/2)$

$$U_0w(0, \dots, 0, x_d) = \alpha_d \left[\int_0^1 r^{d-1} f(r) \left(\int_0^\pi \frac{(\gamma \cos^2 \theta_{d-2} - 1)(\sin \theta_{d-2})^{d-2}}{\sqrt{r^2 + x_d^2 - 2x_d r \cos \theta_{d-2}}^{d-2}} d\theta_{d-2} \right) dr \right] \underbrace{\prod_{j=1}^{d-2} \left(\int_0^\pi (\sin \theta_j)^j d\theta_j \right)}_{=:\beta_d}.$$

Note that $\beta_d > 0$. For brevity we write $x = x_d$ and $\theta = \theta_{d-2}$. From

$$\gamma \cos^2 \theta - 1 = -\gamma \sin^2 \theta - (1 - \gamma)$$

we conclude

$$\begin{aligned} & \int_0^\pi \frac{(\gamma \cos^2 \theta - 1)(\sin \theta)^{d-2}}{\sqrt{r^2 + x^2 - 2xr \cos \theta}^{d-2}} d\theta \\ &= -\gamma \int_0^\pi \frac{(\sin \theta)^d}{\sqrt{r^2 + x^2 - 2xr \cos \theta}^{d-2}} d\theta - (1 - \gamma) \int_0^\pi \frac{(\sin \theta)^{d-2}}{\sqrt{r^2 + x^2 - 2xr \cos \theta}^{d-2}} d\theta \\ &=: -\gamma I_1(r, x) - (1 - \gamma) I_2(r, x). \end{aligned}$$

By (??), there exist constants $b_d, c_d \in \mathbb{R}$ such that

$$I_1(r, x) = \frac{1}{x^{d-2}} \int_0^\pi \frac{(\sin \theta)^d}{\sqrt{\left(\frac{r}{x}\right)^2 + 1 - 2\frac{r}{x} \cos \theta}^{d-2}} d\theta = \frac{1}{x^{d-2}} \left(b_d \left(\frac{r}{x}\right)^2 + c_d \right)$$

for any $0 < r < x$. Similarly,

$$I_2(r, x) = \frac{1}{r^{d-2}} \int_0^\pi \frac{(\sin \theta)^d}{\sqrt{1 + \left(\frac{x}{r}\right)^2 - 2\frac{x}{r} \cos \theta}^{d-2}} d\theta = \frac{1}{r^{d-2}} \left(b_d \left(\frac{x}{r}\right)^2 + c_d \right)$$

for $x < r < 1$. Analogously, we find by (??)

$$I_2(r, x) = \begin{cases} \frac{a_d}{x^{d-2}} & 0 < r < x \\ \frac{a_d}{r^{d-2}} & x < r < 1 \end{cases}$$

for some $a_d \in \mathbb{R}$. It is not difficult to see that $0 < c_d < a_d$. Therefore, we may choose $\gamma = \gamma_d = a_d/(a_d - c_d)$. Then,

$$\int_0^\pi \frac{(\gamma_d \cos^2 \theta - 1)(\sin \theta)^{d-2}}{\sqrt{r^2 + x^2 - 2xr \cos \theta}^{d-2}} d\theta = \begin{cases} -\gamma_d b_d \frac{r^2}{x^d} & 0 < r < x \\ -\gamma_d b_d \frac{x^2}{r^d} & x < r < 1. \end{cases}$$

Hence,

$$U_0w(0, \dots, 0, x) = \underbrace{-\alpha_d \beta_d \gamma_d b_d}_{=: C_d} \left(\frac{1}{x^d} \int_0^x r^{d+1} f(r) dr + x^2 \int_x^1 \frac{f(r)}{r} dr \right).$$

The remaining part of the proof follows as in Example 7.25. Differentiating in x yields

$$\frac{d}{dx} U_0w(0, \dots, 0, x) = C_d \left(-\frac{d}{x^{d+1}} \int_0^x r^{d+1} f(r) dr + 2x \int_x^1 \frac{f(r)}{r} dr \right).$$

It is not hard to show that $\lim_{x \rightarrow 0^+} \frac{d}{dx} U_0 w(0, \dots, 0, x) = 0$. Thus,

$$\frac{\frac{d}{dx} U_0 w(0, \dots, 0, x) - \frac{d}{dx} U_0 w(0, \dots, 0)}{x} = C_d \left(-\frac{d}{x^{d+2}} \int_0^x r^{d+1} f(r) dr + 2 \int_x^1 \frac{f(r)}{r} dr \right).$$

Applying l'Hôpital's rule we obtain

$$\lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} U_0 w(0, \dots, 0, x) - \frac{d}{dx} U_0(0, \dots, 0)}{x} = C_d \left(-\frac{d}{d+2} f(0) + 2 \lim_{x \rightarrow 0^+} \int_x^1 \frac{f(r)}{r} dr \right).$$

This means that the second derivative of $U_0 w$ at $x = 0$ in x_d -direction does not exist if $\int_{0^+}^1 \frac{f(r)}{r} dr$ diverges. A canonical candidate is $f(r) = |\log r|^{-1} \chi(r)$ with a suitable cut-off function $\chi \in \mathcal{C}_c^+([0, 1])$ and $\chi|_{[0, 1/2]} \equiv 1$.

■ ■

Problem 7.18. Solution:

- a) The process (t, B_t) starts at $(0, B_0) = 0$, and if we start at (s, x) we consider the process $(s + t, x + B_t) = (s, x) + (t, B_t)$. Let $f \in \mathcal{B}_b([0, \infty) \times \mathbb{R})$. Since the motion in t is deterministic, we can use the probability space $(\Omega, \mathcal{A}, \mathbb{P} = \mathbb{P})$ generated by the Brownian motion $(B_t)_{t \geq 0}$. Then

$$T_t f(s, x) := \mathbb{E}^{(s, x)} f(t, B_t) := \mathbb{E} f(s + t, x + B_t).$$

T_t preserves $\mathcal{C}_\infty([0, \infty) \times \mathbb{R})$: If $f \in \mathcal{C}_\infty([0, \infty) \times \mathbb{R})$, we see with dominated convergence that

$$\begin{aligned} \lim_{(\sigma, \xi) \rightarrow (s, x)} T_t f(\sigma, \xi) &= \lim_{(\sigma, \xi) \rightarrow (s, x)} \mathbb{E} f(\sigma + t, \xi + B_t) \\ &= \mathbb{E} \lim_{(\sigma, \xi) \rightarrow (s, x)} f(\sigma + t, \xi + B_t) \\ &= \mathbb{E} f(s + t, x + B_t) \\ &= T_t f(s, x) \end{aligned}$$

which shows that T_t preserves $f \in \mathcal{C}_b([0, \infty) \times \mathbb{R})$. In a similar way we see that

$$\lim_{|(\sigma, \xi)| \rightarrow \infty} T_t f(\sigma, \xi) = \mathbb{E} \lim_{|(\sigma, \xi)| \rightarrow \infty} f(\sigma + t, \xi + B_t) = 0,$$

i.e. T_t maps $\mathcal{C}_\infty([0, \infty) \times \mathbb{R})$ into itself.

T_t is a semigroup: Let $f \in \mathcal{C}_\infty([0, \infty) \times \mathbb{R})$. Then, by the independence and stationary increments property of Brownian motion,

$$\begin{aligned} T_{t+\tau} f(s, x) &= \mathbb{E} f(s + t + \tau, x + B_{t+\tau}) \\ &= \mathbb{E} f(s + t + \tau, x + (B_{t+\tau} - B_t) + B_t) \\ &= \mathbb{E} \mathbb{E}^{(t, B_t)} f(s + \tau, x + (B_{t+\tau} - B_t)) \\ &= \mathbb{E} \mathbb{E}^{(t, B_t)} f(s + \tau, x + B_\tau) \end{aligned}$$

$$\begin{aligned} &= \mathbb{E} T_\tau f(s+t, x+B_t) \\ &= T_t T_\tau f(s, x). \end{aligned}$$

T_t is strongly continuous: Since $f \in \mathcal{C}_\infty([0, \infty) \times \mathbb{R})$ is uniformly continuous, we see that for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(s+h, x+y) - f(s, x)| \leq \epsilon \quad \forall h + |y| \leq 2\delta.$$

So, let $t < h < \delta$, then

$$\begin{aligned} |T_t f(s, x) - f(s, x)| &= |\mathbb{E}(f(s+t, x+B_t) - f(s, x))| \\ &\leq \int_{|B_t| \leq \delta} |f(s+t, x+B_t) - f(s, x)| d\mathbb{P} + 2\|f\|_\infty \mathbb{P}(|B_t| > \delta) \\ &\leq \epsilon + 2\|f\|_\infty \frac{1}{\delta^2} \mathbb{E}(B_t^2) \\ &= \epsilon + 2\|f\|_\infty \frac{t}{\delta^2}. \end{aligned}$$

Since the estimate is uniform in (s, x) , this proves strong continuity.

Markov property: this is trivial.

b) The transition semigroup is

$$T_t f(s, x) = \mathbb{E} f(s+t, x+B_t) = (2\pi t)^{-1/2} \int_{\mathbb{R}} f(s+t, x+y) e^{-y^2/(2t)} dy.$$

The resolvent is given by

$$U_\alpha f(s, x) = \int_0^\infty e^{-t\alpha} T_t f(s, x) dt$$

and the generator is, for all $f \in \mathcal{C}^{1,2}([0, \infty) \times \mathbb{R})$

$$\begin{aligned} \frac{T_t f(s, x) - f(s, x)}{t} &= \frac{\mathbb{E} f(s+t, x+B_t) - f(s, x)}{t} \\ &= \frac{\mathbb{E}[f(s+t, x+B_t) - f(s, x+B_t)]}{t} + \frac{\mathbb{E} f(s, x+B_t) - f(s, x)}{t} \\ &\xrightarrow{t \rightarrow 0} \mathbb{E} \partial_t f(s, x+B_0) + \frac{1}{2} \Delta_x f(s, x) \\ &= (\partial_t + \frac{1}{2} \Delta_x) f(s, x). \end{aligned}$$

Note that, in view of Theorem 7.22, pointwise convergence is enough (provided the pointwise limit is a \mathcal{C}_∞ -function).

c) We get for $u \in \mathcal{C}_\infty^{1,2}$ that under $\mathbb{P}^{(s,x)}$

$$M_t^u := u(s+t, x+B_t) - u(s, x) - \int_0^t (\partial_r + \frac{1}{2} \Delta_x) u(s+r, x+B_r) dr$$

is an \mathcal{F}_t -martingale. This is the same assertion as in Theorem 5.6 (up to the choice of u which is restricted here as we need it in the domain of the generator...).

■

Problem 7.19. Solution: Let $u \in \mathfrak{D}(A)$ and σ a stopping time with $\mathbb{E}^x \sigma < \infty$. Use optional stopping (Theorem A.18 in combination with remark A.21) to see that

$$M_{\sigma \wedge t}^u := u(X_{\sigma \wedge t}) - u(x) - \int_0^{\sigma \wedge t} Au(X_r) dr$$

is a martingale (for either \mathcal{F}_t or $\mathcal{F}_{\sigma \wedge t}$). If we take expectations we get

$$\mathbb{E}^x u(X_{\sigma \wedge t}) - u(x) = \mathbb{E}^x \left(\int_0^{\sigma \wedge t} Au(X_r) dr \right).$$

Since $u, Au \in \mathcal{C}_\infty$ we see

$$\left| \mathbb{E}^x \left(\int_0^{\sigma \wedge t} Au(X_r) dr \right) \right| \leq \mathbb{E}^x \left(\int_0^{\sigma \wedge t} \|Au\|_\infty dr \right) \leq \|Au\|_\infty \cdot \mathbb{E}^x \sigma < \infty,$$

i.e. we can use dominated convergence and let $t \rightarrow \infty$. Because of the right continuity of the paths of a Feller process we get Dynkin's formula (7.30). ■ ■

Problem 7.20. Solution: Clearly,

$$\mathbb{P}(X_t \in F \quad \forall t \in \mathbb{R}^+) \leq \mathbb{P}(X_q \in F \quad \forall q \in \mathbb{Q}^+).$$

On the other hand, since F is closed and X_t has continuous paths,

$$X_q \in F \quad \forall q \in \mathbb{Q}^+ \implies X_t = \lim_{\mathbb{Q}^+ \ni q \rightarrow t} X_q \in F \quad \forall t \geq 0$$

and the converse inequality follows. ■ ■

Problem 7.21. Solution:

a) The PMP states that

$$w \in \mathcal{C}_c^\infty(\mathbb{R}^d), \quad w(x_0) = \sup_x w(x) \implies Lw(x_0) \leq 0.$$

Now assume that $u(x) \geq 0$ and $u(x_0) = 0$. Thus, $w := -u$ has a positive maximum at x_0 and, by the linearity of L , $Lu = -Lw$. Thus the PMP implies almost positivity. (The converse direction is wrong.)

b) Let $K = \text{supp } \phi$ and take $x \notin K$. Since K is closed, K^c is open and $\phi(y) \equiv 0$ in $\mathbb{B}(x, r)$ for some small $r > 0$. Thus, by locality, $L(\phi)(x) = L(0)(x)$ at x , but $L(0) \equiv 0$ by linearity. Thus $L\phi(x) = 0$, and we have $\text{supp } L\phi \subset \text{supp } \phi$.

c) We follow the hint, fix some K and some $\chi \in \mathcal{C}_c^\infty$ with $\mathbf{1}_K \leq \chi \leq 1$ and consider the function

$$y \mapsto \phi(y) - \phi(x) - \nabla \phi(x) \cdot (y - x).$$

Using Taylor's theorem gives a constant c such that

$$|\phi(y) - \phi(x) - \nabla \phi(x) \cdot (y - x)| \leq C \|\phi\|_{(2)} |x - y|^2.$$

Now multiply this through with $\chi(y)$, observe that $\text{supp } \phi \subset \{\chi = 1\}$, to get

$$\pm [\phi(y) - \phi(x)\chi(y) - \nabla\phi(x) \cdot (y-x)\chi(y)] \leq C\|\phi\|_{(2)}|x-y|^2\chi(y).$$

Set for fixed x

$$f(y) := C\|\phi\|_{(2)}|x-y|^2\chi(y) \pm [\phi(y) - \phi(x)\chi(y) - \nabla\phi(x) \cdot (y-x)\chi(y)].$$

Then $f(x) = 0$, $f \geq 0$, $f \in \mathcal{C}_c^\infty$ and, by the almost positive property of L , $Lf(x) \geq 0$.

Using the linearity of L now gives

$$C\|\phi\|_{(2)}L(|x-\cdot|^2\chi)(x) \geq \pm L\phi(x) - |\phi(x)L\chi(x)| - |\nabla\phi(x)| \cdot |L((\cdot-x)\chi)(x)|$$

proving the claim.

- d) It is enough to show that L is a second-order differential operator on the sets $\mathcal{C}_c^\infty(K)$ for any compact set $K \subset \mathbb{R}^d$. From Peetre's theorem we already know the structure of L , namely $Lu(x) = \sum_{\text{finite}} a_\alpha(x)\partial^\alpha u(x)$. Since $x \in K$, we can safely assume that a_α are functions. Assume, say that some ∂^α with $|\alpha| > 2$ appears in the sum. Then we cannot control $\|\partial^\alpha u\|_\infty$ by $\|u\|_{(2)}$, thus the claim.

Remark: Almost positivity and the positive maximum principle also give structural results for not necessarily local operators. In fact, one can show the Lévy–Khintchine formula using the PMP. This and other developments are worked out in [17, Theorem 6.8, pp. 44–58] (for constant-coefficient operators) and [1, Theorem 2.21, pp. 47–50] (for variable-coefficient operators).

■■

8 The PDE connection

Problem 8.1. Solution: It is clearly enough to show the exponentially bounded case. From

$$|\mathbb{E} f(B_t + x)| \leq \mathbb{E} |f(B_t + x)| \leq \mathbb{E} C e^{C(|B_t| + |x|)} = C e^{C|x|} \mathbb{E} e^{C|B_t|}$$

we see that $u(t, x)$ is exponentially bounded if f is exponentially bounded. Together with the dominated convergence theorem, this argument also shows continuity in $(t, x) \in (0, \infty) \times \mathbb{R}^d$.

In order to treat the derivatives, we have to go into the structure of $P_t f(x)$, that is, we have to use the fact that $P_t f(x)$ is given through a convolution. Write, for the moment, $g_t(z) = (2\pi t)^{-d/2} e^{-|z|^2/2t}$ for the normal distribution in \mathbb{R}^d . Then we have

$$P_t f(x) = \mathbb{E} f(B_t + x) = \int_{\mathbb{R}^d} f(y) g_t(x - y) dy.$$

Since f is exponentially bounded, we can use the differentiation theorem for parameter dependent integrals. We check this exemplarily for the second mixed x -derivatives $\partial_j \partial_k$, the other calculations are quite similar. We have

$$|\partial_j \partial_k g_t(x - y)| = t^{-2} |(x_j - y_j)(x_k - y_k)| (2\pi t)^{-d/2} e^{-|x-y|^2/2t} \leq C_R t^{-d/2-2} (|y|^2 + 1) e^{-|y|^2/2t}$$

if $|x| < R$. This allows us to swap differentiation and integration, and we get, using $|y_j y_k| \leq \frac{1}{2}(|y_j|^2 + |y_k|^2) \leq |y|^2$

$$\begin{aligned} |\partial_j \partial_k P_t f(x)| &= |(\partial_j \partial_k g_t) * f(x)| \\ &= t^{-2} \left| \int_{\mathbb{R}^d} (x_j - y_j)(x_k - y_k) f(y) g_t(x - y) dy \right| \\ &= t^{-2} \left| \int_{\mathbb{R}^d} y_j y_k f(x - y) g_t(y) dy \right| \\ &\leq t^{-2} \left| \int_{\mathbb{R}^d} |y|^2 f(x - y) g_t(y) dy \right| \\ &\leq t^{-2} \mathbb{E} (|B_t|^2 |f(B_t + x)|), \end{aligned}$$

and (a minute variation of) the argument from the beginning proves exponential boundedness.

Looking once again at the first calculation in the solution reveals that the variation in Part c) leads to an expression of the type

$$\mathbb{E} e^{C|x+B_t|^2} \leq e^{2C|x|^2} \mathbb{E} e^{2C|B_t|^2} \leq e^{2C|x|^2} \mathbb{E} e^{2Ct|B_1|^2}$$

and this expression is finite if $4Ct < 1$, i.e. we get a restriction in the admissible values for t .

■ ■

Problem 8.2. Solution: Write $g_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$ for the heat kernel. Since convolutions are smoothing, one finds easily that $P_\epsilon f = g_\epsilon \star f \in \mathcal{C}_\infty^\infty \subset \mathfrak{D}(\Delta)$. (There is a more general concept behind it: whenever the semigroup is *analytic*—i.e. $z \mapsto P_z$ has an extension to, say, a sector in the complex plane and it is holomorphic there—one has that T_t maps the underlying Banach space into the domain of the generator; cf. e.g. Pazy [12, pp. 60–63].) Thus, if we set $f_\epsilon := P_\epsilon f$, we can apply Lemma 8.1 and find that

$$u_\epsilon(t, x) \stackrel{\text{Lemma 8.1}}{=} P_t f_\epsilon(x) \stackrel{\text{def}}{=} P_t P_\epsilon f(x) \stackrel{\text{semi-group}}{=} P_{t+\epsilon} f(x).$$

By the strong continuity of the heat semigroup, we find that

$$u_\epsilon(t, x) \xrightarrow[\epsilon \rightarrow 0]{\text{uniformly}} P_t f(x).$$

Moreover,

$$\begin{aligned} \frac{\partial}{\partial t} u_\epsilon(t, \cdot) &= \frac{1}{2} \Delta_x P_t P_\epsilon f \\ &= P_\epsilon \left(\underbrace{\frac{1}{2} \Delta_x P_t f}_{\in \mathcal{C}_\infty} \right) \xrightarrow[\epsilon \rightarrow 0]{\text{uniformly}} \frac{1}{2} \Delta_x P_t f. \end{aligned}$$

Since both the sequence and the differentiated sequence converge uniformly, we can interchange differentiation and the limit, cf. [15, Theorem 7.17, p. 152], and we get

$$\frac{\partial}{\partial t} u(t, x) = \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial t} u_\epsilon(t, x) = \frac{1}{2} \Delta_x u(t, x)$$

and

$$u_\epsilon(0, \cdot) = P_\epsilon f \xrightarrow[\epsilon \rightarrow 0]{} f = u(0, \cdot)$$

and we get a solution for the initial value f . The proof of the uniqueness part in Lemma 8.1 is valid.

■ ■

Problem 8.3. Solution: By differentiation we get $\frac{d}{dt} \int_0^t f(B_s) ds = f(B_t)$ so that $f(B_t) = 0$. We can assume that f is positive and bounded, otherwise we could consider $f^\pm(B_t) \wedge c$ for some constant $c > 0$. Now $\mathbb{E} f(B_t) = 0$ and we conclude from this that $f = 0$.

■ ■

Problem 8.4. Solution:

a) Note that

$$\left| \chi_n(B_t) e^{-\alpha \int_0^t g_n(B_s) ds} \right| \leq \left| e^{-\alpha \int_0^t ds} \right| = e^{-\alpha t} \leq 1$$

is uniformly bounded. Moreover,

$$\lim_{n \rightarrow \infty} \chi_n(B_t) e^{-\alpha \int_0^t g_n(B_s) ds} = \mathbb{1}_{\mathbb{R}}(B_t) e^{-\alpha \int_0^t \mathbb{1}_{(0, \infty)}(B_s) ds}$$

which means that, by dominated convergence,

$$v_{n,\lambda}(x) = \int_0^\infty e^{-\lambda t} \mathbb{E} \left(\chi_n(B_t) e^{-\alpha \int_0^t g_n(B_s) ds} \right) dt \xrightarrow[n \rightarrow \infty]{} v_\lambda(x).$$

Moreover, we get that $|v_\lambda(x)| \leq \lambda^{-1}$.

If we rearrange (8.16) we see that

$$v_{n,\lambda}''(x) = 2(\alpha \chi_n(x) + \lambda)v_{n,\lambda}(x) - g_n(x), \quad (*)$$

and since the expression on the right has a limit as $n \rightarrow \infty$, we get that $\lim_{n \rightarrow \infty} v_{n,\lambda}''(x)$ exists.

b) Integrating (*) we find

$$v_{n,\lambda}'(x) - v_{n,\lambda}'(0) = 2 \int_0^x (\alpha \chi_n(y) + \lambda)v_{n,\lambda}(y) dy - \int_0^x g_n(y) dy, \quad (**)$$

and, again by dominated convergence, we conclude that $\lim_{n \rightarrow \infty} [v_{n,\lambda}'(x) - v_{n,\lambda}'(0)]$ exists. In addition, the right-hand side is uniformly bounded (for all $|x| \leq R$):

$$\begin{aligned} \left| 2 \int_0^x (\alpha \chi_n(y) + \lambda)v_{n,\lambda}(y) dy - \int_0^x g_n(y) dy \right| &\leq 2 \int_0^R (\alpha + \lambda) dy + \int_0^R dy \\ &\leq 2(\alpha + \lambda + 1)R. \end{aligned}$$

Integrating (**) reveals

$$v_{n,\lambda}(x) - v_{n,\lambda}(0) - x v_{n,\lambda}'(0) = \int_0^x [v_{n,\lambda}'(z) - v_{n,\lambda}'(0)] dz.$$

Since the expression under the integral converges boundedly and since $\lim_{n \rightarrow \infty} v_{n,\lambda}(x)$ exists, we conclude that $\lim_{n \rightarrow \infty} v_{n,\lambda}'(0)$ exists. Consequently, $\lim_{n \rightarrow \infty} v_{n,\lambda}'(x)$ exists.

c) The above considerations show that

$$\begin{aligned} v_\lambda(x) &= \lim_{n \rightarrow \infty} v_{n,\lambda}(x) \\ v_\lambda'(x) &= \lim_{n \rightarrow \infty} v_{n,\lambda}'(x) \\ v_\lambda''(x) &= \lim_{n \rightarrow \infty} v_{n,\lambda}''(x) \end{aligned}$$

Problem 8.5. Solution: We have to show that $v(t, x) := \int_0^t P_s g(x) ds$ is the unique solution of the initial value problem (8.7) with $g = g(x)$ satisfying $|v(t, x)| \leq C t$.

Existence: The linear growth bound is obvious from $|P_s g(x)| \leq \|P_s g\|_\infty \leq \|g\|_\infty < \infty$. The rest follows from the hint if we take $A = \frac{1}{2} \Delta$ and Lemma 7.10.

Uniqueness: We proceed as in the proof of Lemma 8.1. Set $v_\lambda(x) := \int_0^\infty e^{-\lambda t} v(t, x) dt$. This integral is, for $\lambda > 0$, convergent and it is the Laplace transform of $v(\cdot, x)$. Under the Laplace transform the initial value problem (8.7) with $g = g(x)$ becomes

$$\lambda v_\lambda(x) - Av_\lambda(x) = \lambda^{-1}g(x)$$

and this problem has a unique solution, cf. Proposition 7.13 f). Since the Laplace transform is invertible, we see that v is unique.

Problem 8.6. Solution: Integrating $u''(x) = 0$ twice yields

$$u'(x) = c \quad \text{and} \quad u(x) = cx + d$$

with two integration constants $c, d \in \mathbb{R}$. The boundary conditions $u(0) = a$ and $u(1) = b$ show that

$$d = a \quad \text{and} \quad c = b - a$$

so that

$$u(x) = (b - a)x + a.$$

On the other hand, by Corollary 5.11 (Wald's identities), Brownian motion started in $x \in (0, 1)$ has the probability to exit (at the exit time τ) the interval $(0, 1)$ in the following way:

$$\mathbb{P}^x(B_\tau = 1) = x \quad \text{and} \quad \mathbb{P}^x(B_\tau = 0) = 1 - x.$$

Therefore, if $f : \{0, 1\} \rightarrow \mathbb{R}$ is a function on the boundary of the interval $(0, 1)$ such that $f(0) = a$ and $f(1) = b$, then

$$\mathbb{E}^x f(B_\tau) = (1 - x)f(0) + xf(1) = (b - a)x + a.$$

This means that $u(x) = \mathbb{E}^x f(B_\tau)$, a result which we will see later in Section 8.4 in much greater generality.

Problem 8.7. Solution: The key is to show that *all* points in the open and bounded, hence relatively compact, set D are non-absorbing. Thus the closure of D has an neighbourhood, say $V \supset \bar{D}$ such that $\mathbb{E} \tau_{D^c} \leq \mathbb{E} \tau_{V^c}$. Let us show that $\mathbb{E} \tau_{V^c} < \infty$.

Since D is bounded, there is some $R > 0$ such that $\mathbb{B}(0, R) \supset \bar{D}$. Pick some test function $\chi = \chi_R$ such that $\chi|_{\mathbb{B}^c(0, R)} \equiv 0$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Pick further some function $u \in \mathcal{C}^2(\mathbb{R}^d)$ such that $\Delta u > 0$ in $\mathbb{B}(0, 2R)$. Here are two possibilities to get such a function:

$$u(x) = |x|^2 = \sum_{j=1}^d x_j^2 \implies \frac{1}{2} \Delta u(x) = 1$$

or, if $f \in \mathcal{C}_b(\mathbb{R}^d)$, $f \geq 0$ and $f = f(x_1)$ we set

$$F(x) = F(x_1) := \int_0^{x_1} f(z_1) dz_1$$

and

$$U(x) = U(x_1) := \int_0^{x_1} F(y_1) dy_1 = \int_0^{x_1} \int_0^{y_1} f(z_1) dz_1.$$

Clearly, $\frac{1}{2} \Delta U(x) = \frac{1}{2} \partial_{x_1}^2 U(x_1) = f(x_1)$, and we can arrange things by picking the correct f .

Problem: neither u nor U will be in $\mathfrak{D}(\Delta)$ (unless you are so lucky as in the proof of Lemma 8.10 to pick instantly the right function).

Now observe that

$$\begin{aligned} \chi \cdot u, \chi \cdot U &\in \mathcal{C}_c^2(\mathbb{R}^d) \subset \mathfrak{D}(\Delta) \\ \Delta(\chi \cdot U) &= \chi \cdot \Delta U + U \cdot \Delta \chi + 2\langle \nabla \chi, \nabla U \rangle \end{aligned}$$

which means that

$$\Delta(\chi \cdot U)|_{\mathbb{B}(0,R)} = \Delta U|_{\mathbb{B}(0,R)}.$$

The rest of the proof follows now either as in Lemma 7.33 or Lemma 8.10 (both employ, anyway, the same argument based on Dynkin's formula). ■ ■

Problem 8.8. Solution: We are following the hint. Let $L = \sum_{j,k=1}^d a_{jk}(x) \partial_j \partial_k + \sum_{j=1}^d b_j(x) \partial_j$. Then

$$\begin{aligned} L(\chi f) &= \sum_{j,k} a_{jk} \partial_j \partial_k (\chi f) + \sum_j b_j \partial_j (\chi f) \\ &= \sum_{j,k} a_{jk} (\partial_j \partial_k \chi + \partial_j \partial_k f + \partial_k \chi \partial_j f + \partial_j \chi \partial_k f) + \sum_j b_j (f \partial_j \chi + \chi \partial_j f) \\ &= \chi Lf + f L\chi + \sum_{j,k} (a_{jk} + a_{kj}) \partial_j \chi \partial_k f. \end{aligned}$$

If $|x| < R$ and $\chi|_{\mathbb{B}(0,R)} = 1$, then $L(u\chi)(x) = Lu(x)$. Set $u(x) = e^{-x_1^2/\gamma r^2}$. Then only the derivatives in x_1 -direction give any contribution and we get

$$\partial_1 u(x) = -\frac{2x_1}{\gamma r^2} e^{-\frac{x_1^2}{\gamma r^2}} \quad \text{and} \quad \partial_1^2 u(x) = \frac{2}{\gamma r^2} \left(\frac{2x_1^2}{\gamma r^2} - 1 \right) e^{-\frac{x_1^2}{\gamma r^2}}$$

Thus we get for $L(-u) = -Lu$ and any $|x| < r$

$$\begin{aligned} -Lu(x) &= \frac{2a_{11}(x)}{\gamma r^2} \left(1 - \frac{2x_1^2}{\gamma r^2} \right) e^{-\frac{x_1^2}{\gamma r^2}} + \frac{2b_1(x)x_1}{\gamma r^2} e^{-\frac{x_1^2}{\gamma r^2}} \\ &= \left[\frac{2a_{11}(x)}{\gamma r^2} \left(1 - \frac{2x_1^2}{\gamma r^2} \right) + \frac{2b_1(x)x_1}{\gamma r^2} \right] e^{-\frac{x_1^2}{\gamma r^2}} \\ &\geq \left[\frac{2a_0}{\gamma r^2} \left(1 - \frac{2}{\gamma} \right) - \frac{2b_0}{\gamma r} \right] e^{-\frac{r^2}{\gamma r^2}} \end{aligned}$$

This shows that the drift $b_1(x)$ can make the expression in the bracket negative!

Let us modify the Ansatz. Observe that for $f(x) = f(x_1)$ we have

$$Lf(x) = a_{11}(x)\partial_1^2 f(x) - b_1(x)\partial_1 f(x)$$

and if we know that $\partial_1^2 f, \partial_1 f \geq 0$ we get

$$Lf(x) \geq a_0 \partial_1^2 f(x) - b_0 \partial_1 f(x) \stackrel{!!}{>} 0.$$

This means that $\partial_1^2 f / \partial_1 f > b_0 / a_0$ seems to be natural and a reasonable Ansatz would be

$$f(x) = \int_0^{x_1} e^{\frac{2b_0}{a_0} y} dy.$$

Then

$$\partial_1 f(x) = e^{\frac{2b_0}{a_0} x_1} \quad \text{and} \quad \partial_1^2 f(x) = \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1}$$

and we get

$$\begin{aligned} Lf(x) &= a_{11}(x) \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1} - b_1(x) e^{\frac{2b_0}{a_0} x_1} \\ &\geq a_0 \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1} - b_0 e^{\frac{2b_0}{a_0} x_1} \\ &\geq (2b_0 - b_0) e^{\frac{2b_0}{a_0} x_1} > 0. \end{aligned}$$

With the above localization trick on balls, we are done. ■ ■

Problem 8.9. Solution: Denote by $u(t, x) = \mathbb{E}^x f(B_t)$ the solution to (8.3) and by $v(t, x) := \mathbb{E} \left(\int_0^t g(t-s, B_s) ds \right)$ the solution to (8.7). Define $w(t, x) := u(t, x) + v(t, x)$. Then we get

$$\partial_t w(t, x) = \partial_t u(t, x) + \partial_t v(t, x) = \Delta_x u(t, x) + \Delta_x v(t, x) + g(t, x) = \Delta_x w(t, x) + g(t, x)$$

and

$$w(0, x) = u(0, x) + v(0, x) = f(x) + 0 = f(x)$$

i.e. we get the solution to (8.6). ■ ■

Problem 8.10. Solution: We follow the hint and use the strategy of (the proof of) Lemma 8.7. We write $g_t(x) = (2\pi t)^{-d/2} \exp[-|x|^2/2t]$ and use the formulae for the derivatives $\partial_t g_t, \partial_j g_t$ and $\partial_j \partial_k g_t$ from the proof of Lemma 8.7. By definition

$$\begin{aligned} v(t, x) &:= \mathbb{E}^x \left(\int_0^t g(t-s, B_s) ds \right) \\ &= \mathbb{E} \left(\int_0^t g(t-s, x + B_s) ds \right) \\ &= \int_{\mathbb{R}^d} \int_0^t g(t-s, y) g_s(x-y) ds dy. \end{aligned}$$

1. Since $g \in \mathcal{C}_b([0, \infty) \times \mathbb{R}^d)$ we can use dominated convergence to conclude that $v(t, x) \in \mathcal{C}_b([0, \infty) \times \mathbb{R}^d)$.
2. We have (try to figure out why we can use the differentiation lemma and perform all derivatives under the integral):

$$\begin{aligned} \partial_j v(t, x) &= \int_{\mathbb{R}^d} \int_0^t g(t-s, y) \partial_j g_s(x-y) ds dy \\ &= - \int_{\mathbb{R}^d} \int_0^t g(t-s, y) \frac{x_j - y_j}{s} g_s(x-y) ds dy \\ &= - \int_0^t \mathbb{E} \left(g(t-s, x + B_s) \frac{B_s^{(j)}}{s} \right) ds. \end{aligned}$$

Thus, by scaling $B_s^{(j)} \sim \sqrt{s} B_1^{(j)}$,

$$\begin{aligned} |\partial_j v(t, x)| &\leq \|g\|_\infty \int_0^t \frac{1}{s} \mathbb{E} |B_s^{(j)}| ds \\ &\leq \|g\|_\infty \mathbb{E} |B_1^{(j)}| \int_0^t \frac{1}{\sqrt{s}} ds \\ &= c\sqrt{t}. \end{aligned}$$

In particular, $x \mapsto v(t, x)$ is Lipschitz with the Lipschitz constant of order \sqrt{t} . (Attention: in the proof of Lemma 8.7 we had the Lipschitz constant $1/\sqrt{t}$, due to a further term which is not present in this calculation.)

- 3.
4. We have (try to figure out why we can use the differentiation lemma and perform all derivatives under the integral):

$$\begin{aligned} \partial_j \partial_k v(t, x) &= \int_{\mathbb{R}^d} \int_0^t g(t-s, y) \partial_j \partial_k g_s(x-y) ds dy \\ &= \int_{\mathbb{R}^d} \int_0^t g(t-s, y) \frac{(x_j - y_j)(x_k - y_k) - \delta_{jk}s}{s^2} g_s(x-y) ds dy \\ &= \int_0^t \mathbb{E} \left(g(t-s, B_s) \frac{B_s^{(j)} B_s^{(k)} - \delta_{jk}s}{s^2} \right) ds \\ &= \int_0^t \mathbb{E} \left([g(t-s, B_s) - g(t-s, x)] \frac{B_s^{(j)} B_s^{(k)} - \delta_{jk}s}{s^2} \right) ds. \end{aligned}$$

Here we use that $\mathbb{E} B_s^{(j)} B_s^{(k)} = \delta_{jk}s$ to slip in the term $g(t-s, x)$. Now we use Hölder continuity, and Brownian scaling to get

$$\begin{aligned} |\partial_j \partial_k v(t, x)| &\leq C \int_0^t \mathbb{E} [|B_s|^\kappa] \frac{s}{s^2} ds \\ &= C' \int_0^t s^{\kappa/2-1} ds \\ &< \infty. \end{aligned}$$

5. Before we differentiate in t , we change variables $t-s \rightarrow s$ and get

$$v(t, x) = \int_{\mathbb{R}^d} \int_0^t g(s, y) g_{t-s}(x-y) ds dy.$$

Now we differentiate and see (exactly as in Lemma 8.7)

$$\begin{aligned} \partial_t v(t, x) &= \mathbb{E}^x g(t, B_0) + \int_{\mathbb{R}^d} \int_0^t g(s, y) \partial_t g_{t-s}(x - y) ds dy \\ &= g(t, x) + \frac{1}{2} \sum_{j=1}^d \int_{\mathbb{R}^d} \int_0^t g(s, y) \partial_j^2 g_{t-s}(x - y) ds dy \end{aligned}$$

and this expression is finite as we have seen in the previous step.

Since the solution has the required classical smoothness, it is indeed a solution to the PDE problem No. 3. in Table 8.1.

Problem 8.11. Solution: Assume that $B_0 = 0$. Any other starting point can be reduced to this situation by shifting Brownian motion to $B_0 = 0$. The LIL shows that a Brownian motion satisfies

$$-1 = \liminf_{t \rightarrow 0} \frac{B(t)}{\sqrt{2t \log \log \frac{1}{t}}} < \overline{\lim}_{t \rightarrow 0} \frac{B(t)}{\sqrt{2t \log \log \frac{1}{t}}} = 1$$

i.e. $B(t)$ oscillates for $t \rightarrow 0$ between the curves $\pm \sqrt{2t \log \log \frac{1}{t}}$. Since a Brownian motion has continuous sample paths, this means that it has to cross the level 0 infinitely often.

Problem 8.12. Solution: The idea is to proceed as in Example 8.14 e) where Zaremba's needle plays the role a truncated flat cone in dimension $d = 2$ (but in dimension $d \geq 3$ it has too small dimension). The set-up is as follows: without loss of generality we take $x_0 = 0$ (otherwise we shift Brownian motion) and we assume that the cone lies in the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ (otherwise we rotate things).

Let $B(t) = (b(t), \beta(t))$, $t \geq 0$, be a BM^d where $b(t)$ is a BM^1 and $\beta(t)$ is a $(d - 1)$ -dimensional Brownian motion. Since B is a BM^d , we know that the coordinate processes $b = (b(t))_{t \geq 0}$ and $\beta = (\beta(t))_{t \geq 0}$ are independent processes. Set $\sigma_n = \inf\{t > 1/n : b(t) = 0\}$. Since $0 \in \mathbb{R}$ is regular for $\{0\} \subset \mathbb{R}$, see Example 8.14 e), we get that $\lim_{n \rightarrow \infty} \sigma_n = \tau_{\{0\}} = 0$ almost surely with respect to \mathbb{P}^0 . Since $\beta \perp b$, the random variable $\beta(\sigma_n)$ is rotationally symmetric (see, e.g., the solution to Problem 8.13).

Let C be a flat (i.e. in the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$) cone such that some truncation C' of it lies in D^c . By rotational symmetry, we get

$$\mathbb{P}^0(\beta(\sigma_n) \in C) = \gamma = \frac{\text{opening angle of } C}{\text{full angle}}.$$

By continuity of BM, $\beta(\sigma_n) \rightarrow \beta(0) = 0$, and this gives

$$\mathbb{P}^0(\beta(\sigma_n) \in C') = \gamma.$$

Clearly, $B(\sigma_n) = (b(\sigma_n), \beta(\sigma_n)) = (0, \beta(\sigma_n))$ and $\{\beta(\sigma_n) \in C'\} \subset \{\tau_{D^c} \leq \sigma_n\}$, so

$$\mathbb{P}^0(\tau_{D^c} = 0) = \lim_{n \rightarrow \infty} \mathbb{P}^0(\tau_{D^c} \leq \sigma_n) \geq \liminf_{n \rightarrow \infty} \mathbb{P}^0(\beta(\sigma_n) \in C') \geq \gamma > 0.$$

Now Blumenthal's 0-1-law, Corollary 6.23, applies and gives $\mathbb{P}^0(\tau_{D^c} = 0) = 1$.

Problem 8.13. Solution: Proving that the random variable $\beta(\sigma_n)$ is absolutely continuous with respect to Lebesgue measure is relatively easy: note that, because of the independence of b and β , hence σ_n and β ,

$$\begin{aligned} -\frac{d}{dx} \mathbb{P}^0(\beta(\sigma_n) \geq x) &= -\frac{d}{dx} \int_{\mathbb{R}} \mathbb{P}^0(\beta_t \geq x) \mathbb{P}(\sigma_n \in dt) \\ &= \int_{\mathbb{R}} -\frac{d}{dx} \mathbb{P}^0(\beta_t \geq x) \mathbb{P}(\sigma_n \in dt) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \mathbb{P}(\sigma_n \in dt) \\ &= \int_{1/n}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \mathbb{P}(\sigma_n \in dt). \end{aligned}$$

(observe, for the last equality, that σ_n takes values in $[1/n, \infty)$.) Since the integrand is bounded (even as $t \rightarrow 0$), the interchange of integration and differentiation is clearly satisfied.

$(d-1)$ -dimensional version: Let β be a $(d-1)$ -dimensional version as in Problem 8.12. Proving that the random variable $\beta(\sigma_n)$ is rotationally symmetric is easy: note that, because of the independence of b and β , hence σ_n and β , we have for all Borel sets $A \subset \mathbb{R}^{d-1}$

$$\mathbb{P}^0(\beta(\sigma_n) \in A) = \int_{1/n}^{\infty} \mathbb{P}^0(\beta_t \in A) \mathbb{P}(\sigma_n \in dt)$$

and this shows that the rotational symmetry of β is inherited by $\beta(\sigma_n)$.

We even get a density by formally replacing A by dx :

$$\begin{aligned} \beta(\sigma_n) &\sim \int_{\mathbb{R}} \mathbb{P}^0(\beta_t \in dx) \mathbb{P}(\sigma_n \in dt) \\ &= \int_{1/n}^{\infty} \frac{1}{(2\pi t)^{(d-1)/2}} e^{-|x|^2/(2t)} \mathbb{P}(\sigma_n \in dt) dx. \end{aligned}$$

(here $x \in \mathbb{R}^{d-1}$).

It is a bit more difficult to work out the exact shape of the density. Let us first determine the distribution of σ_n . Clearly,

$$\{\sigma_n > t\} = \left\{ \inf_{1/n \leq s \leq t} |b(s)| > 0 \right\}.$$

By the Markov property of Brownian motion we get

$$\mathbb{P}^0(\sigma_n > t) = \mathbb{P}^0 \left(\inf_{1/n \leq s \leq t} |b(s)| > 0 \right)$$

$$\begin{aligned}
&= \mathbb{E}^0 \mathbb{P}^{b(1/n)} \left(\inf_{s \leq t-1/n} |b(s)| > 0 \right) \\
&= \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^{b(1/n)} \left(\inf_{s \leq t-1/n} b(s) > 0 \right) \right. \\
&\quad \left. + \mathbb{1}_{\{b(1/n) < 0\}} \mathbb{P}^{b(1/n)} \left(\sup_{s \leq t-1/n} b(s) < 0 \right) \right) \\
&= \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(\inf_{s \leq t-1/n} b(s) > -y \right) \right. \\
&\quad \left. + \mathbb{1}_{\{b(1/n) < 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < -y \right) \Big|_{y=b(1/n)} \right) \\
&\stackrel{b \rightsquigarrow -b}{=} \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < y \right) \right. \\
&\quad \left. + \mathbb{1}_{\{b(1/n) < 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < -y \right) \Big|_{y=b(1/n)} \right) \\
&\stackrel{b \rightsquigarrow -b}{=} \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < y \right) \right. \\
&\quad \left. + \mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < -y \right) \Big|_{-y=b(1/n)} \right) \\
&= 2 \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(\sup_{s \leq t-1/n} b(s) < y \right) \Big|_{y=b(1/n)} \right) \\
&\stackrel{(6.13)}{=} 2 \mathbb{E}^0 \left(\mathbb{1}_{\{b(1/n) > 0\}} \mathbb{P}^0 \left(|b(t-1/n)| < y \right) \Big|_{y=b(1/n)} \right) \\
&= 4 \int_0^\infty \mathbb{P}^0 \left(b(t-1/n) < y \right) \mathbb{P}^0 \left(b(1/n) \in dy \right) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{t-\frac{1}{n}} \sqrt{\frac{1}{n}}} \int_0^\infty \int_0^y e^{-z^2/2(t-1/n)} dz e^{-ny^2/2} dy
\end{aligned}$$

change of variables: $\zeta = z/\sqrt{t-\frac{1}{n}}$

$$= \frac{2\sqrt{n}}{\pi} \int_0^\infty \int_0^{y/\sqrt{t-\frac{1}{n}}} e^{-\zeta^2/2} d\zeta e^{-ny^2/2} dy.$$

For the density we differentiate in t :

$$\begin{aligned}
-\frac{d}{dt} \mathbb{P}^0(\sigma_n > t) &= -\frac{2\sqrt{n}}{\pi} \frac{d}{dt} \int_0^\infty \int_0^{y/\sqrt{t-\frac{1}{n}}} e^{-\zeta^2/2} d\zeta e^{ny^2/2} dy \\
&= \frac{\sqrt{n}}{\pi} \left(t-\frac{1}{n}\right)^{-3/2} \int_0^\infty y e^{-y^2/2(t-\frac{1}{n})} e^{-ny^2/2} dy \\
&= \frac{\sqrt{n}}{\pi} \left(t-\frac{1}{n}\right)^{-3/2} \int_0^\infty y e^{-\frac{y^2}{2} \frac{nt}{t-1/n}} dy \\
&= \frac{\sqrt{n}}{\pi} \left(t-\frac{1}{n}\right)^{-3/2} \frac{t-\frac{1}{n}}{nt} \left[-e^{-\frac{y^2}{2} \frac{nt}{t-1/n}} \right]_{y=0}^\infty \\
&= \frac{\sqrt{n}}{\pi} \left(t-\frac{1}{n}\right)^{-1/2} \frac{1}{nt} \\
&= \frac{1}{\pi} \frac{1}{t\sqrt{nt-1}}.
\end{aligned}$$

Now we proceed with the d -dimensional case. We have for all $x \in \mathbb{R}^{d-1}$

$$\begin{aligned}
\beta(\sigma_n) &\sim \int_{1/n}^\infty \frac{1}{(2\pi t)^{(d-1)/2}} e^{-|x|^2/(2t)} \mathbb{P}(\sigma_n \in dt) dx \\
&= \frac{1}{\pi^{(d+1)/2} 2^{(d-1)/2}} \int_{1/n}^\infty \frac{1}{t^{(d+1)/2} \sqrt{nt-1}} e^{-|x|^2/(2t)} dt
\end{aligned}$$

$$\begin{aligned}
 &= \frac{n^{(d-1)/2}}{\pi^{(d+1)/2} 2^{(d-1)/2}} \int_1^\infty \frac{1}{s^{(d+1)/2} \sqrt{s-1}} e^{-n|x|^2/(2s)} ds \\
 &\stackrel{(*)}{=} \frac{n^{(d-1)/2}}{\pi^{(d+1)/2} 2^{(d-1)/2}} B\left(\frac{d}{2}, \frac{1}{2}\right) {}_1F_1\left(\frac{d}{2}, \frac{d+1}{2}; -\frac{n}{2}|x|^2\right)
 \end{aligned}$$

where $B(\cdot, \cdot)$ is Euler's Beta-function and ${}_1F_1$ is the degenerate hypergeometric function, cf. Gradshteyn–Ryzhik [8, Section 9.20, 9.21] and, for (*), [8, Entry 3.471.5, p. 340].

■ ■

9 The variation of Brownian paths

Problem 9.1. Solution: Let $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \dots < t_m = 1\}$ be any partition of $[0, 1]$. As a continuous function on a compact space, f is uniformly continuous, i.e. there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2m}$ for all $x, y \in [0, 1]$ with $|x - y| < \delta$. Pick $n_0 \in \mathbb{N}$ so that $|\Pi_n| < \delta' := \delta \wedge \frac{1}{2} \min_{1 \leq i \leq m} |t_i - t_{i-1}|$ for all $n \geq n_0$.

Now, the balls $\mathbb{B}(t_j, \delta')$ for $0 \leq j \leq m$ are disjoint as $\delta' \leq \frac{1}{2} \min_{1 \leq i \leq m} |t_i - t_{i-1}|$. Therefore the sets $\mathbb{B}(t_j, \delta') \cap \Pi_{n_0}$ for $0 \leq j \leq m$ are also disjoint, and non-empty as $|\Pi_{n_0}| < \delta'$. In particular, there exists a subpartition $\Pi' = \{q_0 = 0 < q_1 < \dots < q_m = 1\}$ of Π_{n_0} such that $|t_j - q_j| < \delta' \leq \delta$ for all $0 \leq j \leq m$. This implies

$$\begin{aligned} \left| \sum_{j=1}^m |f(t_j) - f(t_{j-1})| - \sum_{j=1}^m |f(q_j) - f(q_{j-1})| \right| &\leq \sum_{j=1}^m \left| |f(t_j) - f(t_{j-1})| - |f(q_j) - f(q_{j-1})| \right| \\ &\leq \sum_{j=1}^m |f(t_j) - f(q_j) + f(t_{j-1}) - f(q_{j-1})| \\ &\leq 2 \cdot \sum_{j=0}^m |f(t_j) - f(q_j)| \\ &\leq \epsilon. \end{aligned}$$

Because adding points to a partition increases the corresponding variation sum, we have $S_1^\Pi(f; [0, 1]) \leq S_1^{\Pi'}(f; [0, 1]) + \epsilon \leq S_1^{\Pi_{n_0}}(f; [0, 1]) + \epsilon \leq \lim_{n \rightarrow \infty} S_1^{\Pi_n}(f; [0, 1]) + \epsilon \leq \text{VAR}_1(f; [0, 1]) + \epsilon$ and since Π was arbitrarily chosen, we deduce

$$\text{VAR}_1(f; [0, 1]) \leq \lim_{n \rightarrow \infty} S_1^{\Pi_n}(f; [0, 1]) + \epsilon \leq \text{VAR}_1(f; [0, 1]) + \epsilon$$

for every $\epsilon > 0$. Letting ϵ tend to zero completes the proof. ■ ■

Remark: The continuity of the function f is essential. A counterexample would be Dirichlet's discontinuous function $f = \mathbf{1}_{\mathbb{Q} \cap [0, 1]}$ and Π_n a refining sequence of partitions made up of rational points.

Problem 9.2. Solution: Note that the problem is straightforward if $\|x\|$ stands for the maximum norm: $\|x\| = \max_{1 \leq j \leq d} |x_j|$.

Remember that all norms on \mathbb{R}^d are equivalent. One quick way of showing this is the following: Denote by e_j with $j \in \{1, \dots, d\}$ the usual basis of \mathbb{R}^d . Then

$$\|x\| \leq \left(d \cdot \max_{1 \leq j \leq d} \|e_j\| \right) \cdot \max_{1 \leq j \leq d} |x_j| = \left(d \cdot \max_{1 \leq j \leq d} \|e_j\| \right) \cdot \|x\|_\infty =: B \cdot \|x\|_\infty$$

for every $x = \sum_{j=1}^d x_j e_j$ in \mathbb{R}^d using the triangle inequality and the positive homogeneity of norms. In particular, $x \mapsto \|x\|$ is a continuous mapping from \mathbb{R}^d equipped with the supremum-norm $\|\cdot\|_\infty$ to \mathbb{R} , since

$$\left| \|x\| - \|y\| \right| \leq \|x - y\| \leq B \cdot \|x - y\|_\infty$$

holds for every x, y in \mathbb{R}^d . Hence, the extreme value theorem claims that $x \mapsto \|x\|$ attains its minimum on the compact set $\{x \in \mathbb{R}^d : \|x\|_\infty = 1\}$. Finally, this implies $A := \min\{\|x\| : \|x\|_\infty = 1\} > 0$ and hence

$$\|x\| = \left\| \frac{x}{\|x\|_\infty} \right\| \cdot \|x\|_\infty \geq A \cdot \|x\|_\infty$$

for every $x \neq 0$ in \mathbb{R}^d as required.

As a result of the equivalence of norms on \mathbb{R}^d , it suffices to consider the supremum-norm to determine the finiteness of variations. In particular, $\text{VAR}_p(f; [0, t]) < \infty$ if, and only if,

$$\sup \left\{ \sum_{t_{j-1}, t_j \in \Pi} |g(t_j) - g(t_{j-1})|^p \vee |h(t_j) - h(t_{j-1})|^p : \Pi \text{ finite partition of } [0, 1] \right\}$$

is finite. But this term is bounded from below by $\text{VAR}_p(g; [0, t]) \vee \text{VAR}_p(h; [0, t])$ and from above by $\text{VAR}_p(g; [0, t]) + \text{VAR}_p(h; [0, t])$, which proves the desired result. ■ ■

Problem 9.3. Solution: Let $p > 0$, $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = 1\}$ a partition of $[0, 1]$. Since f is continuous and the rational numbers are dense in \mathbb{R} , there exist $0 < q_1 < \dots < q_{n-1} < 1$ such that q_j is rational and $|f(t_j) - f(q_j)| < n^{-1/p} \epsilon^{1/p}$ for every $1 \leq j \leq n-1$. In particular, $\Pi' = \{q_0 = 0 < q_1 < \dots < q_n = 1\}$ is a rational partition of $[0, 1]$ such that $\sum_{j=0}^n |f(t_j) - f(q_j)|^p \leq \epsilon$.

Some preliminary considerations: If $\phi : [0, \infty) \rightarrow \mathbb{R}$ is concave and $\phi(0) \geq 0$ then $\phi(ta) = \phi(ta + (1-t)0) \geq t\phi(a) + (1-t)\phi(0) \geq t\phi(a)$ for all $a \geq 0$ and $t \in [0, 1]$. Hence

$$\phi(a+b) = \frac{a}{a+b} \phi(a+b) + \frac{b}{a+b} \phi(a+b) \leq \phi(a) + \phi(b)$$

for all $a, b \geq 0$, i.e. ϕ is subadditive. In particular, we have $|x+y|^p \leq (|x|+|y|)^p \leq |x|^p + |y|^p$ and thus

$$\left| |x|^p - |y|^p \right| \leq |x-y|^p \quad \text{for all } p \leq 1 \quad \text{and } x, y \in \mathbb{R}. \quad (*)$$

For $p > 1$, on the other hand, and $x, y \in \mathbb{R}$ such that $|x| < |y|$ we find

$$\left| |y|^p - |x|^p \right| = \int_{|x|}^{|y|} p t^{p-1} dt \leq p \cdot (|x| \vee |y|)^{p-1} \cdot (|y| - |x|) \leq p \cdot (|x| \vee |y|)^{p-1} \cdot |y - x|$$

and hence

$$\left| |y|^p - |x|^p \right| \leq p \cdot (|x| \vee |y|)^{p-1} \cdot |y - x| \quad \text{for all } p > 1 \quad \text{and } x, y \in \mathbb{R} \quad (**)$$

using the symmetry of the inequality.

Let $p > 0$ and $\epsilon > 0$. For every partition $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = 1\}$ there exists a rational partition $\Pi' = \{q_0 = 0 < q_1 < \dots < q_n = 1\}$ such that $\sum_{j=0}^n |f(t_j) - f(q_j)|^{1 \wedge p} \leq \epsilon$ and hence

$$\begin{aligned} & \left| \sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p - \sum_{j=1}^n |f(q_j) - f(q_{j-1})|^p \right| \\ & \leq \sum_{j=1}^n \left| |f(t_j) - f(t_{j-1})|^p - |f(q_j) - f(q_{j-1})|^p \right| \\ & \stackrel{(*)}{\leq} \max \{1, (p \cdot 2^{p-1} \cdot \|f\|_\infty^{p-1})\} \cdot \sum_{j=1}^n |f(t_j) - f(q_j) + f(t_{j-1}) - f(q_{j-1})|^{1 \wedge p} \\ & \stackrel{(**)}{\leq} C \cdot \sum_{j=0}^n |f(t_j) - f(q_j)|^{1 \wedge p} \\ & \leq C \cdot \epsilon \end{aligned}$$

with a finite constant $C > 0$.

In particular, we have $\text{VAR}_p(f; [0, 1]) - C \cdot \epsilon \leq \text{VAR}_p^{\mathbb{Q}}(f; [0, 1]) \leq \text{VAR}_p(f; [0, 1])$ where

$$\text{VAR}_p^{\mathbb{Q}}(f; [0, 1]) := \sup \left\{ \sum_{q_{j-1}, q_j \in \Pi'} |f(q_j) - f(q_{j-1})|^p : \Pi' \text{ finite, rational partition of } [0, 1] \right\}$$

and hence the desired result as ϵ tends to zero.

Alternative Approach: Note that $(\xi_1, \dots, \xi_n) \mapsto \sum_{j=1}^n |f(\xi_j) - f(\xi_{j-1})|^p$ is a continuous map since it is the finite sum and composition of continuous maps, and that the rational numbers are dense in \mathbb{R} .

■ ■

Problem 9.4. Solution: Obviously, we have $\text{VAR}_p^{\circ}(f; [0, t]) \leq \text{VAR}_p(f; [0, t])$ with

$$\text{VAR}_p^{\circ}(f; [0, t]) := \sup \left\{ \sum_{j=1}^n |f(s_j) - f(s_{j-1})|^p : n \in \mathbb{N} \text{ and } 0 < s_0 < s_1 < \dots < s_n < t \right\}$$

because there are less (non-negative) summands in the definition of $\text{VAR}_p^{\circ}(f; [0, t])$.

Let $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = t\}$ a partition of $[0, t]$. Set $s_j = t_j$ for $1 \leq j \leq n-1$ and note that $\xi \mapsto |f(\xi_0) - f(\xi)|^p$ is a continuous map for every $\xi_0 \in [0, t]$ since it is the composition of continuous maps. Hence we can pick $s_0 \in (t_0, t_1)$ and $s_n \in (t_{n-1}, t_n)$ with

$$\begin{aligned} & \left| |f(s_1) - f(t_0)|^p - |f(s_1) - f(s_0)|^p \right| < \frac{\epsilon}{2} \\ & \left| |f(t_n) - f(t_{n-1})|^p - |f(s_n) - f(t_{n-1})|^p \right| < \frac{\epsilon}{2} \end{aligned}$$

and so that $0 < s_0 < s_1 < \dots < s_n < t$. This implies

$$\sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p = |f(s_1) - f(t_0)|^p + \sum_{j=2}^{n-1} |f(s_j) - f(s_{j-1})|^p + |f(t_n) - f(s_{n-1})|^p$$

$$\begin{aligned} &\leq \frac{\epsilon}{2} + \sum_{j=1}^n |f(s_j) - f(s_{j-1})|^p + \frac{\epsilon}{2} \\ &\leq \epsilon + \text{VAR}_p^\circ(f; [0, t]) \end{aligned}$$

and thus $\text{VAR}_p(f; [0, t]) \leq \epsilon + \text{VAR}_p^\circ(f; [0, t])$ since the partition $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = t\}$ was arbitrarily chosen. Consequently, $\text{VAR}_p(f; [0, t]) \leq \text{VAR}_p^\circ(f; [0, t])$ as ϵ tends to zero, as required.

The same argument shows that $\text{var}_p(f; [0, t])$ does not change its value (if it exists).

Problem 9.5. Solution:

a) Use $B(t) - B(s) \sim \mathbf{N}(0, |t - s|)$ to find

$$\begin{aligned} \mathbb{E} Y_n &= \sum_{k=1}^n \mathbb{E} \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right)^2 \\ &= \sum_{k=1}^n \mathbb{V} \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right) \\ &= \sum_{k=1}^n \left(\frac{k}{n} - \frac{k-1}{n} \right) \\ &= \sum_{k=1}^n \frac{1}{n} \\ &= 1 \end{aligned}$$

and the independence of increments to get

$$\begin{aligned} \mathbb{V} Y_n &= \sum_{k=1}^n \mathbb{V} \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right)^2 \\ &= \sum_{k=1}^n \mathbb{E} \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right)^4 - \left(\mathbb{E} \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right) \right)^2 \\ &= \sum_{k=1}^n 3 \cdot \left(\frac{k}{n} - \frac{k-1}{n} \right)^2 - \left(\frac{k}{n} - \frac{k-1}{n} \right)^2 \\ &= 2 \cdot \sum_{k=1}^n \frac{1}{n^2} \\ &= 2 \cdot \frac{1}{n} \end{aligned}$$

where we also used that $\mathbb{E}(X^4) = 3 \cdot \sigma^4$ for $X \sim \mathbf{N}(0, \sigma^2)$.

b) Note that the increments $B(\frac{k}{n}) - B(\frac{k-1}{n}) \sim \mathbf{N}(0, \frac{1}{n})$ are iid random variables. By a standard result the sum of squares $n \sum_{k=1}^n \left(B(\frac{k}{n}) - B(\frac{k-1}{n}) \right)^2$ has a χ_n^2 -distribution, i.e. its density is given by

$$2^{-n/2} \frac{1}{\Gamma(\frac{n}{2})} s^{\frac{n}{2}-1} e^{-\frac{s}{2}} \mathbb{1}_{[0, \infty)}(s).$$

and we get

$$\sum_{k=1}^n \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right)^2 \sim 2^{-n/2} \frac{n}{\Gamma(\frac{n}{2})} (ns)^{\frac{n}{2}-1} e^{-\frac{ns}{2}} \mathbb{1}_{[0, \infty)}(s).$$

Here is the calculation: (in case you do not know this standard result...): If $X \sim \mathbf{N}(0, 1)$ and $x > 0$, we have

$$\begin{aligned} \mathbb{P}(X^2 \leq x) &= \mathbb{P}(X \leq \sqrt{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} \exp\left(-\frac{t^2}{2}\right) dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{x}} \exp\left(-\frac{t^2}{2}\right) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{s}{2}\right) \cdot s^{-1/2} ds \end{aligned}$$

using the change of variable $s = t^2$. Hence, X^2 has density

$$f_{X^2}(s) = \mathbb{1}_{(0, \infty)}(s) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{-1/2}.$$

Let X_1, X_2, \dots be independent and identically distributed random variables with $X_1 \sim \mathbf{N}(0, 1)$. We want to prove by induction that for $n \geq 1$

$$f_{X_1^2 + \dots + X_n^2}(s) = C_n \cdot \mathbb{1}_{(0, \infty)}(s) \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1}$$

with some normalizing constants $C_n > 0$. Assume that this is true for $1, \dots, n$. Since X_{n+1}^2 is independent of $X_1^2 + \dots + X_n^2$ and distributed like X_1^2 , we know that the density of the sum is a convolution. This leads to

$$\begin{aligned} f_{X_1^2 + \dots + X_{n+1}^2}(s) &= \int_{-\infty}^{\infty} f_{X_1^2 + \dots + X_n^2}(t) \cdot f_{X_{n+1}^2}(s-t) dt \\ &= C_n \cdot C_1 \cdot \int_0^s \exp\left(-\frac{t}{2}\right) \cdot t^{n/2-1} \cdot \exp\left(-\frac{s-t}{2}\right) \cdot (s-t)^{-1/2} dt \\ &= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot \int_0^s t^{n/2-1} \cdot (s-t)^{-1/2} dt \\ &= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1} \cdot s^{-1/2} \cdot \int_0^s \left(\frac{t}{s}\right)^{n/2-1} \cdot \left(1 - \frac{t}{s}\right)^{-1/2} dt \\ &= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{(n+1)/2-1} \cdot \int_0^1 x^{n/2-1} \cdot (1-x)^{-1/2} dx \\ &= C_{n+1} \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{(n+1)/2-1} \end{aligned}$$

using the change of variable $x = t/s$. Since probability distribution functions integrate to one, we find

$$\begin{aligned} 1 &= C_n \cdot \int_0^{\infty} \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1} ds = C_n \cdot 2^{n/2} \int_0^{\infty} \exp(-t) \cdot t^{n/2-1} dt \\ &= C_n \cdot 2^{n/2} \cdot \Gamma(n/2) \end{aligned}$$

and thus

$$f_{X_1^2 + \dots + X_n^2}(s) = (2^{n/2} \cdot \Gamma(n/2))^{-1} \cdot \mathbb{1}_{(0, \infty)}(s) \cdot e^{-s/2} \cdot s^{n/2-1}$$

which is usually called *chi-squared* or χ^2 -distribution with n degrees of freedom. Now, remember that $B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \sim \mathbf{N}(0, 1/n) \sim n^{-1/2} \cdot X_k$ for $1 \leq k \leq n$. Hence

$$\begin{aligned} f_{Y_n}(s) &= n \cdot f_{X_1^2 + \dots + X_n^2}(n \cdot s) \\ &= n \cdot (2^{n/2} \cdot \Gamma(n/2))^{-1} \cdot \mathbb{1}_{(0, \infty)}(s) \cdot e^{-n \cdot s/2} \cdot (n \cdot s)^{n/2-1}. \end{aligned}$$

c) For $X \in \mathbf{N}(0, 1)$ and $\xi < 1/2$, we find

$$\begin{aligned} \mathbb{E}(e^{\xi \cdot X^2}) &= (2 \cdot \pi)^{-1/2} \int_{-\infty}^{\infty} e^{\xi \cdot x^2} e^{-x^2/2} dx = \frac{2}{\sqrt{2 \cdot \pi}} \int_0^{\infty} e^{-1/2 \cdot (1-2\xi) \cdot x^2} dx \\ &= (1-2\xi)^{-1/2} \frac{2}{\sqrt{2 \cdot \pi}} \int_0^{\infty} e^{-y^2/2} dy \\ &= (1-2\xi)^{-1/2} \end{aligned}$$

using the change of variable $x^2 = (1-2\xi)y^2$. Since the moment generating function $\xi \mapsto (1-2\xi)^{-1/2}$ has a unique analytic extension to an open strip around the imaginary axis, the characteristic function is of the form

$$\mathbb{E}(e^{i\xi \cdot X^2}) = (1-2i\xi)^{-1/2}.$$

Using the independence and $B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \sim \mathbf{N}(0, 1/n)$, we obtain

$$\mathbb{E}(e^{i\xi \cdot Y_n}) = \prod_{k=1}^n \mathbb{E}(e^{i\xi \cdot (B_{k/n} - B_{(k-1)/n})^2}) = \prod_{k=1}^n \mathbb{E}(e^{i(\xi/n) \cdot X^2}) = (1-2i(\xi/n))^{-n/2}$$

and hence

$$\lim_{n \rightarrow \infty} \phi_n(\xi) = \lim_{n \rightarrow \infty} (1-2i(\xi/n))^{-n/2} = \left(\lim_{n \rightarrow \infty} \left(1 - \frac{2i\xi}{n}\right)^n \right)^{-1/2} = (e^{-2i\xi})^{-1/2} = e^{i\xi}.$$

d) We have shown in a) that $\mathbb{E}((Y_n - 1)^2) = \mathbb{V}(Y_n) = 2/n$ which tends to zero as $n \rightarrow \infty$.

■ ■

Problem 9.6. Solution:

a)

$$\begin{aligned} \sqrt{2\pi} \cdot \mathbb{P}(Z > x) &= \int_x^{\infty} e^{-y^2/2} dy > \int_x^{\infty} \frac{y}{x} \cdot e^{-y^2/2} dy = \frac{1}{x} \cdot \left[-e^{-y^2/2} \right]_x^{\infty} = \frac{1}{x} \cdot e^{-x^2/2} \\ \implies \mathbb{P}(Z > x) &< \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \end{aligned}$$

On the other hand

$$\begin{aligned} \sqrt{2\pi} \cdot \mathbb{P}(Z > x) &= \int_x^{\infty} e^{-y^2/2} dy \\ &< \int_x^{\infty} \frac{x^2}{y^2} \cdot e^{-y^2/2} dy \\ &= x^2 \cdot \left(\left[-\frac{1}{y} \cdot e^{-y^2/2} \right]_x^{\infty} - \int_x^{\infty} e^{-y^2/2} dy \right) \\ &= x^2 \cdot \left(\left[-\frac{1}{y} \cdot e^{-y^2/2} \right]_x^{\infty} - \sqrt{2\pi} \cdot \mathbb{P}(Z > x) \right) \\ \implies (1+x^2) \cdot \sqrt{2\pi} \cdot \mathbb{P}(Z > x) &\geq x \cdot e^{-x^2/2} \\ \implies \mathbb{P}(Z > x) &> \frac{1}{\sqrt{2\pi}} \frac{x e^{-x^2/2}}{x^2 + 1} \end{aligned}$$

b) Using the independence of $A_{k,n}$ for $1 \leq k \leq 2^n$, we find

$$\begin{aligned} \mathbb{P}\left(\overline{\lim}_{n \rightarrow \infty} \bigcup_{k=1}^{2^n} A_{k,n}\right) &= 1 - \mathbb{P}\left(\liminf_{n \rightarrow \infty} \bigcap_{k=1}^{2^n} A_{k,n}^c\right) \\ &\geq 1 - \liminf_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{k=1}^{2^n} A_{k,n}^c\right) \\ &= 1 - \liminf_{n \rightarrow \infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) \end{aligned}$$

and hence it suffices to prove $\liminf_{n \rightarrow \infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) = 0$.

Since $1 - x \leq e^{-x}$ for $x \geq 0$, we obtain

$$\prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) = \left(1 - \mathbb{P}(A_{1,n})\right)^{2^n} \leq e^{-2^n \cdot \mathbb{P}(A_{1,n})}$$

and a) implies

$$\begin{aligned} 2^n \cdot \mathbb{P}(A_{1,n}) &= 2^n \cdot \mathbb{P}(\sqrt{2^{-n}} \cdot |Z| > c\sqrt{n2^{-n}}) \\ &= 2^{n+1} \cdot \mathbb{P}(Z > c\sqrt{n}) \\ &\geq \frac{2^{n+1}}{\sqrt{2\pi}} \cdot \frac{c\sqrt{n}}{c^2n+1} \cdot e^{-c^2n/2}. \end{aligned}$$

Now, $(c^2n)/(c^2n+1) \rightarrow 1$ as $n \rightarrow \infty$ and thus there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{c^2n}{c^2n+1} \geq \frac{1}{2} \iff \frac{c\sqrt{n}}{c^2n+1} \geq \frac{1}{2c\sqrt{n}}$$

for all $n \geq n_0$. Therefore, we have

$$2^n \cdot \mathbb{P}(A_{1,n}) \geq \frac{2^n}{\sqrt{2\pi}} \cdot \frac{1}{c\sqrt{n}} \cdot e^{-c^2n/2} = \frac{1}{\sqrt{2\pi c}} \cdot \frac{1}{\sqrt{n}} \cdot e^{(\log(2)-c^2/2)n}$$

for $n \geq n_0$. Since $\ln(2) - c^2/2 > 0$ if, and only if, $c < \sqrt{2\log(2)}$, we have $2^n \cdot \mathbb{P}(A_{1,n}) \rightarrow \infty$ and thus $\liminf_{n \rightarrow \infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) = 0$ if $c < \sqrt{2\log(2)}$.

c) With $c < \sqrt{2\log(2)}$ we deduce

$$\begin{aligned} 1 &= \mathbb{P}\left(\limsup_{n \rightarrow \infty} \bigcup_{k=1}^{2^n} A_{k,n}\right) \\ &= \mathbb{P}\left(\left\{\omega \in \Omega : \text{for infinitely many } n \in \mathbb{N} \exists k \in \{1, \dots, 2^n\} \right. \right. \\ &\quad \left. \left. \text{such that } |B(k2^{-n})(\omega) - B((k-1)2^{-n})(\omega)| > c\sqrt{n2^{-n}}\right\}\right) \\ &= \mathbb{P}\left(\left\{\omega \in \Omega : \text{for infinitely many } n \in \mathbb{N} \exists k \in \{1, \dots, 2^n\} \right. \right. \\ &\quad \left. \left. \text{such that } \frac{|B(k2^{-n})(\omega) - B((k-1)2^{-n})(\omega)|}{\sqrt{2^{-n}}} > c\sqrt{n}\right\}\right) \\ &\leq \mathbb{P}\left(\left\{\omega \in \Omega : t \mapsto B_t(\omega) \text{ is NOT } 1/2\text{-H\"older continuous}\right\}\right). \end{aligned}$$

■

Problem 9.7. Solution: From Problem 9.5 we know that

$$\Phi(\lambda) = \mathbb{E}(e^{\lambda(X^2-1)}) = e^{-\lambda} \mathbb{E}(e^{\lambda X^2}) = e^{-\lambda}(1-2\lambda)^{-1/2} \quad \text{for all } 0 < \lambda < 1/2.$$

Using $(a-b)^2 \leq 2(a^2+b^2)$, we get

$$|(X^2-1)^2 e^{\lambda(X^2-1)}| \leq |X^2-1|^2 \cdot e^{\lambda X^2} \leq 2(X^4+1) \cdot e^{\lambda_0 X^2}.$$

Since $\lambda < \lambda_0 < 1/2$ there is some $\epsilon > 0$ such that $\lambda < \lambda_0 < \lambda_0 + \epsilon < 1/2$. Thus,

$$|(X^2-1)^2 e^{\lambda(X^2-1)}| \leq 2(X^4+1)e^{-\epsilon X^2} \cdot e^{(\lambda_0+\epsilon)X^2}.$$

It is straightforward to see that

$$2(X^4+1)e^{-\epsilon X^2} \leq C_\epsilon = C(\lambda_0) < \infty,$$

and the claim follows. ■ ■

Problem 9.8. Solution: We follow the hint. Note that

$$e^{i\eta \mathbf{1}_F} = \begin{cases} e^{i\eta}, & x \in F \\ 1, & x \notin F \end{cases} = e^{i\eta} \mathbf{1}_F + \mathbf{1}_{F^c}.$$

Since $F, F^c \in \mathcal{F}$, we get $\forall \xi, \eta \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}(e^{i\xi X} e^{i\eta \mathbf{1}_F}) &= \mathbb{E}(e^{i\xi X} e^{i\eta} \mathbf{1}_F) + \mathbb{E}(e^{i\xi X} \mathbf{1}_{F^c}) \\ &= \mathbb{E}(e^{i\xi X}) \mathbb{E}(e^{i\eta} \mathbf{1}_F) + \mathbb{E}(e^{i\xi X}) \mathbb{E}(\mathbf{1}_{F^c}) \\ &= \mathbb{E}(e^{i\xi X}) \mathbb{E}(e^{i\eta \mathbf{1}_F}). \end{aligned}$$

This is, however, M. Kac's characterization of independence and we conclude that $X \perp\!\!\!\perp \mathbf{1}_F$, hence, $X \perp\!\!\!\perp F$ for all sets $F \in \mathcal{F}$.

The converse is obvious. ■ ■

10 Regularity of Brownian paths

Problem 10.1. Solution:

a) Note that for $t, h \geq 0$ and any integer $k = 0, 1, 2, \dots$

$$\mathbb{P}(N_{t+h} - N_t = k) = \mathbb{P}(N_h = k) = \frac{(\lambda h)^k}{k!} e^{-\lambda h}.$$

This shows that we have for any $\alpha > 0$

$$\begin{aligned} \mathbb{E}(|N_{t+h} - N_t|^\alpha) &= \sum_{k=0}^{\infty} k^\alpha \frac{(\lambda h)^k}{k!} e^{-\lambda h} \\ &= \lambda h e^{-\lambda h} + \sum_{k=2}^{\infty} k^\alpha \frac{(\lambda h)^k}{k!} e^{-\lambda h} \\ &= \lambda h e^{-\lambda h} + \lambda h \sum_{k=2}^{\infty} k^\alpha \frac{(\lambda h)^{k-1}}{k!} e^{-\lambda h} \\ &= \lambda h e^{-\lambda h} + o(h) \end{aligned}$$

and, thus,

$$\lim_{h \rightarrow 0} \frac{\mathbb{E}(|N_{t+h} - N_t|^\alpha)}{h} = \lambda$$

which means that (10.1) cannot hold for any $\alpha > 0$ and $\beta > 0$.

b) Part a) shows also $\mathbb{E}(|N_{t+h} - N_t|^\alpha) \leq ch$, i.e. condition (10.1) holds for $\alpha > 0$ and $\beta = 0$.

The fact that $\beta = 0$ is needed for the convergence of the dyadic series (with the power $\gamma < \beta/\alpha$) in the proof of Theorem 10.1.

c) We have

$$\begin{aligned} \mathbb{E}(N_t) &= \sum_{k=0}^{\infty} k \frac{t^k}{k!} e^{-t} = \sum_{k=1}^{\infty} k \frac{t^k}{k!} e^{-t} = t \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} e^{-t} = t \sum_{j=0}^{\infty} \frac{t^j}{j!} e^{-t} = t \\ \mathbb{E}(N_t^2) &= \sum_{k=0}^{\infty} k^2 \frac{t^k}{k!} e^{-t} = \sum_{k=1}^{\infty} k^2 \frac{t^k}{k!} e^{-t} = t \sum_{k=1}^{\infty} k \frac{t^{k-1}}{(k-1)!} e^{-t} \\ &= t \sum_{k=1}^{\infty} (k-1) \frac{t^{k-1}}{(k-1)!} e^{-t} + t \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} e^{-t} \\ &= t^2 \sum_{k=2}^{\infty} \frac{t^{k-2}}{(k-2)!} e^{-t} + t \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} e^{-t} = t^2 + t \end{aligned}$$

and this shows that

$$\mathbb{E}(N_t - t) = \mathbb{E} N_t - t = 0$$

$$\mathbb{E}((N_t - t)^2) = \mathbb{E}(N_t^2) - 2t \mathbb{E} N_t + t^2 = t$$

and, finally, if $s \leq t$

$$\begin{aligned} \text{Cov}((N_t - t)(N_s - s)) &= \mathbb{E}((N_t - t)(N_s - s)) \\ &= \mathbb{E}((N_t - N_s - t + s)(N_s - s)) + \mathbb{E}((N_s - s)^2) \\ &= \mathbb{E}((N_t - N_s - t + s)) \mathbb{E}((N_s - s)) + s \\ &= s = s \wedge t \end{aligned}$$

where we used the independence of $N_t - N_s \perp\!\!\!\perp N_s$.

Alternative Solution: One can show, as for a Brownian motion (Example 5.2 a)), that N_t is a martingale for the canonical filtration $\mathcal{F}_t^N = \sigma(N_s : s \leq t)$. The proof only uses stationary and independent increments. Thus, by the tower property, pull out and the martingale property,

$$\begin{aligned} \mathbb{E}((N_t - t)(N_s - s)) &= \mathbb{E}(\mathbb{E}((N_t - t)(N_s - s) \mid \mathcal{F}_s^N)) \\ &= \mathbb{E}((N_s - s) \mathbb{E}((N_t - t) \mid \mathcal{F}_s^N)) \\ &= \mathbb{E}((N_s - s)^2) \\ &= s = s \wedge t. \end{aligned}$$

Problem 10.2. Solution: We have

$$\max_{1 \leq j \leq n} |x_j|^p \leq \max_{1 \leq j \leq n} (|x_1|^p + \dots + |x_n|^p) = \sum_{j=1}^n |x_j|^p \leq \sum_{j=1}^n \max_{1 \leq k \leq n} |x_k|^p = n \max_{1 \leq k \leq n} |x_k|^p.$$

Since $\max_{1 \leq j \leq n} |x_j|^p = (\max_{1 \leq j \leq n} |x_j|)^p$ the claim follows (actually with $n^{1/p}$ which is smaller than $n \dots$)

Problem 10.3. Solution: Let $\alpha \in (0, 1)$. Since

$$|x + y|^\alpha \leq (|x| + |y|)^\alpha$$

it is enough to show that

$$(|x| + |y|)^\alpha \leq |x|^\alpha + |y|^\alpha$$

and, without loss of generality

$$(s + t)^\alpha \leq s^\alpha + t^\alpha \quad \forall s, t > 0.$$

This follows from

$$s^\alpha + t^\alpha = s \cdot s^{\alpha-1} + t \cdot t^{\alpha-1} \geq s \cdot (s + t)^{\alpha-1} + t \cdot (s + t)^{\alpha-1} = (s + t)(s + t)^{\alpha-1} = (s + t)^\alpha.$$

Since the expectation is linear, this proves that

$$\mathbb{E}(|X + Y|^\alpha) \leq \mathbb{E}(|X|^\alpha) + \mathbb{E}(|Y|^\alpha).$$

In the proof of Theorem 10.1 (page 154, line 1 from above and onwards) we get:

This entails for $\alpha \in (0, 1)$ because of the subadditivity of $x \mapsto |x|^\alpha$

$$\begin{aligned} \left(\sup_{x, y \in D, x \neq y} \frac{|\xi(x) - \xi(y)|}{|x - y|^\gamma} \right)^\alpha &= \sup_{m \geq 0} \sup_{\substack{x, y \in D \\ 2^{-m-1} \leq |x-y| < 2^{-m}}} \frac{|\xi(x) - \xi(y)|^\alpha}{2^{-(m+1)\gamma\alpha}} \\ &\leq \sup_{m \geq 0} \left(2^\alpha \cdot 2^{(m+1)\gamma\alpha} \sum_{j \geq m} \sigma_j^\alpha \right) \\ &= 2^{(1+\gamma)\alpha} \sup_{m \geq 0} \sum_{j \geq m} 2^{m\gamma\alpha} \sigma_j^\alpha \\ &\leq 2^{(1+\gamma)\alpha} \sum_{j=0}^{\infty} 2^{j\gamma\alpha} \sigma_j^\alpha. \end{aligned}$$

For $\alpha \in (0, 1)$ and $\alpha\gamma < \beta$ we get

$$\begin{aligned} \mathbb{E} \left[\left(\sup_{x \neq y, x, y \in D} \frac{|\xi(x) - \xi(y)|}{|x - y|^\gamma} \right)^\alpha \right] &\leq 2^{(1+\gamma)\alpha} \sum_{j=0}^{\infty} 2^{j\gamma\alpha} \mathbb{E} [\sigma_j^\alpha] \\ &\leq c 2^{(1+\gamma)\alpha} \sum_{j=0}^{\infty} 2^{j\gamma\alpha} 3^n 2^{-j\beta} \\ &= c 2^{(1+\gamma)\alpha} 3^n \sum_{j=0}^{\infty} 2^{j(\gamma\alpha - \beta)} < \infty. \end{aligned}$$

The rest of the proof continues literally as on page 154, line 10 onwards.

Alternative Solution: use the subadditivity of $Z \mapsto \mathbb{E}(|Z|^\alpha)$ directly in the second part of the calculation, replacing $\|Z\|_{L^\alpha}$ by $\mathbb{E}(|Z|^\alpha)$.

■ ■

Problem 10.4. Solution: We show the following

Theorem. Let $(B_t)_{t \geq 0}$ be a BM¹. Then $t \mapsto B_t(\omega)$ is for almost all $\omega \in \Omega$ nowhere Hölder continuous of any order $\alpha > 1/2$.

Proof. Set for every $n \geq 1$

$$A_n := A_{n,\alpha} = \left\{ \omega \in \Omega : B(\cdot, \omega) \text{ is in } [0, n] \text{ nowhere Hölder continuous of order } \alpha > \frac{1}{2} \right\}.$$

It is not clear if the set $A_{n,\alpha}$ is measurable. We will show that $\Omega \setminus A_{n,\alpha} \subset N_{n,\alpha}$ for a measurable null set $N_{n,\alpha}$.

Assume that the function f is α -Hölder continuous of order α at the point $t_0 \in [0, n]$. Then

$$\exists \delta > 0 \exists L > 0 \forall t \in \mathbb{B}(t_0, \delta) : |f(t) - f(t_0)| \leq L |t - t_0|^\alpha.$$

Since $[0, n]$ is compact, we can use a covering argument to get a uniform Hölder constant. Consider for sufficiently large values of $k \geq 1$ the grid $\{\frac{j}{k} : j = 1, \dots, nk\}$. Then there exists a smallest index $j = j(k)$ such that for $\nu \geq 3$ and, actually, $1 - \nu\alpha + \nu/2 < 0$

$$t_0 \leq \frac{j}{k} \quad \text{and} \quad \frac{j}{k}, \dots, \frac{j+\nu}{k} \in \mathbb{B}(t_0, \delta).$$

For $i = j+1, j+2, \dots, j+\nu$ we get therefore

$$\begin{aligned} |f(\frac{i}{k}) - f(\frac{i-1}{k})| &\leq |f(\frac{i}{k}) - f(t_0)| + |f(t_0) - f(\frac{i-1}{k})| \\ &\leq L(|\frac{i}{k} - t_0|^\alpha + |\frac{i-1}{k} - t_0|^\alpha) \\ &\leq L\left(\frac{(\nu+1)^\alpha}{k^\alpha} + \frac{\nu^\alpha}{k^\alpha}\right) = \frac{2L(\nu+1)^\alpha}{k^\alpha}. \end{aligned}$$

If f is a Brownian path, this implies that for the sets

$$C_m^{L,\nu,\alpha} := \bigcap_{k=m}^{\infty} \bigcup_{j=1}^{kn} \bigcap_{i=j+1}^{j+\nu} \left\{ |B(\frac{i}{k}) - B(\frac{i-1}{k})| \leq \frac{2L(\nu+1)^\alpha}{k^\alpha} \right\}$$

we have

$$\Omega \setminus A_{n,\alpha} \subset \bigcup_{L=1}^{\infty} \bigcup_{m=1}^{\infty} C_m^{L,\nu,\alpha}.$$

Our assertion follows if we can show that $\mathbb{P}(C_m^{L,\nu,\alpha}) = 0$ for all $m, L \geq 1$ and all rational $\alpha > 1/2$. If $k \geq m$,

$$\begin{aligned} \mathbb{P}(C_m^{L,\nu,\alpha}) &\leq \mathbb{P}\left(\bigcup_{j=1}^{kn} \bigcap_{i=j+1}^{j+\nu} \left\{ |B(\frac{i}{k}) - B(\frac{i-1}{k})| \leq \frac{2L(\nu+1)^\alpha}{k^\alpha} \right\}\right) \\ &\leq \sum_{j=1}^{kn} \mathbb{P}\left(\bigcap_{i=j+1}^{j+\nu} \left\{ |B(\frac{i}{k}) - B(\frac{i-1}{k})| \leq \frac{2L(\nu+1)^\alpha}{k^\alpha} \right\}\right) \\ &\stackrel{(B1)}{=} \sum_{j=1}^{kn} \mathbb{P}\left(\left\{ |B(\frac{i}{k}) - B(\frac{i-1}{k})| \leq \frac{2L(\nu+1)^\alpha}{k^\alpha} \right\}^\nu\right) \\ &\stackrel{(B2)}{=} kn \mathbb{P}\left(\left\{ |B(\frac{1}{k})| \leq \frac{2L(\nu+1)^\alpha}{k^\alpha} \right\}^\nu\right) \\ &\leq kn \left(\frac{c}{k^{\alpha-1/2}}\right)^\nu = c^\nu n k^{1-\nu\alpha+\nu/2} \xrightarrow[k \rightarrow \infty]{1-\nu\alpha+\nu/2 < 0} 0. \end{aligned}$$

For the last estimate we use $B(\frac{1}{k}) \sim k^{-1/2}B(1)$, cf. 2.16, and therefore

$$\mathbb{P}\left(|B(\frac{1}{k})| \leq x\right) = \mathbb{P}\left(|B(1)| \leq x\sqrt{k}\right) = \frac{1}{\sqrt{2\pi}} \int_{-x\sqrt{k}}^{x\sqrt{k}} \underbrace{e^{-y^2/2}}_{\leq 1} dy \leq cx\sqrt{k}.$$

This proves that a Brownian path is almost surely nowhere not Hölder continuous of a fixed order $\alpha > 1/2$. Call the set where this holds Ω_α . Then $\Omega_0 := \bigcap_{\alpha > 1/2} \Omega_\alpha$ is a set with $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ we know that BM is nowhere Hölder continuous of any order $\alpha > 1/2$.

The last conclusion uses the following simple remark. Let $0 < \alpha < q < \infty$. Then we have for $f : [0, n] \rightarrow \mathbb{R}$ and $x, y \in [0, n]$ with $|x - y| < 1$ that

$$|f(x) - f(y)| \leq L|x - y|^q \leq L|x - y|^\alpha.$$

Thus q -Hölder continuity implies α -Hölder continuity. □



Problem 10.5. Solution: Fix $\epsilon > 0$, fix a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and $h_0 = h_0(2, \omega)$ such that (10.6) holds for all $\omega \in \Omega_0$, i.e. for all $h \leq h_0$ we have

$$\sup_{0 \leq t \leq 1-h} |B(t+h, \omega) - B(t, \omega)| \leq 2\sqrt{2h \log \frac{1}{h}}.$$

Pick a partition $\Pi = \{t_0 = 0 < t_1 < \dots < t_n\}$ of $[0, 1]$ with mesh size $h = \max_j(t_j - t_{j-1}) \leq h_0$ and assume that $h_0/2 \leq h \leq h_0$. Then we get

$$\begin{aligned} \sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} &\leq 2^{2+2\epsilon} \cdot 2^{1+\epsilon} \sum_{j=1}^n \left((t_j - t_{j-1}) \log \frac{1}{t_j - t_{j-1}} \right)^{1+\epsilon} \\ &\leq c_\epsilon \sum_{j=1}^n (t_j - t_{j-1}) = c_\epsilon. \end{aligned}$$

This shows that

$$\sup_{|\Pi| \leq h_0} \sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} \leq c_\epsilon.$$

Since we have $|x - y|^p \leq 2^{p-1}(|x - z|^p + |z - y|^p)$ and since we can refine any partition Π of $[0, 1]$ in finitely many steps to a partition of mesh $< h_0$, we get

$$\text{VAR}_{2+2\epsilon}(B; [0, 1]) = \sup_{\Pi \subset [0, 1]} \sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} < \infty$$

for all $\omega \in \Omega_0$.



11 Brownian motion as a random fractal

Problem 11.1. Solution: The idea is to show that \mathcal{H}_δ^s for every $\delta > 0$ is an outer measure.

This solves the problem, since these properties are retained by taking the supremum over $\delta > 0$.

It is easy to see that $E_j := \emptyset$ for $j \in \mathbb{N}$ is a δ -cover of $E = \emptyset$ and hence $\mathcal{H}_\delta^s(\emptyset) = 0$. Moreover, if $(E_j)_{j \in \mathbb{N}}$ is a δ -cover of $F \subset \mathbb{R}^d$ and $E \subset F$, then it is also a δ -cover of E and therefore $\mathcal{H}_\delta^s(E) \leq \mathcal{H}_\delta^s(F)$.

Let $\epsilon > 0$ and suppose that $(E^k)_{k \in \mathbb{N}}$ is a sequence of subsets of \mathbb{R}^d . Due to the definition of \mathcal{H}_δ^s , there exists a δ -cover $(E_j^k)_{j \in \mathbb{N}}$ for every $k \in \mathbb{N}$ such that

$$\sum_{j \in \mathbb{N}} |E_j^k|^s \leq \mathcal{H}_\delta^s(E^k) + 2^{-k} \epsilon$$

holds. Since the double sequence $(E_j^k)_{j,k \in \mathbb{N}}$ is obviously a δ -cover of $\bigcup_{k \in \mathbb{N}} E^k$, we find that

$$\mathcal{H}_\delta^s\left(\bigcup_{k \in \mathbb{N}} E^k\right) \leq \sum_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} |E_j^k|^s \leq \sum_{k \in \mathbb{N}} (\mathcal{H}_\delta^s(E^k) + 2^{-k} \epsilon) \leq \sum_{k \in \mathbb{N}} \mathcal{H}_\delta^s(E^k) + \epsilon.$$

holds. This implies the σ -subadditivity as $\epsilon \rightarrow 0$.

■ ■

Problem 11.2. Solution: Let U be open. Then U contains an open ball $B \subset U$ and B contains a cube $Q \subset B \subset U$. On the other hand, since U is bounded, it is contained in a large cube $Q' \supset U$. Since Hausdorff measure is monotone, we have $\mathcal{H}^d(Q) \leq \mathcal{H}^d(U) \leq \mathcal{H}^d(Q')$ and it is, thus, enough to show the claim for cubes.

The following argument is easily adapted to a general cube. Assume that $Q = [0, 1]^d$ and cover Q by n^d non-overlapping cubes which are shifted copies of $[0, 1/n]^d$. Clearly, if $n > 1/\delta$,

$$\mathcal{H}_\delta^d(Q) \leq \sum_{j=1}^{n^d} |[0, 1/n]^d|^d = n^d (\sqrt{d}n^{-1})^d = (\sqrt{d})^d.$$

This shows that $\mathcal{H}^d(Q) \leq (\sqrt{d})^d < \infty$.

For the lower bound we take any δ -cover $(E_j)_{j \geq 1}$ of Q . For each j there is a closed cube C_j such that $E_j \subset C_j$ and the lengths of the edges of C_j are less or equal to $2|E_j| \leq 2\delta$. If λ^d is Lebesgue measure, we get

$$1 = \lambda^d(Q) \leq \lambda^d\left(\bigcup_j E_j\right) \leq \lambda^d\left(\bigcup_j C_j\right) \leq \sum_j \lambda^d(C_j) = \sum_j (2|E_j|)^d.$$

This gives

$$\sum_{j=1}^{\infty} |E_j|^d \geq 2^{-d} > 0 \implies \mathcal{H}^d(Q) \geq 2^{-d}.$$

Problem 11.3. Solution: It is enough to show the following two assertions: Let $0 \leq \alpha < \beta < \infty$ and $E \subset \mathbb{R}^d$. Then

a) $\mathcal{H}^\alpha(E) < \infty \implies \mathcal{H}^\beta(E) = 0$.

b) $\mathcal{H}^\beta(E) > 0 \implies \mathcal{H}^\alpha(E) = \infty$.

Claim a) is just Lemma 11.4. Part b) is just the contraposition of a).

Problem 11.4. Solution: Since $E_j \subset E$, we have $\dim E_j \leq \dim E$ and $\sup_{j \geq 1} \dim E_j \leq \dim E$. Conversely, if $\alpha < \dim E$, then $\mathcal{H}^\alpha(E) = \infty$ and by the σ -subadditivity of the outer measure \mathcal{H}^α , we get $\mathcal{H}^\alpha(E_{j_0}) > 0$ for at least one index j_0 . Thus, $\alpha \leq \dim E_{j_0} \leq \sup_{j \geq 1} \dim E_j$. This proves $\dim E \leq \sup_{j \geq 1} \dim E_j$. (Indeed, if we had $\dim E > \sup_{j \geq 1} \dim E_j$, we could find some λ such that $\dim E > \lambda > \sup_{j \geq 1} \dim E_j$ contradicting our previous calculation.)

Problem 11.5. Solution: It is possible to show that $\dim(E \times F) \geq \dim(E) + \dim(F)$ holds for arbitrary $E \subset \mathbb{R}^d$ and $F \subset \mathbb{R}^n$, cf. [6, Theorem 5.12]. Unfortunately, the opposite direction only holds under certain restriction on the sets E and F , cf. for example [7, Corollary 7.4]. In fact, one can show that there exist Borel sets $E, F \subset \mathbb{R}$ with $\dim(E) = \dim(F) = 0$ and $\dim(E \times F) \geq 1$, cf. [6, Theorem 5.11].

We are going to prove the other direction (that does not hold in general) for this special case: Let $t > \dim(E)$ and $\delta > 0$. According to the Definition 11.5, there exists a δ -cover $(E_j)_{j \in \mathbb{N}}$ of $E \subset \mathbb{R}^d$ with $\sum_{j \in \mathbb{N}} |E_j|^t \leq \delta$. Let $m \in \mathbb{N}$ so that $\sqrt{n}/m \leq \delta$, and $(F_k)_k$ be a disjoint tessellation of $[0, 1]^n$ by m^n -many cubes with side-length $1/m$. Now, $(E_j \times F_k)_{j,k}$ is a δ^2 -cover of $E \times [0, 1]^n$ and hence

$$\mathcal{H}_{\delta^2}^{t+n}(E \times [0, 1]^n) \leq \sum_{k=1}^{m^n} \sum_{j \in \mathbb{N}} |E_j \times F_k|^{t+n} \leq \sum_{k=1}^{m^n} \sum_{j \in \mathbb{N}} |E_j|^t n^{n/2} m^{-n} \leq n^{n/2} \delta$$

holds. In particular, $\mathcal{H}^{t+n}(E \times [0, 1]^n) = 0$ as $\delta \rightarrow 0$ and thus $\dim(E \times [0, 1]^n) \leq t + n$. Since \mathbb{R}^n can be represented as countable union of cubes with unit side-length, Problem 11.4 tells us that we also have $\dim(E \times \mathbb{R}^n) = \dim(E \times [0, 1]^n) \leq t + n$. This proves that $\dim(E \times \mathbb{R}^n) \leq \dim(E) + n$, as required.

Problem 11.6. Solution: Remark 11.6.c) says that $\dim f(E) \leq \dim E$ holds for a Lipschitz map $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$. Therefore, we also have $\dim E = \dim f^{-1}(f(E)) \leq \dim f(E)$ for a bi-Lipschitz map f and hence the desired result.

Moreover, Remark 11.6.c) tells us that $\dim f(E) \leq \gamma^{-1} \dim E$ holds for a Hölder continuous map $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$ with index $\gamma \in (0, 1]$. Note that this inequality can be strict, e.g. take $f \equiv 0$ and any $E \subset \mathbb{R}^d$ with $\dim E > 0$.

Note that there is no bi-Lipschitz $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$ that is also Hölder continuous with index $\gamma \in (0, 1)$: Suppose f had these properties, then there would exist a constant $C > 0$ such that

$$|x - y| = |f(f^{-1}(x)) - f(f^{-1}(y))| \leq C|x - y|^\gamma$$

holds for all $x, y \in \mathbb{R}^d$. This leads to a contradiction to the boundedness of $C > 0$. Hence, there is no bi-Lipschitz map that is also Hölder continuous with index $\gamma \in (0, 1)$.

Problem 11.7. Solution: Let $C_0 := [0, 1]$. It is easy to see that $C_n = f_1(C_{n-1}) \cup f_2(C_{n-1})$ for $n \in \mathbb{N}$ and $C := \bigcap_{n \in \mathbb{N}} C_n$ models the recursive definition of Cantor's discontinuum in the description of the problem. Now, note that

$$\begin{aligned} f_1\left(\sum_{j=1}^{\infty} t_j 3^{-j}\right) &= \sum_{j=1}^{\infty} t_j 3^{-(j+1)} = 0 \cdot 3^{-1} + \sum_{j=2}^{\infty} t_{j-1} 3^{-j} \\ f_2\left(\sum_{j=1}^{\infty} t_j 3^{-j}\right) &= \sum_{j=1}^{\infty} t_j 3^{-(j+1)} + 2/3 = 2 \cdot 3^{-1} + \sum_{j=2}^{\infty} t_{j-1} 3^{-j} \end{aligned}$$

holds for sequences $(t_j)_{j \in \mathbb{N}}$ with $t_j \in \{0, 1, 2\}$ and that

$$C_0 = \left\{ \sum_{j=1}^{\infty} t_j 3^{-j} : t_j \in \{0, 1, 2\} \text{ for } j \in \mathbb{N} \right\}$$

reflects the triadic representation of real numbers. This representation implies that

$$C_n = \left\{ \sum_{j=1}^{\infty} t_j 3^{-j} : t_j \in \{0, 2\} \text{ for } j \leq n \text{ and } t_j \in \{0, 1, 2\} \text{ for } j > n \right\}$$

holds for every $n \in \mathbb{N}$ using mathematical induction. Hence,

$$C = \left\{ \sum_{j=1}^{\infty} t_j 3^{-j} : t_j \in \{0, 2\} \text{ for } j \in \mathbb{N} \right\}$$

and therefore $C = f_1(C) \cup f_2(C)$. The results from Remark 11.6.e) solve the rest of the problem.

Problem 11.8. Solution: Denote by $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$ the surface volume of the $(d-1)$ -dimensional unit sphere in \mathbb{R}^d . Using polar coordinates, we find

$$\begin{aligned} \mathbb{E}(|B_1|^{-\lambda}) &= \int_{\mathbb{R}^d} |x|^{-\lambda} \mathbb{P}(B_1 \in dx) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |x|^{-\lambda} e^{-\frac{1}{2}|x|^2} dx \\ &= \sigma_d (2\pi)^{-\frac{d}{2}} \int_0^{\infty} r^{d-\lambda-1} e^{-\frac{1}{2}r^2} dr \end{aligned}$$

$$\begin{aligned}
 &= \sigma_d(2\pi)^{-\frac{d}{2}} \int_0^\infty (2u)^{\frac{d-\lambda-2}{2}} e^{-u} du \\
 &= \sigma_d(2\pi)^{-\frac{d}{2}} 2^{\frac{d-\lambda-2}{2}} \Gamma\left(\frac{d-\lambda}{2}\right) \\
 &= \frac{\Gamma\left(\frac{d-\lambda}{2}\right)}{2^{\frac{\lambda}{2}} \Gamma\left(\frac{d}{2}\right)}.
 \end{aligned}$$

Problem 11.9. Solution: Following the hint we have

$$B^{-1}(A) = W^{-1}(A \times \mathbb{R}) \leq \frac{1}{2} \dim(A \times \mathbb{R}) \leq \frac{1}{2}(1 + \dim A),$$

where we used the result of Problem 11.5.

Problem 11.10. Solution: (F. Hausdorff) We show that a perfect set contains a Cantor-type set. Since Cantor sets are uncountable, we are done.

Pick $a_1, a_2 \in F$ and disjoint closed balls $F_j, j = 1, 2$ with centre a_j . Now take open balls such that $U_j \subset F_j$. Since $\bar{U}_j \cap F, j = 1, 2$, are again perfect sets, we can repeat this construction, i.e. pick $a_{j1}, a_{j2} \in U_j \cap F$ and disjoint closed balls $F_{jk} \subset U_j$ with centre a_{jk} and open balls $U_{jk} \subset F_{jk}, k = 1, 2$. Each of the four sets $A \cap \bar{U}_{jk}, j, k = 1, 2$, is perfect. Again we find points $a_{jk1}, a_{jk2} \in U_{jk}$ etc. Without loss of generality we can arrange things such that the diameters of the balls $F_j, F_{jk}, F_{jkl}, \dots$ are smaller than $1, \frac{1}{2}, \frac{1}{3}, \dots$. This construction yields a discontinuum set $D \subset F$: Any $x \in D$ which is contained in $F_j, F_{jk}, F_{jkl}, \dots$ is the limit of the centres $a_j, a_{jk}, a_{jkl}, \dots$. It is now obvious how to make a correspondence between the points $a_{jkl} \dots \in F$ and the Cantor ternary set.

Problem 11.11. Solution:

a) We have

$$\begin{aligned}
 \mathbb{P}(g_t \in ds) &= \int_0^\infty \mathbb{P}(g_t \in ds, |B_t| \in dy) = \frac{ds}{\pi \sqrt{s(t-s)^3}} \int_0^\infty y e^{-y^2/(2(t-s))} dy \\
 &= \frac{ds}{\pi \sqrt{s(t-s)}} \int_0^\infty \frac{y}{(t-s)} e^{-y^2/(2(t-s))} dy \\
 &= \frac{ds}{\pi \sqrt{s(t-s)}} \left[-e^{-y^2/(2(t-s))} \right]_{y=0}^\infty = \frac{ds}{\pi \sqrt{s(t-s)}}.
 \end{aligned}$$

Using $\sin^2 \phi + \cos^2 \phi = 1$ we see

$$\begin{aligned}
 \mathbb{P}(g_t < s) &= \int_0^s \frac{du}{\pi \sqrt{u(t-u)}} \\
 &\stackrel{u=tv}{=} \int_0^{s/t} \frac{dv}{\pi \sqrt{v(1-v)}} \\
 &\stackrel{v=\sin^2 \phi}{=} \int_0^{\arcsin \sqrt{s/t}} \frac{2 \sin \phi \cos \phi}{\pi \sin \phi \cos \phi} d\phi = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}.
 \end{aligned}$$

b) From part a) we know that the density of g_t is

$$f_{g_t}(s) = \frac{1}{\pi\sqrt{s(t-s)}}, \quad 0 < s < t.$$

The joint density of $(|B_t|, g_t)$ is, by Theorem 11.25

$$f_{(|B_t|, g_t)}(y, s) = \frac{y}{\pi\sqrt{s(t-s)^3}} e^{-y^2/(2(t-s))}, \quad 0 < s < t, y > 0.$$

Using standard formulae for the conditional density, we find

$$f_{|B_t||g_t}(y | s) = \frac{f_{(|B_t|, g_t)}(y, s)}{\int_0^\infty f_{(|B_t|, g_t)}(y, s) dy} = \frac{f_{(|B_t|, g_t)}(y, s)}{f_{g_t}(s)} = \frac{y}{t-s} e^{-y^2/(2(t-s))}.$$

c) Write

$$p_t(x, y) = f_{B_t}(x - y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}$$

for the law of B_t and set

$$g_t(x, y) := \frac{|x-y|}{\sqrt{2\pi t^3}} e^{-(x-y)^2/(2t)} \stackrel{y>x}{=} \frac{\partial}{\partial x} p_t(x, y).$$

As a function of t , this is the density of τ_{x-y} , see (6.14). Then the identity reads (after cancelling the factor 2)

$$p_t(0, y) = \int_0^t p_s(0, 0) g_{t-s}(0, y) ds = \int_0^t g_{t-s}(0, y) p_s(y, y) ds.$$

The first identity is a “last exit decomposition” of the density $p_t(0, y)$ while the last identity is a “first entrance decomposition”.

Problem 11.12. Solution: Following the hint we find

$$\frac{\partial}{\partial s} \left(1 - \frac{2}{\pi} \arccos \sqrt{\frac{s}{u}} \right) = \frac{2}{\pi} \frac{1}{\sqrt{1-\frac{s}{u}}} \frac{1}{2\sqrt{\frac{s}{u}}} \frac{1}{u} = \frac{1}{\pi} \frac{1}{\sqrt{1-\frac{s}{u}}} \frac{1}{\sqrt{s}\sqrt{u}} = \frac{1}{\pi} \frac{1}{\sqrt{s}\sqrt{u-s}}.$$

and so

$$\frac{\partial}{\partial u} \frac{\partial}{\partial s} \left(1 - \frac{2}{\pi} \arccos \sqrt{\frac{s}{u}} \right) = \frac{\partial}{\partial u} \left(\frac{1}{\pi} \frac{1}{\sqrt{s}\sqrt{u-s}} \right) = \frac{-1}{2\pi\sqrt{s}(u-s)^{3/2}}.$$

This is (up to the minus sign) just the density from Corollary 11.26. From Lemma 11.23 we know, however, that for $s < t < u$

$$\mathbb{P}(g_t \leq s, d_t \geq u) = \mathbb{P}(B_\bullet \text{ has no zero in } (s, u)) = 1 - \frac{2}{\pi} \arccos \sqrt{\frac{s}{u}}.$$

This proves the claim.

Problem 11.13. Solution: Notation: We write f_X for the density of the random variable X .

Using Corollary 11.26 we have, with some obvious changes in the integration variables,

$$\begin{aligned}
 \mathbb{E} \phi(L_t^-, L_t) &= \mathbb{E} \phi(t - g_t, d_t - g_t) \\
 &= \iint \phi(t - s, u - s) \mathbb{P}(g_t \in ds, d_t \in du) \\
 &= \int_{s=0}^t \int_{u=t}^{\infty} \phi(t - s, u - s) \frac{du ds}{2\pi\sqrt{s(u-s)^3}} \\
 &= \int_{r=0}^t \int_{u=t}^{\infty} \phi(r, u - t + r) \frac{du dr}{2\pi\sqrt{(t-r)(u-t+r)^3}} \\
 &= \int_{r=0}^t \int_{l=r}^{\infty} \phi(r, l) \frac{dl dr}{2\pi\sqrt{(t-r)l^3}}
 \end{aligned}$$

which shows that the joint density satisfies

$$f_{(L_t^-, L_t)}(r, l) = \frac{1}{2\pi\sqrt{(t-r)l^3}}, \quad 0 < r < t, l > r \quad \left(\iff 0 < r < t \wedge l \right)$$

Integrating out $L_t \in dl$ now yields

$$f_{L_t^-}(r) = \frac{1}{\pi\sqrt{r(t-r)}}, \quad 0 < r < t$$

(this is just the arc-sine density, rewritten for $L_t^- = t - g_t$).

Integrating out $L_t^- \in dr$ now yields

$$f_{L_t}(l) = \frac{1}{2\pi\sqrt{l^3}} \int_0^{t \wedge l} \frac{dr}{\sqrt{t-r}} = \begin{cases} \frac{\sqrt{t}}{\pi\sqrt{l^3}}, & l \in [t, \infty) \\ \frac{\sqrt{t} - \sqrt{t-l}}{\pi\sqrt{l^3}}, & l \in (0, t). \end{cases}$$

Using standard formulae for the conditional densities, we get

$$f_{L_t|L_t^-}(l | r) = \frac{f_{(L_t^-, L_t)}(r, l)}{f_{L_t^-}(r)} = \frac{1}{2} \sqrt{\frac{r}{l^3}}, \quad 0 < r < t \wedge l.$$

From this we get

$$\mathbb{P}(L_t > r + s | L_t^- = r) = \int_{r+s}^{\infty} \frac{1}{2} \sqrt{\frac{r}{l^3}} dl = \frac{1}{2} \sqrt{r} \left[-2l^{-\frac{1}{2}} \right]_{l=r+s}^{\infty} = \sqrt{\frac{r}{r+s}}.$$

■ ■

12 The growth of Brownian paths

Problem 12.1. Solution: Fix $C > 2$ and define $A_n := \{M_n > C\sqrt{n \log n}\}$. By the reflection principle we find

$$\begin{aligned} \mathbb{P}(A_n) &= \mathbb{P}\left(\sup_{s \leq n} B_s > C\sqrt{n \log n}\right) \\ &= 2 \mathbb{P}\left(B_n > C\sqrt{n \log n}\right) \\ &\stackrel{\text{scaling}}{=} 2 \mathbb{P}\left(\sqrt{n} B_1 > C\sqrt{n \log n}\right) \\ &= 2 \mathbb{P}\left(B_1 > C\sqrt{\log n}\right) \\ &\stackrel{(12.1)}{\leq} \frac{2}{\sqrt{2\pi}} \frac{1}{C\sqrt{\log n}} \exp\left(-\frac{C^2}{2} \log n\right) \\ &= \frac{2}{\sqrt{2\pi}} \frac{1}{C\sqrt{\log n}} \frac{1}{n^{C^2/2}}. \end{aligned}$$

Since $C^2/2 > 2$, the series $\sum_n \mathbb{P}(A_n)$ converges and, by the Borel–Cantelli lemma we see that

$$\exists \Omega_C \subset \Omega, \mathbb{P}(\Omega_C) = 1, \quad \forall \omega \in \Omega_C \quad \exists n_0(\omega) \quad \forall n \geq n_0(\omega) : M_n(\omega) \leq C\sqrt{n \log n}.$$

This shows that

$$\forall \omega \in \Omega_C : \overline{\lim}_{n \rightarrow \infty} \frac{M_n}{\sqrt{n \log n}} \leq C.$$

Since every t is in some interval $[n-1, n]$ and since $t \mapsto \sqrt{t \log t}$ is increasing, we see that

$$\frac{M_t}{\sqrt{t \log t}} \leq \frac{M_n}{\sqrt{(n-1) \log(n-1)}} = \frac{M_n}{\sqrt{n \log n}} \underbrace{\frac{\sqrt{n \log n}}{\sqrt{(n-1) \log(n-1)}}}_{\rightarrow 1 \text{ as } n \rightarrow \infty}$$

and the claim follows.

Remark: We can get the exceptional set in a uniform way: On the set $\Omega_0 := \bigcap_{C \geq 2} \Omega_C$ we have $\mathbb{P}(\Omega_0) = 1$ and

$$\forall \omega \in \Omega_0 : \overline{\lim}_{n \rightarrow \infty} \frac{M_n}{\sqrt{n \log n}} \leq 2.$$

■ ■

Problem 12.2. Solution: One should assume that $\xi > 0$. Since $y \mapsto \exp(\xi y)$ is monotone increasing, we see

$$\mathbb{P}\left(\sup_{s \leq t} (B_s - \frac{1}{2}\xi s) > x\right) = \mathbb{P}\left(e^{\sup_{s \leq t} (\xi B_s - \frac{1}{2}\xi^2 s)} > e^{\xi x}\right)$$

$$\stackrel{\text{Doob}}{\leq} e^{-x\xi} \mathbb{E} e^{\xi B_t - \frac{1}{2}\xi^2 t} = e^{-x\xi}. \quad (\text{A.13})$$

(Remark: we have shown (A.13) only for $\sup_{D \ni s \leq t} M_s^\xi$ where D is a dense subset of $[0, \infty)$. Since $s \mapsto M_s^\xi$ has continuous paths, it is easy to see that $\sup_{D \ni s \leq t} M_s^\xi = \sup_{s \leq t} M_s^\xi$ almost surely.)

Usage in step 1° of the Proof of Theorem 12.1: With the notation of the proof we set

$$t = q^n \quad \text{and} \quad \xi = q^{-n}(1 + \epsilon)\sqrt{2q^n \log \log q^n} \quad \text{and} \quad x = \frac{1}{2}\sqrt{2q^n \log \log q^n}.$$

Since $\sup_{s \leq t} (B_s - \frac{1}{2}\xi s) \geq \sup_{s \leq t} B_s - \frac{1}{2}\xi t$ the above inequality becomes

$$\mathbb{P}\left(\sup_{s \leq t} B_s > x + \frac{1}{2}\xi t\right) \leq e^{-x\xi}$$

and if we plug in t, x, ξ we see

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t} B_s > x + \frac{1}{2}\xi t\right) &= \mathbb{P}\left(\sup_{s \leq q^n} B_s > \frac{1}{2}\sqrt{2q^n \log \log q^n} + \frac{1}{2}(1 + \epsilon)\sqrt{2q^n \log \log q^n}\right) \\ &= \mathbb{P}\left(\sup_{s \leq q^n} B_s > (1 + \frac{\epsilon}{2})\sqrt{2q^n \log \log q^n}\right) \\ &\leq \exp\left(-\frac{1}{2}\sqrt{2q^n \log \log q^n} q^{-n}(1 + \epsilon)\sqrt{2q^n \log \log q^n}\right) \\ &= \exp\left(-(1 + \epsilon) \log \log q^n\right) \\ &= \frac{1}{(\log q^n)^{1+\epsilon}} \\ &= \frac{1}{(\log q)^{1+\epsilon}} \frac{1}{n^{1+\epsilon}}. \end{aligned}$$

Now we can argue as in the proof of Theorem 12.1. ■ ■

Problem 12.3. Solution: Actually, the hint is not needed, the present proof can be adapted in an easier way. We perform the following changes at the beginning of page 166: *Since every $t > 1$ is in some interval of the form $[q^{n-1}, q^n]$ and since the function $\Lambda(t) = \sqrt{2t \log \log t}$ is increasing for $t > 3$, we find for all $t \geq q^{n-1} > 3$*

$$\frac{|B(t)|}{\sqrt{2t \log \log t}} \leq \frac{\sup_{s \leq q^n} |B(s)|}{\sqrt{2q^n \log \log q^n}} \frac{\sqrt{2q^n \log \log q^n}}{\sqrt{2q^{n-1} \log \log q^{n-1}}}.$$

Therefore

$$\overline{\lim}_{t \rightarrow \infty} \frac{|B(t)|}{\sqrt{2t \log \log t}} \leq (1 + \epsilon)\sqrt{q} \quad \text{a.s.}$$

Letting $\epsilon \rightarrow 0$ and $q \rightarrow 1$ along countable sequences, we find the upper bound. □

Remark: The interesting paper by Dupuis [4] shows LILs for processes $(X_t)_{t \geq 0}$ with stationary and independent increments. It is shown there that the important ingredient are estimates of the type $\mathbb{P}(X_t > x)$. Thus, if we know that $\mathbb{P}(X_t > x) \asymp \mathbb{P}\left(\sup_{s \leq t} X_s > x\right)$, we get a LIL for X_t if, and only if, we have a LIL for $\sup_{s \leq t} X_s$. ■ ■

Problem 12.4. Solution: Direct calculation using (12.6). ■ ■

Problem 12.5. Solution: Denote by $W(t) = tB(1/t)$ and $W(0) = 0$ the projective reflection of $(B_t)_{t \geq 0}$. This is again a BM¹. Thus

$$\begin{aligned} \mathbb{P}(B(t) < \kappa(t) \text{ as } t \rightarrow \infty) &= \mathbb{P}(W(t) < \kappa(t) \text{ as } t \rightarrow \infty) \\ &= \mathbb{P}(tB(1/t) < \kappa(t) \text{ as } t \rightarrow \infty) \\ &= \mathbb{P}(B(1/t) < \kappa(t)/t \text{ as } t \rightarrow \infty) \\ &= \mathbb{P}(B(s) < s\kappa(1/s) \text{ as } s \rightarrow 0). \end{aligned}$$

Set $K(s) := s\kappa(1/s)$. In order to apply Kolmogorov's test we need (always $s \rightarrow 0, t = 1/s \rightarrow \infty$, \uparrow =increasing, \downarrow =decreasing) that

$$K(s) \uparrow \iff s\kappa(1/s) \uparrow \iff \kappa(t)/t \downarrow$$

and

$$K(s)/\sqrt{s} \downarrow \iff \sqrt{s}\kappa(1/s) \downarrow \iff \kappa(t)/\sqrt{t} \uparrow.$$

Finally, by a change of variables and using the integral test,

$$\int_0^1 s^{-3/2} K(s) e^{-K^2(s)/2s} ds \stackrel{s=1/t}{=} \int_1^\infty t^{-3/2} \kappa(t) e^{-\kappa^2(t)/2t} dt,$$

and the claim follows. ■ ■

Problem 12.6. Solution:

a) By the LIL for Brownian motion we find

$$\frac{B_t}{b\sqrt{a+t}} = \underbrace{\frac{B_t}{\sqrt{2t \log \log t}}}_{\overline{\lim}_{t \rightarrow \infty} (\dots) = 1} \cdot \underbrace{\frac{\sqrt{2t \log \log t}}{b\sqrt{a+t}}}_{\lim_{t \rightarrow \infty} (\dots) = \infty}$$

which shows that

$$\overline{\lim}_{t \rightarrow \infty} \frac{B_t}{b\sqrt{a+t}} = \infty$$

almost surely. Therefore, $\mathbb{P}(\tau < \infty) = 1$.

b) Let $b \geq 1$ and assume, to the contrary, that $\mathbb{E}\tau < \infty$. Then we can use the second Wald identity, cf. Theorem 5.10, and get

$$\mathbb{E}\tau = \mathbb{E}B^2(\tau) = \mathbb{E}(b^2(a + \tau)) = ab^2 + b^2 \mathbb{E}\tau > b^2 \mathbb{E}\tau \geq \mathbb{E}\tau,$$

leading to a contradiction. Thus, $\mathbb{E}\tau = \infty$.

c) Consider the stopping time $\tau \wedge n$. As in b) we get for all $b > 0$

$$\mathbb{E}(\tau \wedge n) = \mathbb{E}B^2(\tau \wedge n) \leq \mathbb{E}(b^2(a + \tau \wedge n)).$$

This gives, if $b < 1$,

$$(1 - b^2) \mathbb{E}(\tau \wedge n) \leq ab^2 \xrightarrow{b^2 < 1} \mathbb{E}(\tau \wedge n) \leq \frac{ab^2}{1 - b^2} \xrightarrow[\text{convergence}]{\text{monotone}} \mathbb{E}\tau \leq \frac{ab^2}{1 - b^2} < \infty.$$

Problem 12.7. Solution: Using symmetry we get

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t} X_s \geq x\right) &\leq \mathbb{P}\left(\sup_{s \leq t} |X_s| \geq x\right) \\ &\leq 3\mathbb{P}(|X_t| \geq x/3) \\ &= 3\mathbb{P}(X_t \geq x/3) + 3\mathbb{P}(-X_t \geq x/3) \\ &= 6\mathbb{P}(X_t \geq x/3). \end{aligned}$$

(We could also use the slightly sharper Lévy inequality which, however, holds only for symmetric Lévy processes.)

Pick $a > 1$ and set $A_k := \{\sup_{t \leq a^k} X(t) > g(a^{k+1})\}$. Using the above inequality, we get

$$\mathbb{P}(A_k) \leq 6\mathbb{P}(X(a^k) > g(a^{k+1})/3).$$

For $t \in [a^k, a^{k+1}]$ we have

$$g(t) \leq g(a^{k+1}) \quad \text{and} \quad 2\mathbb{P}(X(t) - X(a^k) \geq 0) = 2\mathbb{P}(X(t - a^k) \geq 0) \geq 1.$$

Thus,

$$\begin{aligned} \mathbb{P}(A_k) &\leq 6\mathbb{P}(X(a^k) > g(t)/3) \quad (g \text{ monotone}) \\ &\leq 12\mathbb{P}(X(a^k) > g(t)/3) \mathbb{P}(X(t) - X(a^k) \geq 0) \quad (\text{symmetry}) \\ &\leq 12\mathbb{P}(X(a^k) > g(t)/3, X(t) - X(a^k) \geq 0) \quad (\text{indep. increments}) \\ &\leq 12\mathbb{P}(X(t) \geq g(t)/3). \end{aligned}$$

Therefore,

$$\log a \mathbb{P}(A_k) = \int_{a^k}^{a^{k+1}} \frac{1}{t} \mathbb{P}(A_k) dt \leq 12 \int_{a^k}^{a^{k+1}} \frac{1}{t} \mathbb{P}(X(t) \geq g(t)/3) dt$$

and summing over $k = 1, 2, \dots$ we get

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) \leq \frac{1}{\log a} \int_1^{\infty} \frac{1}{t} \mathbb{P}(X(t) > g(t)/3) dt < \infty.$$

We can now apply (the easy direction of) the Borel–Cantelli lemma to see that a.s. only finitely many of the A_k happen.

$$\mathbb{P}\left(\lim_{k \rightarrow \infty} \frac{1}{g(a^{k+1})} \sup_{a^{k-1} \leq t \leq a^k} X(t) \leq 1\right) = 1.$$

Now we use the growth assumption on g : For $t \in [a^{k-1}, a^k]$ and $k \geq n_0(\omega)$ (this number comes from the limsup or the Borel–Cantelli lemma) we have

$$\frac{X(t)}{\kappa_2(a^2)g(t)} \leq \frac{1}{\kappa_2(a^2)g(a^{k-1})} \sup_{a^{k-1} \leq t \leq a^k} X(t) \leq \frac{1}{g(a^{k+1})} \sup_{a^{k-1} \leq t \leq a^k} X(t).$$

Since $a > 1$ is arbitrary and $\kappa_2(a^2) \rightarrow 1$ as $a \rightarrow 1$ we see that $g(t)$ is an upper function for $X(t)$.

Remark: With some minor variations, this proof still works for $|X(t)|$. The main change is needed in the step where we use that $2\mathbb{P}(X(t) - X(a^k) \geq 0) \geq 1$. This must be aptly replaced by an Ottaviani–Skorokhod type bound. See [9, p. 295] for details.

■ ■

13 Strassen's functional law of the iterated logarithm

Problem 13.1. Solution: We construct a counterexample.

The function $w(t) = \sqrt{t}$, $0 \leq t \leq 1$, is a limit point of the family

$$Z_s(t) = \frac{B(st)}{\sqrt{2s \log \log s}}$$

where $t > 0$ is fixed and for $s \rightarrow \infty$.

By the Khintchine's LIL (cf. Theorem 11.1) we obtain

$$\overline{\lim}_{s \rightarrow \infty} \frac{B(st)}{\sqrt{2st \log \log(st)}} = 1 \quad (\text{almost surely } \mathbb{P})$$

and so

$$\overline{\lim}_{s \rightarrow \infty} \frac{B(st)}{\sqrt{2s \log \log(st)}} = \sqrt{t} \quad (\text{almost surely } \mathbb{P})$$

which implies

$$\overline{\lim}_{s \rightarrow \infty} \frac{B(st)}{\sqrt{2s \log \log s}} = \overline{\lim}_{s \rightarrow \infty} \frac{B(st)}{\sqrt{2s \log \log(st)}} \cdot \underbrace{\sqrt{\frac{\log \log(st)}{\log \log s}}}_{\rightarrow 1 \text{ for } s \rightarrow \infty} = \sqrt{t}.$$

On the other hand, the function $w(t) = \sqrt{t}$ cannot be a limit point of $Z_s(\cdot)$ in $\mathcal{C}_{(0)}[0, 1]$ for $s \rightarrow \infty$. We prove this indirectly: Let $s_n = s_n(\omega)$ be a sequence, such that $\lim_{n \rightarrow \infty} s_n = \infty$. Then

$$\|Z_{s_n}(\cdot) - w(\cdot)\|_{\infty} \xrightarrow{n \rightarrow \infty} 0$$

implies that for every $\epsilon > 0$ the inequality

$$(\sqrt{t} - \epsilon) \cdot \sqrt{2s_n \log \log s_n} \leq B(s_n \cdot t) \leq (\sqrt{t} + \epsilon) \sqrt{2s_n \log \log s_n} \quad (*)$$

holds for all sufficiently large n and every $t \in [0, 1]$. This, however, contradicts

$$(1 - \epsilon) \sqrt{2t_k \log \left(\log \frac{1}{t_k} \right)} \leq B(t_k) \leq (1 + \epsilon) \sqrt{2t_k \log \left(\log \frac{1}{t_k} \right)}, \quad (**)$$

for a sequence $t_k = t_k(\omega) \rightarrow 0$, $k \rightarrow \infty$, cf. Corollary 12.2.

Indeed: fix some n , then the right side of (*) is in contradiction with the left side of (**).

Remark: Note that

$$\int_0^1 w'(s)^2 ds = \frac{1}{4} \int_0^1 \frac{ds}{s} = +\infty.$$

■ ■

Problem 13.2. Solution: For any $w \in \mathcal{K}$ we have

$$|w(t)|^2 = \left| \int_0^t w'(s) ds \right|^2 \leq \int_0^t w'(s)^2 ds \cdot \int_0^t 1 ds \leq \int_0^1 w'(s)^2 ds \cdot t \leq t.$$

■ ■

Problem 13.3. Solution: Since u is absolutely continuous (w.r.t. Lebesgue measure), for almost all $t \in [0, 1]$, the derivative $u'(t)$ exists almost everywhere.

Let t be a point where u' exists and let $(\Pi_n)_{n \geq 1}$ be a sequence of partitions of $[0, 1]$ such that $|\Pi_n| \rightarrow 0$ as $n \rightarrow \infty$. We denote the points in Π_n by $t_k^{(n)}$. Clearly, there exists a sequence $(t_{j_n}^{(n)})_{n \geq 1}$ such that $t_{j_n}^{(n)} \in \Pi_n$ and $t_{j_{n-1}}^{(n)} \leq t \leq t_{j_n}^{(n)}$ for all $n \in \mathbb{N}$ and $t_{j_n}^{(n)} - t_{j_{n-1}}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. We obtain

$$f_n(t) = \left[\frac{1}{t_{j_n}^{(n)} - t_{j_{n-1}}^{(n)}} \int_{t_{j_{n-1}}^{(n)}}^{t_{j_n}^{(n)}} u'(s) ds \right]^2$$

to simplify notation, we set $t_j := t_{j_n}^{(n)}$ and $t_{j-1} := t_{j_{n-1}}^{(n)}$, then

$$\begin{aligned} &= \left[\frac{1}{t_j - t_{j-1}} \cdot (u(t_j) - u(t_{j-1})) \right]^2 \\ &= \left[\frac{1}{t_j - t_{j-1}} \cdot (u(t_j) - u(t) + u(t) - u(t_{j-1})) \right]^2 \\ &= \left[\frac{t_j - t}{t_j - t_{j-1}} \cdot \underbrace{\frac{u(t_j) - u(t)}{t_j - t}}_{\rightarrow u'(t)} + \frac{t - t_{j-1}}{t_j - t_{j-1}} \cdot \underbrace{\frac{u(t) - u(t_{j-1})}{t - t_{j-1}}}_{\rightarrow u'(t)} \right]^2 \\ &\xrightarrow{n \rightarrow \infty} [u'(t)]^2. \end{aligned}$$

■ ■

Problem 13.4. Solution: We use the notation of Chapter 4: $\Omega = \mathcal{C}_{(o)}[0, 1]$, $w = \omega$, $\mathcal{A} = \mathcal{B}(\mathcal{C}_{(o)}[0, 1])$, $\mathbb{P} = \mu$, $B(t, \omega) = B_t(\omega) = w(t)$, $t \in [0, \infty)$.

Linearity of G^ϕ is clear. Let Π_n , $n \geq 1$, be a sequence of partitions of $[0, 1]$ such that $\lim_{n \rightarrow \infty} |\Pi_n| = 0$,

$$\Pi_n = \left\{ s_k^{(n)} : 0 = s_0^{(n)} < s_1^{(n)} < \dots < s_{l_n}^{(n)} = 1 \right\};$$

by $\tilde{s}_k^{(n)}$, $k = 1, \dots, l_n$ we denote arbitrary intermediate points, i.e. $s_{k-1}^{(n)} \leq \tilde{s}_k^{(n)} \leq s_k^{(n)}$ for all k . Then we have

$$\begin{aligned} G^\phi(\omega) &= \phi(1)B_1(\omega) - \int_0^1 B_s(\omega) d\phi(s) \\ &= \phi(1)B_1(\omega) - \lim_{|\Pi_n| \rightarrow 0} \sum_{k=1}^{l_n} B_{\tilde{s}_k^{(n)}}(\omega) (\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)})). \end{aligned}$$

Write

$$G_n^\phi := \phi(1)B_1 - \sum_{k=1}^{l_n} B_{\tilde{s}_k^{(n)}}(\omega) (\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)}))$$

$$= \sum_{k=1}^{l_n} (B_1 - B_{\tilde{s}_k^{(n)}}) (\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)})) + B_1 \phi(0).$$

Then $G^\phi(\omega) = \lim_{n \rightarrow \infty} G_n^\phi(\omega)$ for all $\omega \in \Omega$. Moreover, the elementary identity

$$\sum_{k=1}^l a_k (b_k - b_{k-1}) = \sum_{k=1}^{l-1} (a_k - a_{k+1}) b_k + a_l b_l - a_1 b_0$$

implies

$$\begin{aligned} G_n^\phi &= \sum_{k=1}^{l_n-1} (B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}}) \phi(s_k^{(n)}) + (B_1 - B_{\tilde{s}_{l_n}^{(n)}}) \phi(1) - (B_1 - B_{\tilde{s}_1^{(n)}}) \phi(0) + B_1 \phi(0) \\ &= \sum_{k=0}^{l_n} (B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}}) \phi(s_k^{(n)}) + B_{\tilde{s}_1^{(n)}} \phi(0), \end{aligned}$$

where $\tilde{s}_{l_n+1}^{(n)} := 1$, $\tilde{s}_0^{(n)} := 0$.

a) G_n^ϕ is a Gaussian random variable with mean $\mathbb{E} G_n^\phi = 0$ and variance

$$\begin{aligned} \mathbb{V} G_n^\phi &= \sum_{k=0}^{l_n} \phi^2(s_k^{(n)}) \mathbb{V} (B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}}) + \phi^2(0) \mathbb{V} B_{\tilde{s}_1^{(n)}} \\ &= \sum_{k=0}^{l_n} \phi^2(s_k^{(n)}) (\tilde{s}_{k+1}^{(n)} - \tilde{s}_k^{(n)}) + \phi^2(0) \tilde{s}_1^{(n)} \\ &\xrightarrow{n \rightarrow \infty} \int_0^1 \phi^2(s) ds. \end{aligned}$$

This and $\lim_{n \rightarrow \infty} G_n^\phi = G^\phi$ (\mathbb{P} -a.s.) imply that G^ϕ is a Gaussian random variable with $\mathbb{E} G^\phi = 0$ and $\mathbb{V} G^\phi = \int_0^1 \phi^2(s) ds$.

b) Without loss of generality we use for ϕ and ψ the same sequence of partitions.

Clearly, $G_n^\phi \cdot G_n^\psi \rightarrow G^\phi \cdot G^\psi$ for $n \rightarrow \infty$ (\mathbb{P} -a.s.) Using the elementary inequality $2ab \leq a^2 + b^2$ and the fact that for a Gaussian random variable $\mathbb{E}(G^4) = 3(\mathbb{E}(G^2))^2$, we get

$$\begin{aligned} \mathbb{E} ((G_n^\phi G_n^\psi)^2) &\leq \frac{1}{2} [\mathbb{E} ((G_n^\phi)^4) + \mathbb{E} ((G_n^\psi)^4)] \\ &= \frac{3}{2} [(\mathbb{E} (G_n^\phi)^2)^2 + (\mathbb{E} (G_n^\psi)^2)^2] \\ &\leq \frac{3}{2} \left[\left(\int_0^1 \phi^2(s) ds \right)^2 + \left(\int_0^1 \psi^2(s) ds \right)^2 \right] + \epsilon \quad (n \geq n_\epsilon). \end{aligned}$$

This implies

$$\mathbb{E} (G_n^\phi G_n^\psi) \xrightarrow{n \rightarrow \infty} \mathbb{E} (G^\phi G^\psi).$$

Moreover,

$$\begin{aligned} \mathbb{E} (G_n^\phi G_n^\psi) &= \mathbb{E} \left[\left(\sum_{k=0}^{l_n} (B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}}) \phi(s_k^{(n)}) \right) \cdot \left(\sum_{j=0}^{l_n} (B_{\tilde{s}_{j+1}^{(n)}} - B_{\tilde{s}_j^{(n)}}) \psi(s_j^{(n)}) \right) \right] \\ &\quad + \phi(0) \psi(0) \mathbb{E} (B_{\tilde{s}_1^{(n)}}^2) + \phi(0) \mathbb{E} \left[B_{\tilde{s}_1^{(n)}} \sum_{j=0}^{l_n} (B_{\tilde{s}_{j+1}^{(n)}} - B_{\tilde{s}_j^{(n)}}) \psi(s_j^{(n)}) \right] \end{aligned}$$

$$\begin{aligned}
 & + \psi(0) \mathbb{E} \left[B_{\bar{s}_1^{(n)}} \sum_{k=0}^{l_n} (B_{\bar{s}_{k+1}^{(n)}} - B_{\bar{s}_k^{(n)}}) \psi(s_k^{(n)}) \right] \\
 & = \sum_{k=0}^{l_n} \underbrace{\mathbb{E} \left((B_{\bar{s}_{k+1}^{(n)}} - B_{\bar{s}_k^{(n)}})^2 \right)}_{= \bar{s}_{k+1}^{(n)} - \bar{s}_k^{(n)}} \phi(s_k^{(n)}) \psi(s_k^{(n)}) + \dots \\
 & \xrightarrow{n \rightarrow \infty} \int_0^1 \phi(s) \psi(s) ds.
 \end{aligned}$$

This proves

$$\mathbb{E}(G^\phi G^\psi) = \int_0^1 \phi(s) \psi(s) ds.$$

c) Using a) and b) we see

$$\begin{aligned}
 \mathbb{E}[(G_n^\phi - G_n^\psi)^2] &= \mathbb{E}[(G_n^\phi)^2] - 2\mathbb{E}[G_n^\phi G_n^\psi] + \mathbb{E}[(G_n^\psi)^2] \\
 &= \int_0^1 \phi_n^2(s) ds - 2 \int_0^1 \phi_n(s) \psi_n(s) ds + \int_0^1 \psi_n^2(s) ds \\
 &= \int_0^1 (\phi_n(s) - \psi_n(s))^2 ds.
 \end{aligned}$$

This and $\phi_n \rightarrow \phi$ in L^2 imply that $(G_n^\phi)_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Consequently, the limit $X = \lim_{n \rightarrow \infty} G_n^\phi$ exists in L^2 . Moreover, as $\phi_n \rightarrow \phi$ in L^2 , we also obtain that $\int_0^1 \phi_n^2(s) ds \rightarrow \int_0^1 \phi^2(s) ds$.

Since G_n^ϕ is a Gaussian random variable with mean 0 and variance $\int_0^1 \phi_n^2(s) ds$, we see that G^ϕ is Gaussian with mean 0 and variance $\int_0^1 \phi^2(s) ds$.

Finally, we have $\phi_n \rightarrow \phi$ and $\psi_n \rightarrow \psi$ in $L_2([0, 1])$ implying

$$\mathbb{E}(G_n^\phi G_n^\psi) \rightarrow \mathbb{E}(G^\phi G^\psi)$$

—see part b)—and

$$\int_0^1 \phi_n(s) \psi_n(s) ds \rightarrow \int_0^1 \phi(s) \psi(s) ds.$$

Thus,

$$\mathbb{E}(G^\phi G^\psi) = \int_0^1 \phi(s) \psi(s) ds.$$

Problem 13.5. Solution:

a) It is clear that \mathcal{H}^1 is a normed vector space with a scalar product. (The definiteness of the norm in \mathcal{H}^1 follows from the absolute continuity! Note that $h'(s) ds = dh(s)$ in the scalar product since $h \in \mathcal{H}^1$ is absolutely continuous.) Let us show that \mathcal{H}^1 is closed. Assume that $(u_n)_{n \geq 1} \subset \mathcal{H}^1$ converges. This means that $u'_n \xrightarrow[n \rightarrow \infty]{L^2(ds)} w$ where $w \in L^2(ds)$. Thus,

$$W(t) := \int_0^t w(s) ds \quad \text{exists and} \quad W \in \mathcal{H}^1, \quad W'(t) = w(t).$$

Moreover, by construction $u_n \xrightarrow[n \rightarrow \infty]{\mathcal{H}^1} W$.

- b) See the last part of Paragraph 13.5.
 c) With the Cauchy–Schwarz inequality we see

$$\begin{aligned} \langle \phi, h \rangle_{\mathcal{H}^1} &\leq \|\phi\|_{\mathcal{H}^1} \|h\|_{\mathcal{H}^1} \\ \implies \left\langle \frac{\phi}{\|\phi\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1} &\leq \|h\|_{\mathcal{H}^1} \\ \implies \sup_{\phi \in \mathcal{H}_\circ^1, \|\phi\|_{\mathcal{H}^1}=1} \langle \phi, h \rangle_{\mathcal{H}^1} &\leq \sup_{\phi \in \mathcal{H}^1, \|\phi\|_{\mathcal{H}^1}=1} \langle \phi, h \rangle_{\mathcal{H}^1} \leq \|h\|_{\mathcal{H}^1}. \end{aligned}$$

Conversely, the supremum is attained if we take $\phi = h / \|h\|_{\mathcal{H}^1}$. Thus,

$$\|h\|_{\mathcal{H}^1} = \left\langle \frac{h}{\|h\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1} \leq \sup_{\phi \in \mathcal{H}^1} \left\langle \frac{\phi}{\|\phi\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1}.$$

If we approximate $h \in \mathcal{H}^1$ by a sequence $(\phi_n)_{n \geq 1} \subset \mathcal{H}_\circ^1$, we get

$$\|h\|_{\mathcal{H}^1} = \left\langle \frac{h}{\|h\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1} = \lim_{n \rightarrow \infty} \left\langle \frac{\phi_n}{\|\phi_n\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1} \leq \sup_{\phi \in \mathcal{H}_\circ^1} \left\langle \frac{\phi}{\|\phi\|_{\mathcal{H}^1}}, h \right\rangle_{\mathcal{H}^1}.$$

- d) Assume that there is some $(\phi_n)_{n \geq 1} \subset \mathcal{H}_\circ^1$ with $\|\phi_n\|_{\mathcal{H}^1} = 1$ and $\langle \phi_n, h \rangle_{\mathcal{H}^1} \geq 2n$. Then

$$\|h\|_{\mathcal{H}^1} = \sup_{\|\phi\|_{\mathcal{H}^1}=1} \langle h, \phi \rangle_{\mathcal{H}^1} \geq \sup_{n \geq 1} \langle \phi_n, h \rangle_{\mathcal{H}^1} = \infty$$

which means that $h \notin \mathcal{H}^1$.

Conversely, assume that for every sequence $(\phi_n)_{n \geq 1} \subset \mathcal{H}_\circ^1$ with $\|\phi_n\|_{\mathcal{H}^1} = 1$ we have $\langle \phi_n, h \rangle_{\mathcal{H}^1} \leq C$. (Think! Why this is the proper negation of the condition in the problem?) Since the supremum can be realized by a sequence, we get for a suitable sequence of ϕ_n 's

$$\|h\|_{\mathcal{H}^1} = \sup_{\|\phi\|_{\mathcal{H}^1}=1, \phi \in \mathcal{H}_\circ^1} \langle h, \phi \rangle_{\mathcal{H}^1} = \lim_{n \rightarrow \infty} \langle \phi_n, h \rangle_{\mathcal{H}^1} \leq C.$$

This means that $h \in \mathcal{H}^1$.

Remark. An alternative, and more elementary argument for part (d) can be based on step functions and Lemmas 13.2 and 13.3. ■■

Problem 13.6. Solution: The vectors (X, Y) in a) – d) are a.s. limits of two-dimensional Gaussian distributions. Therefore, they are also Gaussian. Their mean is clearly 0. The general density of a two-dimensional Gaussian law (with mean zero) is given by

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} - \frac{2\rho xy}{\sigma_1\sigma_2} \right) \right\}.$$

In order to solve the problems we have to determine the variances $\sigma_1^2 = \mathbb{V} X$, $\sigma_2^2 = \mathbb{V} Y$ and the correlation coefficient $\rho = \frac{\mathbb{E} XY}{\sigma_1\sigma_2}$. We will use the results of Problem 13.4.

$$\begin{aligned} \text{a) } \sigma_1^2 &= \mathbb{V} \left(\int_{1/2}^t s^2 dw(s) \right) = \int_0^1 \mathbb{1}_{[1/2, t]}(s) s^4 ds = \frac{1}{5} \left(t^5 - \frac{1}{32} \right), \\ \sigma_2^2 &= \mathbb{V} w(1/2) = 1/2 \quad (= \mathbb{V} B_{1/2} \text{ cf. canonical model}), \\ \mathbb{E} \left(\int_{1/2}^t s^2 dw(s) \cdot w(1/2) \right) &= \int_0^1 \mathbb{1}_{[1/2, t]}(s) s^2 \cdot \mathbb{1}_{[0, 1/2]}(s) ds = 0 \\ &\implies \rho = 0. \end{aligned}$$

$$\begin{aligned} \text{b) } \sigma_1^2 &= \frac{1}{5} \left(t^5 - \frac{1}{32} \right) \\ \sigma_2^2 &= \mathbb{V} w(u + 1/2) = u + 1/2 \\ \mathbb{E} \left(\int_{1/2}^t s^2 dw(s) \cdot w(u + 1/2) \right) &= \int_0^1 \mathbb{1}_{[1/2, t]}(s) s^2 \cdot \mathbb{1}_{[0, u+1/2]}(s) ds \\ &= \int_{1/2}^{(1/2+u) \wedge t} s^2 ds \\ &= \frac{1}{3} \left(\left(\left(\frac{1}{2} + u \right) \wedge t \right)^3 - \frac{1}{8} \right). \\ &\implies \rho = \frac{\frac{1}{3} \left(\left(\left(\frac{1}{2} + u \right) \wedge t \right)^3 - \frac{1}{8} \right)}{\left[\frac{1}{5} \left(t^5 - \frac{1}{32} \right) \cdot \left(u + \frac{1}{2} \right) \right]^{1/2}}. \end{aligned}$$

$$\begin{aligned} \text{c) } \sigma_1^2 &= \mathbb{V} \left(\int_{1/2}^t s^2 dw(s) \right) = \frac{1}{5} \left(t^5 - \frac{1}{32} \right), \\ \sigma_2^2 &= \mathbb{V} \left(\int_{1/2}^t s dw(s) \right) = \frac{1}{3} \left(t^3 - \frac{1}{8} \right) \\ \mathbb{E} \left(\int_{1/2}^t s^2 dw(s) \cdot \int_{1/2}^t s dw(s) \right) &= \int_{1/2}^t s^3 ds = \frac{1}{4} \left(t^4 - \frac{1}{16} \right) \\ &\implies \rho = \frac{\frac{1}{4} \left(t^4 - \frac{1}{16} \right)}{\left[\frac{1}{5} \left(t^5 - \frac{1}{32} \right) \cdot \frac{1}{3} \left(t^3 - \frac{1}{8} \right) \right]^{1/2}}. \end{aligned}$$

$$\begin{aligned} \text{d) } \sigma_1^2 &= \mathbb{V} \left(\int_{1/2}^1 e^s dw(s) \right) = \int_{1/2}^1 e^{2s} ds = \frac{1}{2} (e^2 - e), \\ \sigma_2^2 &= \mathbb{V} (w(1) - w(1/2)) = 1/2, \\ \mathbb{E} \left(\int_{1/2}^1 e^s dw(s) \cdot (w(1) - w(1/2)) \right) &= \int_{1/2}^1 e^s \cdot 1 ds = e - e^{1/2}. \\ &\implies \rho = \frac{e - e^{1/2}}{\left(\frac{1}{4} (e^2 - e) \right)^{1/2}}. \end{aligned}$$

■ ■

Problem 13.7. Solution: Let $w_n \in F$, $n \geq 1$, and $w_n \rightarrow v$ in $\mathcal{C}_{(o)}[0, 1]$. We have to show that $v \in F$.

Now:

$$w_n \in F \implies \exists (c_n, r_n) \in [q^{-1}, 1] \times [0, 1] : |w_n(c_n r_n) - w_n(r_n)| \geq 1.$$

Observe that the function $(c, r) \mapsto w(cr) - w(r)$ with $(c, r) \in [q^{-1}, 1] \times [0, 1]$ is continuous for every $w \in \mathcal{C}_{(o)}[0, 1]$.

Since $[q^{-1}, 1] \times [0, 1]$ is compact, there exists a subsequence $(n_k)_{k \geq 1}$ such that $c_{n_k} \rightarrow \tilde{c}$ and $r_{n_k} \rightarrow \tilde{r}$ as $k \rightarrow \infty$ and $(\tilde{c}, \tilde{r}) \in [q^{-1}, 1] \times [0, 1]$.

By assumption, $w_{n_k} \rightarrow v$ uniformly and this implies

$$w_{n_k}(c_{n_k} r_{n_k}) \rightarrow v(\tilde{c}\tilde{r}) \quad \text{and} \quad w_{n_k}(r_{n_k}) \rightarrow v(\tilde{r}).$$

Finally,

$$|v(\tilde{c}\tilde{r}) - v(\tilde{r})| = \lim_{k \rightarrow \infty} |w_{n_k}(c_{n_k} r_{n_k}) - w_{n_k}(r_{n_k})| \geq 1,$$

and $v \in F$ follows. ■ ■

Problem 13.8. Solution: Set $L(t) = \sqrt{2t \log \log t}$, $t \geq e$ and $s_n = q^n$, $n \in \mathbb{N}$, $q > 1$. Then:

a) for the first inequality:

$$\begin{aligned} \mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4}\right) &= \mathbb{P}\left(\underbrace{\left|\frac{B(s_{n-1})}{\sqrt{s_{n-1}}}\right|}_{\sim N(0,1)} \cdot \frac{1}{\sqrt{2q \log \log s_n}} > \frac{\epsilon}{4}\right) \\ &= \mathbb{P}\left(|B(1)| > \frac{\epsilon}{4} \sqrt{2q \log \log q^n}\right) \end{aligned}$$

using Problem 9.6 and $\mathbb{P}(|Z| > x) = 2\mathbb{P}(Z > x)$ for $x \geq 0$

$$\begin{aligned} &\leq \sqrt{\frac{2}{\pi}} \frac{4}{\epsilon \sqrt{2q \log \log q^n}} \cdot \exp\left\{-\frac{\epsilon^2}{32} \cdot 2q \log \log q^n\right\} \\ &\leq \frac{C}{n^2} \end{aligned}$$

if q is sufficiently large.

b) for the second inequality:

$$\begin{aligned} \sup_{t \leq q^{-1}} |w(t)| &= \sup_{t \leq q^{-1}} \left| \int_0^t w'(s) ds \right| \\ &\leq \int_0^{1/q} |w'(s)| ds \\ &\leq \left[\int_0^{1/q} w'(s)^2 ds \cdot \int_0^{1/q} ds \right]^{1/2} \\ &\leq \left[\int_0^1 w'(s)^2 ds \cdot \frac{1}{q} \right]^{1/2} \\ &\leq \sqrt{\frac{2r}{q}} < \frac{\epsilon}{4} \end{aligned}$$

for all sufficiently large q .

c) for the third inequality: Brownian scaling $\frac{B(\cdot s_n)}{\sqrt{s_n}} \sim B(\cdot)$ yields

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq q^{-1}} \frac{|B(ts_n)|}{\sqrt{2s_n \log \log s_n}} > \frac{\epsilon}{4}\right) &= \mathbb{P}\left(\sup_{0 \leq t \leq q^{-1}} \frac{|B(t)|}{\sqrt{2 \log \log s_n}} > \frac{\epsilon}{4}\right) \\ &= \mathbb{P}\left(\sup_{0 \leq t \leq q^{-1}} |B(t)| > \frac{\epsilon}{4} \sqrt{2 \log \log s_n}\right) \\ &\stackrel{(*)}{\leq} 2 \mathbb{P}\left(|B(1/q)| > \frac{\epsilon}{4} \sqrt{2 \log \log s_n}\right) \\ &= 2 \mathbb{P}\left(\frac{|B(1/q)|}{\sqrt{1/q}} > \frac{\epsilon}{4} \sqrt{2q \log \log q^n}\right) \\ &\leq \frac{C}{n^2} \end{aligned}$$

for all q sufficiently large. In the estimate marked with $(*)$ we used

$$\mathbb{P}\left(\sup_{0 \leq t \leq t_0} |B(t)| > x\right) \leq 2 \mathbb{P}\left(\sup_{0 \leq t \leq t_0} B(t) > x\right) \stackrel{\text{Thm. 6.10}}{=} 2 \mathbb{P}(M(t_0) > x) = 2 \mathbb{P}(|B(t_0)| > x).$$

d) for the last inequality:

$$\begin{aligned} &\mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} + \sup_{t \leq q^{-1}} |w(t)| + \sup_{0 \leq t \leq q^{-1}} \frac{|B(ts_n)|}{L(s_n)} > \frac{3\epsilon}{4}\right) \\ &\leq \mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4} \text{ or } \sup_{t \leq q^{-1}} |w(t)| > \frac{\epsilon}{4} \text{ or } \sup_{0 \leq t \leq q^{-1}} \frac{|B(ts_n)|}{L(s_n)} > \frac{\epsilon}{4}\right) \\ &\leq \mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4}\right) + \mathbb{P}\left(\sup_{t \leq q^{-1}} |w(t)| > \frac{\epsilon}{4}\right) + \mathbb{P}\left(\sup_{0 \leq t \leq q^{-1}} \frac{|B(ts_n)|}{L(s_n)} > \frac{\epsilon}{4}\right) \\ &\leq \frac{C}{n^2} + 0 + \frac{C}{n^2} \end{aligned}$$

for all sufficiently large q . Using the Borel–Cantelli lemma we see that

$$\overline{\lim}_{n \rightarrow \infty} \left(\frac{|B(s_{n-1})|}{L(s_n)} + \sup_{t \leq q^{-1}} |w(t)| + \sup_{0 \leq t \leq q^{-1}} \frac{|B(ts_n)|}{L(s_n)}\right) \leq \frac{3}{4}\epsilon.$$

■ ■

14 Skorokhod representation

Problem 14.1. Solution: Clearly, $\mathcal{F}_t^B := \sigma(B_r : r \leq t) \subset \sigma(B_r : r \leq t, U, V) = \mathcal{F}_t$. It remains to show that $B_t - B_s \perp \mathcal{F}_s$ for all $s \leq t$. Let A, A', C be Borel sets in \mathbb{R}^d . Then we find for $F \in \mathcal{F}_s^B$

$$\begin{aligned} & \mathbb{P}(\{B_t - B_s \in C\} \cap F \cap \{U \in A\} \cap \{V \in A'\}) \\ &= \mathbb{P}(\{B_t - B_s \in C\} \cap F) \cdot \mathbb{P}(\{U \in A\} \cap \{V \in A'\}) \quad (\text{since } U, V \perp \mathcal{F}_s^B) \\ &= \mathbb{P}(\{B_t - B_s \in C\}) \cdot \mathbb{P}(F) \cdot \mathbb{P}(\{U \in A\} \cap \{V \in A'\}) \quad (\text{since } B_t - B_s \perp \mathcal{F}_s^B) \\ &= \mathbb{P}(\{B_t - B_s \in C\}) \cdot \mathbb{P}(F \cap \{U \in A\} \cap \{V \in A'\}) \quad (\text{since } U, V \perp \mathcal{F}_s^B) \end{aligned}$$

and this shows that $B_t - B_s$ is independent of the family $\mathcal{E}_s = \{F \cap G : F \in \mathcal{F}_s^B, G \in \sigma(U, V)\}$. This family is stable under finite intersections, so $B_t - B_s \perp \sigma(\mathcal{E}_s) = \mathcal{F}_s$. ■

Problem 14.2. Solution: Following the hint we use the martingale $M_t = B_t^4 - 6tB_t^2 + 3t^2$, $t \geq 0$, see Problem 5.6. Since τ is an a.s. finite stopping time, $\tau \wedge n$, $n \geq 1$, are bounded stopping times such that $\tau \wedge n \uparrow \tau$. Using optional stopping we get

$$0 = \mathbb{E} M_{\tau \wedge n} = \mathbb{E}(B_{\tau \wedge n}^4) - 6 \mathbb{E}((\tau \wedge n)B_{\tau \wedge n}^2) + 3 \mathbb{E}((\tau \wedge n)^2).$$

We can rearrange this identity and use the Cauchy–Schwarz inequality to get

$$\begin{aligned} \mathbb{E}(B_{\tau \wedge n}^4) + 3 \mathbb{E}((\tau \wedge n)^2) &= 6 \mathbb{E}((\tau \wedge n)B_{\tau \wedge n}^2) \\ &\leq 6 \sqrt{\mathbb{E}((\tau \wedge n)^2)} \sqrt{\mathbb{E}(B_{\tau \wedge n}^4)}. \end{aligned}$$

Divide by $\mathbb{E}(B_{\tau \wedge n}^4)$ and set $x^2 = \mathbb{E}((\tau \wedge n)^2) / \mathbb{E}(B_{\tau \wedge n}^4)$ (this is the quantity we want to bound). We get

$$1 + 3x^2 \leq 6x \implies 3(x-1)^2 \leq 2 \implies x \leq 1 + \sqrt{\frac{2}{3}} < 2.$$

This means that

$$\mathbb{E}((\tau \wedge n)^2) \leq 4 \mathbb{E}(B_{\tau \wedge n}^4).$$

Now we can use monotone convergence/Beppo Levi on the left-hand side and dominated convergence on the right-hand side to finish the proof.

As in Theorem 14.2 we have

$$4 \int x^4 dF(x) = 4 \mathbb{E}[B^4(\tau_{(U,W)^c}^\circ)] = \iint 4 \mathbb{E}[B^4(\tau_{(u,w)^c}^\circ)] \nu(du, dw)$$

$$\geq \iint \mathbb{E} \left[\left(\tau_{(u,w)^c}^\circ \right)^2 \right] \nu(du, dw) = \mathbb{E} \left[\left(\tau_{(U,W)^c}^\circ \right)^2 \right].$$

Since, under the integral, u, w are fixed, the stopping time $\tau = \tau_{(u,w)^c}^\circ$ satisfies our assumptions from above as $B_{t \wedge \tau}$ is bounded by $\max(|u|, |w|)$ (it stays in the interval (u, w)). Note that these calculations include the case that one or both sides are infinite! We may lose finiteness in the ν -integral. Of course, if the left-hand side $\mathbb{E} X^4$ is finite, then so is $\mathbb{E}(\tau^2)$.

Problem 14.3. Solution:

- a) We use Mark Kac's characterization of independence using characteristic functions, see e.g. [21, Corollary 27.8]. For all $\xi, \eta, \theta \in \mathbb{R}$ we have

$$\begin{aligned} & \mathbb{E} e^{i\xi B + i\eta H + i\theta T} \stackrel{T \perp (B, H)}{=} \mathbb{E} e^{i\xi B + i\eta H} \mathbb{E} e^{i\theta T} \\ & \stackrel{B \perp H}{=} \mathbb{E} e^{i\xi B} \mathbb{E} e^{i\eta H} \mathbb{E} e^{i\theta T} \stackrel{T \perp (B, H)}{=} \mathbb{E} e^{i\xi B + i\theta T} \mathbb{E} e^{i\eta H} \end{aligned}$$

and we conclude that $(B, T) \perp H$.

- b) Without loss of generality we can assume that H is a set (otherwise we would use $\{H \leq h\}$ in the following calculation). We have for all $\xi, \theta \in \mathbb{R}$

$$\begin{aligned} & \mathbb{P}(H) \mathbb{E} \left(e^{i\xi B + i\theta T} \mid H \right) = \mathbb{E} \left(e^{i\xi B} e^{i\theta T} \mathbf{1}_H \right) \stackrel{B \perp (H, T)}{=} \mathbb{E} \left(e^{i\xi B} \right) \mathbb{E} \left(e^{i\theta T} \mathbf{1}_H \right) \\ & = \frac{\mathbb{P}(H) \mathbb{E} \left(e^{i\xi B} \right)}{\mathbb{P}(H)} \mathbb{E} \left(e^{i\theta T} \mathbf{1}_H \right) \stackrel{B \perp (H, T)}{=} \frac{\mathbb{E} \left(e^{i\xi B} \mathbf{1}_H \right)}{\mathbb{P}(H)} \mathbb{E} \left(e^{i\theta T} \mathbf{1}_H \right) = \mathbb{E} \left(e^{i\xi B} \mid H \right) \mathbb{E} \left(e^{i\theta T} \mathbf{1}_H \right) \end{aligned}$$

and dividing by $\mathbb{P}(H)$ proves

$$\mathbb{E} \left(e^{i\xi B + i\theta T} \mid H \right) = \mathbb{E} \left(e^{i\xi B} \mid H \right) \mathbb{E} \left(e^{i\theta T} \mid H \right)$$

which is what is claimed.

15 Stochastic integrals: L^2 -theory

Problem 15.1. Solution: By definition of the angle bracket,

$$M^2 - \langle M \rangle \quad \text{and} \quad N^2 - \langle N \rangle$$

are martingales. Moreover, $M \pm N$ are L^2 martingales, i.e.

$$(M + N)^2 - \langle M + N \rangle \quad \text{and} \quad (M - N)^2 - \langle M - N \rangle$$

are martingales. So, we subtract them to get a new martingale:

$$(M + N)^2 - (M - N)^2 = 4MN \quad \text{and} \quad \langle M + N \rangle - \langle M - N \rangle \stackrel{\text{def}}{=} 4\langle M, N \rangle$$

which shows that $4MN - 4\langle MN \rangle$ is a martingale.

Problem 15.2. Solution: Let $\lambda \in \mathbb{R}$. Clearly, $\lambda M + N$ is again an L^2 martingale, and we have by bilinearity

$$0 \leq \langle \lambda M + N \rangle = \langle \lambda M + N, \lambda M + N \rangle = \lambda^2 \langle M, M \rangle + 2\lambda \langle M, N \rangle + \langle N \rangle.$$

We can minimize this expression in λ if we take $\lambda = -\langle M, N \rangle / \langle M \rangle$. Inserting this into the above inequality yields the estimate.

Problem 15.3. Solution: Note that

$$[a, b] \cap [c, d] = [a \vee c, b \wedge d] \quad (\text{with the convention } [M, m] = \emptyset \text{ if } M \geq m).$$

Then assume that we have any two representations for a simple process

$$f = \sum_j \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} = \sum_k \psi_{k-1} \mathbb{1}_{[t_{k-1}, t_k)}$$

Then

$$f = \sum_j \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[0, T)} = \sum_{j,k} \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[t_{k-1}, t_k)}$$

and, similarly,

$$f = \sum_{k,j} \psi_{k-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[t_{k-1}, t_k)}.$$

Then we get, since $\phi_{j-1} = \psi_{k-1}$ whenever $[s_{j-1}, s_j) \cap [t_{k-1}, t_k) \neq \emptyset$

$$\begin{aligned}
 \sum_j \phi_{j-1}(B(s_j) - B(s_{j-1})) &= \sum_{(j,k): [s_{j-1}, s_j) \cap [t_{k-1}, t_k) \neq \emptyset} \phi_{j-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
 &= \sum_{(j,k): [s_{j-1}, s_j) \cap [t_{k-1}, t_k) \neq \emptyset} \psi_{k-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
 &= \sum_{(k,j): [s_{j-1}, s_j) \cap [t_{k-1}, t_k) \neq \emptyset} \psi_{k-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
 &= \sum_k \psi_{k-1}(B(t_k) - B(t_{k-1}))
 \end{aligned}$$

■ ■

Problem 15.4. Solution:

- Positivity is clear, finiteness follows with Doob's maximal inequality

$$\mathbb{E} \left[\sup_{s \leq T} |M_s|^2 \right] \leq 4 \sup_{s \leq T} \mathbb{E} [|M_s|^2] = 4 \mathbb{E} [|M_T|^2].$$

- Triangle inequality:

$$\begin{aligned}
 \|M + N\|_{\mathcal{M}_T^2} &= \left(\mathbb{E} \left[\sup_{s \leq T} |M_s + N_s|^2 \right] \right)^{\frac{1}{2}} \\
 &\leq \left(\mathbb{E} \left[\left(\sup_{s \leq T} |M_s| + \sup_{s \leq T} |N_s| \right)^2 \right] \right)^{\frac{1}{2}} \\
 &\leq \left(\mathbb{E} \left[\sup_{s \leq T} |M_s|^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\sup_{s \leq T} |N_s|^2 \right] \right)^{\frac{1}{2}}
 \end{aligned}$$

where we used in the first estimate the subadditivity of the supremum and in the second inequality the Minkowski inequality (triangle inequality) in L^2 .

- Positive homogeneity

$$\|\lambda M\|_{\mathcal{M}_T^2} = \left(\mathbb{E} \left[\sup_{s \leq T} |\lambda M_s|^2 \right] \right)^{\frac{1}{2}} = |\lambda| \left(\mathbb{E} \left[\sup_{s \leq T} |M_s|^2 \right] \right)^{\frac{1}{2}} = |\lambda| \cdot \|M\|_{\mathcal{M}_T^2}.$$

- Definiteness

$$\|M\|_{\mathcal{M}_T^2} = 0 \iff \sup_{s \leq T} |M_s|^2 = 0 \quad (\text{almost surely}).$$

■ ■

Problem 15.5. Solution: Let $f_n \rightarrow f$ and $g_n \rightarrow f$ be two sequences which approximate f in the norm of $L^2(\lambda_T \otimes \mathbb{P})$. Then we have

$$\begin{aligned}
 \mathbb{E} \left(|f_n \bullet B_T - g_n \bullet B_T|^2 \right) &= \mathbb{E} \left(|(f_n - g_n) \bullet B_T|^2 \right) \\
 &= \mathbb{E} \left(\int_0^T |f_n(s) - g_n(s)|^2 ds \right) \\
 &= \|f_n - g_n\|_{L^2(\lambda_T \otimes \mathbb{P})}^2
 \end{aligned}$$

$$\xrightarrow[n \rightarrow \infty]{} 0.$$

This means that

$$L^2(\mathbb{P})\text{-}\lim_{n \rightarrow \infty} f_n \bullet B_T = L^2(\mathbb{P})\text{-}\lim_{n \rightarrow \infty} g_n \bullet B_T.$$

Problem 15.6. Solution: Solution 1: Let τ be a stopping time and consider the sequence of discrete stopping times

$$\tau_m := \frac{\lfloor 2^m \tau \rfloor + 1}{2^m} \wedge T.$$

Let $t_0 = 0 < t_1 < t_2 < \dots < t_n = T$ and, without loss of generality, $\tau_m(\Omega) \subset \{t_0, \dots, t_n\}$. Then $(B_{t_j}^2 - t_j)_j$ is again a discrete martingale and by optional stopping we get that $(B_{\tau_m \wedge t_j}^2 - \tau_m \wedge t_j)_j$ is a discrete martingale. This means that for each $m \geq 1$

$$\langle B^{\tau_m} \rangle_{t_j} = \tau_m \wedge t_j \quad \text{for all } j$$

and this indicates that we can set $\langle B^\tau \rangle_t = t \wedge \tau$. This process makes $B_{t \wedge \tau}^2 - t \wedge \tau$ into a martingale. Indeed: fix $0 \leq s \leq t \leq T$ and add them to the partition, if necessary. Then

$$B_{\tau_m \wedge t}^2 \xrightarrow[m \rightarrow \infty]{\text{a.e.}} B_{\tau \wedge t}^2 \quad \text{and} \quad B_{\tau_m \wedge t}^2 \xrightarrow[m \rightarrow \infty]{L^1(\mathbb{P})} B_{\tau \wedge t}^2$$

by dominated convergence, since $\sup_{r \leq T} B_r^2$ is an integrable majorant. Thus,

$$\begin{aligned} \int_F (B_{\tau \wedge s}^2 - \tau \wedge s) d\mathbb{P} &= \lim_{m \rightarrow \infty} \int_F (B_{\tau_m \wedge s}^2 - \tau_m \wedge s) d\mathbb{P} \\ &= \lim_{m \rightarrow \infty} \int_F (B_{\tau_m \wedge t}^2 - \tau_m \wedge t) d\mathbb{P} \\ &= \int_F (B_{\tau \wedge t}^2 - \tau \wedge t) d\mathbb{P} \quad \text{for all } F \in \mathcal{F}_s \end{aligned}$$

and we conclude that $(B_{\tau \wedge t}^2 - \tau \wedge t)_t$ is a martingale.

Solution 2: Observe that

$$B_t^\tau = \int_0^t \mathbb{1}_{[0, \tau)} dB_s$$

and by Theorem 15.15 b) we get

$$\left\langle \int_0^\bullet \mathbb{1}_{[0, \tau)} dB_s \right\rangle_t = \int_0^t \mathbb{1}_{[0, \tau)}^2 ds = \int_0^t \mathbb{1}_{[0, \tau)} ds = \tau \wedge t.$$

(Of course, one should make sure that $\mathbb{1}_{[0, \tau)} \in \mathcal{L}_T^2$, see e.g. Problem 15.20 below or Problem 16.2 in combination with Theorem 15.23.)

Problem 15.7. Solution: We begin with a general remark: if $f = 0$ on $[0, s] \times \Omega$, we can use Theorem 15.15 f) and deduce $f \bullet B_s = 0$.

a) We have

$$\mathbb{E}[(f \bullet B_t)^2 \mid \mathcal{F}_s] = \mathbb{E}[(f \bullet B_t - f \bullet B_s)^2 \mid \mathcal{F}_s] \stackrel{15.15 \text{ b)}}{\stackrel{(15.21)}}{=} \mathbb{E}\left[\int_s^t f^2(r) dr \mid \mathcal{F}_s\right].$$

If both f and g vanish on $[0, s]$, the same is true for $f \pm g$. We get

$$\mathbb{E}[(f \pm g \bullet B_t)^2 \mid \mathcal{F}_s] = \mathbb{E}\left[\int_s^t (f \pm g)^2(r) dr \mid \mathcal{F}_s\right].$$

Subtracting the ‘minus’ version from the ‘plus’ version and gives

$$\mathbb{E}[(f + g \bullet B_t)^2 - (f - g \bullet B_t)^2 \mid \mathcal{F}_s] = \mathbb{E}\left[\int_s^t (f + g)^2(r) - (f - g)^2(r) dr \mid \mathcal{F}_s\right].$$

or

$$4 \mathbb{E}[(f \bullet B_t) \cdot (g \bullet B_t) \mid \mathcal{F}_s] = 4 \mathbb{E}\left[\int_s^t (f \cdot g)(r) dr \mid \mathcal{F}_s\right].$$

b) Since $f \bullet B_t$ is a martingale, we get for $t \geq s$

$$\mathbb{E}(f \bullet B_t \mid \mathcal{F}_s) \stackrel{\text{martingale}}{=} f \bullet B_s \stackrel{\text{see above}}{=} 0$$

since f vanishes on $[0, s]$.

c) By Theorem 15.15 f) we have for all $t \leq T$

$$f \bullet B_t(\omega) \mathbb{1}_A(\omega) = 0 \bullet B_t(\omega) \mathbb{1}_A(\omega) = 0.$$

■ ■

Problem 15.8. Solution: Because of Lemma 15.12 it is enough to show that $f_n \bullet B_T \xrightarrow{n \rightarrow \infty} f \bullet B_T$ in $L^2(\mathbb{P})$. This follows immediately from Theorem 15.15 c):

$$\begin{aligned} \mathbb{E}\left[|f_n \bullet B_T - f \bullet B_T|^2\right] &= \mathbb{E}\left[|(f_n - f) \bullet B_T|^2\right] \\ &= \mathbb{E}\left[\int_0^T |f_n(s) - f(s)|^2 ds\right] \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

■ ■

Problem 15.9. Solution: Without loss of generality we assume that $f(0) = 0$. Fix $c > 0$. Then we have for all $\epsilon > 0$, using the Markov inequality and the Hölder inequality with $p = 4$ and $q = 4/3$

$$\begin{aligned} \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right| > c\right) &= \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right|^{1/2} > \sqrt{c}\right) \\ &\leq c^{-1/2} \mathbb{E}\left[\frac{1}{|B_\epsilon|^{1/2}} \left|\int_0^\epsilon f(s) dB_s\right|^{1/2}\right] \\ &\leq c^{-1/2} \left(\mathbb{E}\left[\frac{1}{|B_\epsilon|^{2/3}}\right]\right)^{3/4} \left(\mathbb{E}\left[\left(\int_0^\epsilon f(s) dB_s\right)^2\right]\right)^{1/4}. \end{aligned}$$

Using Itô's isometry and Brownian scaling yields

$$\begin{aligned} \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right| > c\right) &\leq c^{-1/2} \left(\mathbb{E}[|B_1|^{-2/3}]\right)^{3/4} \epsilon^{-1/4} \left(\int_0^\epsilon \mathbb{E}[|f(s)|^2] ds\right)^{1/4} \\ &= c^{-1/2} \left(\mathbb{E}[|B_1|^{-2/3}]\right)^{3/4} \left(\frac{1}{\epsilon} \int_0^\epsilon \mathbb{E}[|f(s)|^2] ds\right)^{1/4} \\ &\leq c^{-1/2} \left(\mathbb{E}[|B_1|^{-2/3}]\right)^{3/4} \left(\sup_{s \leq \epsilon} \mathbb{E}[|f(s)|^2]\right)^{1/4}. \end{aligned}$$

Since $\mathbb{E}[|B_1|^{-2/3}] < \infty$, see (the solution of) Problem 11.8, we find

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right| > c\right) &\leq C \left(\underbrace{\limsup_{\epsilon \rightarrow 0} \sup_{s \leq \epsilon} \mathbb{E}[|f(s)|^2]}_{=\limsup_{\epsilon \rightarrow 0}}\right)^{1/4} \\ &= C \left(\lim_{\epsilon \rightarrow 0} \mathbb{E}[|f(\epsilon)|^2]\right)^{1/4} = 0. \end{aligned}$$

Alternative solution: We fix $c, K > 0$ and observe

$$\begin{aligned} \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right| > c\right) &\leq \mathbb{P}\left(\left|\frac{1}{B_\epsilon} \int_0^\epsilon f(s) dB_s\right| > c, \left|\frac{\sqrt{\epsilon}}{B_\epsilon}\right| \leq K\right) + \mathbb{P}\left(\left|\frac{\sqrt{\epsilon}}{B_\epsilon}\right| > K\right) \\ &\leq \mathbb{P}\left(\left|\frac{1}{\sqrt{\epsilon}} \int_0^\epsilon f(s) dB_s\right| > \frac{c}{K}\right) + \mathbb{P}\left(\frac{|B_\epsilon|}{\sqrt{\epsilon}} < \frac{1}{K}\right) \end{aligned}$$

The first expression can now be estimated using the Markov inequality and Itô's isometry, for the second expression we use Brownian scaling. ■ ■

Problem 15.10. Solution: Do not try to argue like this:

$$Y_t \int_0^t X_s dB_s = \int_0^t Y_t X_s dB_s = \int_0^t Y_t Y_s X_s dB_s = Y_t \int_0^t Y_s X_s dB_s$$

since the pull-in and pull-out of Y_t under the stochastic integral cannot be justified. The correct argument goes like this: The condition $Y_t Y_s = Y_t$ means that $Y_t \in \{0, 1\}$. Define $F = \{Y_t = 1\}$ and use Theorem 15.15.f): The condition $Y_t Y_s = Y_t$ ensures that on F the processes X_s and $Y_s X_s$ coincide. Thus, the claim. ■ ■

Problem 15.11. Solution: We have to show that for all $s, t \geq 0$ one has $\lim_{s \rightarrow t} \mathbb{E}\left[|\mathbb{1}_{\{B_s > 0\}} - \mathbb{1}_{\{B_t > 0\}}|^2\right] = 0$. Since the integrand can have only the value 1 or 0, we see

$$\mathbb{E}\left[|\mathbb{1}_{\{B_s > 0\}} - \mathbb{1}_{\{B_t > 0\}}|^2\right] = \mathbb{E}\left[|\mathbb{1}_{\{B_s > 0\}} - \mathbb{1}_{\{B_t > 0\}}|\right] = \mathbb{P}(B_s > 0, B_t \leq 0) + \mathbb{P}(B_s \leq 0, B_t > 0).$$

Since s, t play symmetric roles, it is enough to consider one of these probabilities. Moreover, we can assume that $s \leq t$, the other calculation is similar. Since $\mathbb{P}(B_t = 0) = 0$ and $B_s \perp\!\!\!\perp B_t - B_s \sim B_{t-s}$ we get

$$\mathbb{P}(B_s > 0, B_t \leq 0) = \mathbb{P}(B_s > 0, B_t < 0) = \mathbb{P}(B_s > 0, B_t - B_s + B_s < 0)$$

$$= \int_0^\infty \mathbb{P}(B_{t-s} < -x) \mathbb{P}(B_s \in dx) = \int_0^\infty \mathbb{P}(B_{t-s} < -\sqrt{sy}) \mathbb{P}(B_1 \in dy)$$

In the last step we use $B_s \sim \sqrt{s}B_1$ and a change of variables. Now we can use dominated convergence and the fact that, because of stochastic continuity, $\mathbb{P}(B_{t-s} < -\sqrt{sy}) \rightarrow 0$ as $s \rightarrow t$ (here we need that $t > 0$).

Problem 15.12. Solution: Assume that $(f \bullet B)^2 - A$ is a martingale where A_t is continuous and increasing. Since $(f \bullet B)^2 - f^2 \bullet \langle B \rangle$ is a martingale, we conclude that

$$((f \bullet B)^2 - f^2 \bullet \langle B \rangle) - ((f \bullet B)^2 - A) = f^2 \bullet \langle B \rangle - A$$

is a continuous martingale with BV paths. Hence, it is a.s. constant.

Problem 15.13. Solution: By the Cauchy–Schwarz inequality we see that

$$\mathbb{E} \left[\int_0^t f(s)g(s) ds \right] \leq \sqrt{\mathbb{E} \left[\int_0^t |f(s)|^2 ds \right]} \sqrt{\mathbb{E} \left[\int_0^t |g(s)|^2 ds \right]} < \infty$$

which means that $\int_0^t f(s)g(s) ds$ is well-defined. Since $f \pm g \in \mathcal{L}_t^2$, we get by polarization

$$\begin{aligned} 4\langle f \bullet B, g \bullet B \rangle_t &= \langle (f+g) \bullet B \rangle_t - \langle (f-g) \bullet B \rangle_t \\ &\stackrel{\text{Thm. 15.15b)}}{=} \int_0^t (f+g)^2(s) ds - \int_0^t (f-g)^2(s) ds \\ &= \int_0^t [(f+g)^2(s) - (f-g)^2(s)] ds \\ &= \int_0^t 4f(s)g(s) ds, \end{aligned}$$

and the claim follows.

Problem 15.14. Solution: Observe that for $\epsilon \downarrow 0$ we have $\mathbb{1}_{[s_{j-1}+\epsilon, s_j+\epsilon)}(t) \rightarrow \mathbb{1}_{(s_{j-1}, s_j]}(t)$. If we define f_ϵ with the weights ϕ_j and the shifted steps $\mathbb{1}_{[s_{j-1}+\epsilon, s_j+\epsilon)}(t)$, then $f_\epsilon \rightarrow g$ as $\epsilon \rightarrow 0$. By Itô's isometry, it is clear that

$$\begin{aligned} \mathbb{E} [(f \bullet B - f_\epsilon \bullet B)^2] &= \mathbb{E} \left(\int_0^T |f(s, \omega) - f_\epsilon(s, \omega)| d\langle B \rangle_s(\omega) \right) \\ &\xrightarrow{\epsilon \rightarrow 0} \mathbb{E} \left(\int_0^T |f(s, \omega) - g(s, \omega)|^2 d\langle B \rangle_s(\omega) \right). \end{aligned}$$

Since f and g differ, for each ω at at most finitely many points, and since the measure induced by $s \mapsto \langle B \rangle_s$ does not have atoms, we see that

$$\mathbb{E} [(f \bullet B - f_\epsilon \bullet B)^2] \xrightarrow{\epsilon \rightarrow 0} \mathbb{E} (0) = 0.$$

Problem 15.15. Solution: If $X_n \xrightarrow{L^2} X$ then $\sup_n \mathbb{E}(X_n^2) < \infty$ and the claim follows from the fact that

$$\begin{aligned} \mathbb{E}|X_n^2 - X_m^2| &= \mathbb{E}[|X_n - X_m||X_n + X_m|] \\ &\leq \sqrt{\mathbb{E}|X_n + X_m|^2} \sqrt{\mathbb{E}|X_n - X_m|^2} \\ &\leq (\sqrt{\mathbb{E}|X_n|^2} + \sqrt{\mathbb{E}|X_m|^2}) \sqrt{\mathbb{E}|X_n - X_m|^2}. \end{aligned}$$

■ ■

Problem 15.16. Solution: Let $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = T\}$ be a partition of $[0, T]$. Then we get

$$\begin{aligned} B_T^3 &= \sum_{j=1}^n (B_{t_j}^3 - B_{t_{j-1}}^3) \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}}) [B_{t_j}^2 + B_{t_j} B_{t_{j-1}} + B_{t_{j-1}}^2] \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}}) [B_{t_j}^2 - 2B_{t_j} B_{t_{j-1}} + B_{t_{j-1}}^2 + 3B_{t_j} B_{t_{j-1}}] \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}}) [(B_{t_j} - B_{t_{j-1}})^2 + 3B_{t_j} B_{t_{j-1}}] \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}}) [(B_{t_j} - B_{t_{j-1}})^2 + 3B_{t_{j-1}}^2 + 3B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})] \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})^3 + 3 \sum_{j=1}^n B_{t_{j-1}}^2 (B_{t_j} - B_{t_{j-1}}) + 3 \sum_{j=1}^n B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}})^2 \\ &= \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})^3 + 3 \sum_{j=1}^n B_{t_{j-1}}^2 (B_{t_j} - B_{t_{j-1}}) + 3 \sum_{j=1}^n B_{t_{j-1}} (t_j - t_{j-1}) \\ &\quad + 3 \sum_{j=1}^n B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Clearly,

$$I_2 \xrightarrow{|\Pi| \rightarrow 0} 3 \int_0^T B_s^2 dB_s \quad \text{and} \quad I_3 \xrightarrow{|\Pi| \rightarrow 0} 3 \int_0^T B_s ds$$

by Proposition 15.18 and by the construction of the stochastic resp. Riemann-Stieltjes integral. The latter also converges in L^2 since I_2 and, as we will see in a moment, I_1 and I_4 converge in L^2 -sense.

Let us show that $I_1, I_4 \rightarrow 0$.

$$\begin{aligned} \mathbb{V} I_1 &= \mathbb{V} \left(\sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})^3 \right) \stackrel{(B1)}{=} \sum_{j=1}^n \mathbb{V} \left((B_{t_j} - B_{t_{j-1}})^3 \right) \\ &\stackrel{(B2)}{=} \sum_{j=1}^n \mathbb{V} \left(B_{t_j - t_{j-1}}^3 \right) \\ &\stackrel{\text{scaling}}{=} \sum_{j=1}^n (t_j - t_{j-1})^3 \mathbb{V} (B_1^3) \end{aligned}$$

$$\begin{aligned} &\leq |\Pi|^2 \sum_{j=1}^n (t_j - t_{j-1}) \mathbb{V}(B_1^3) \\ &= |\Pi|^2 T \mathbb{V}(B_1^3) \xrightarrow{|\Pi| \rightarrow 0} 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}(I_4^2) &= \mathbb{E} \left(\left(3 \sum_{j=1}^n B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] \right)^2 \right) \\ &= 9 \mathbb{E} \left(\sum_{j=1}^n \sum_{k=1}^n B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] B_{t_{k-1}} [(B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})] \right) \\ &= 9 \mathbb{E} \left(\sum_{j=1}^n B_{t_{j-1}}^2 [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})]^2 \right) \end{aligned}$$

since the mixed terms break away, see below.

$$\begin{aligned} &= 9 \sum_{j=1}^n \mathbb{E} \left(B_{t_{j-1}}^2 [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})]^2 \right) \\ &\stackrel{(B1)}{=} 9 \sum_{j=1}^n \mathbb{E} \left(B_{t_{j-1}}^2 \right) \mathbb{E} \left([(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})]^2 \right) \\ &\stackrel{(B2)}{=} 9 \sum_{j=1}^n \mathbb{E} \left(B_{t_{j-1}}^2 \right) \mathbb{E} \left([B_{t_j - t_{j-1}}^2 - (t_j - t_{j-1})]^2 \right) \\ &\stackrel{\text{scaling}}{=} 9 \sum_{j=1}^n t_{j-1} \mathbb{E} \left(B_1^2 \right) (t_j - t_{j-1})^2 \mathbb{E} \left([B_1^2 - 1]^2 \right) \\ &= 9 \sum_{j=1}^n t_{j-1} (t_j - t_{j-1})^2 \mathbb{V}(B_1^2) \\ &\leq 9T |\Pi| \sum_{j=1}^n (t_j - t_{j-1}) \mathbb{V}(B_1^2) \\ &\leq 9T^2 |\Pi| \mathbb{V}(B_1^2) \xrightarrow{|\Pi| \rightarrow 0} 0. \end{aligned}$$

Now for the argument with the mixed terms. Let $j < k$; then $t_{j-1} < t_j \leq t_{k-1} < t_k$, and by the tower property,

$$\begin{aligned} &\mathbb{E} \left(B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] B_{t_{k-1}} [(B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})] \right) \\ &\stackrel{\text{tower}}{=} \mathbb{E} \left(\mathbb{E} \left[B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] B_{t_{k-1}} [(B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})] \mid \mathcal{F}_{t_{k-1}} \right] \right) \\ &\stackrel{\text{pull out}}{=} \mathbb{E} \left(B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] B_{t_{k-1}} \mathbb{E} \left[[(B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})] \mid \mathcal{F}_{t_{k-1}} \right] \right) \\ &\stackrel{(B1)}{=} \mathbb{E} \left(B_{t_{j-1}} [(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})] B_{t_{k-1}} \underbrace{\mathbb{E} \left[(B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1}) \right]}_{=0} \right) \\ &= 0. \end{aligned}$$

Problem 15.17. Solution: Let $\Pi = \{t_0 = 0 < t_1 < \dots < t_n = T\}$ be a partition of $[0, T]$. Then we get

$$f(t_j)B_{t_j} - f(t_{j-1})B_{t_{j-1}}$$

$$= f(t_{j-1})(B_{t_j} - B_{t_{j-1}}) + B_{t_{j-1}}(f(t_j) - f(t_{j-1})) + (B_{t_j} - B_{t_{j-1}})(f(t_j) - f(t_{j-1})).$$

If we sum over $j = 1, \dots, n$ we get

$$\begin{aligned} & f(T)B_T - f(0)B_0 \\ &= \sum_{j=1}^n f(t_{j-1})(B_{t_j} - B_{t_{j-1}}) + \sum_{j=1}^n B_{t_{j-1}}(f(t_j) - f(t_{j-1})) + \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})(f(t_j) - f(t_{j-1})) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Clearly,

$$\begin{aligned} I_1 &\xrightarrow{L^2} \int_0^T f(s) dB_s && \text{(stochastic integral)} \\ I_2 &\xrightarrow{\text{a.s.}} \int_0^T B_s df(x) && \text{(Riemann-Stieltjes integral)} \end{aligned}$$

and if we can show that $I_3 \rightarrow 0$ in L^2 , then we are done (as this also implies the L^2 -convergence of I_2). Now we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})(f(t_j) - f(t_{j-1})) \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{j=1}^n \sum_{k=1}^n (B_{t_j} - B_{t_{j-1}})(f(t_j) - f(t_{j-1}))(B_{t_k} - B_{t_{k-1}})(f(t_k) - f(t_{k-1})) \right] \end{aligned}$$

the mixed terms break away because of the independent increments property of Brownian motion

$$\begin{aligned} &= \sum_{j=1}^n \mathbb{E} [(B_{t_j} - B_{t_{j-1}})^2 (f(t_j) - f(t_{j-1}))^2] \\ &= \sum_{j=1}^n (f(t_j) - f(t_{j-1}))^2 \mathbb{E} [(B_{t_j} - B_{t_{j-1}})^2] \\ &= \sum_{j=1}^n (t_j - t_{j-1})(f(t_j) - f(t_{j-1}))^2 \\ &\leq 2|\Pi| \cdot \|f\|_\infty \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \\ &\leq 2|\Pi| \cdot \|f\|_\infty \text{VAR}_1(f; [0, T]) \xrightarrow{|\Pi| \rightarrow 0} 0 \end{aligned}$$

where we used the fact that a BV-function is necessarily bounded:

$$|f(t)| \leq |f(t) - f(0)| + |f(0)| \leq \text{VAR}_1(f; [0, t]) + \text{VAR}_1(f; \{0\}) \leq 2\text{VAR}_1(f; [0, T])$$

for all $t \in [0, T]$.

Problem 15.18. Solution: Replace, starting in the fourth line of the proof of Proposition 15.18, the argument as follows:

By the maximal inequalities (15.23) for Itô integrals we get

$$\begin{aligned}
 & \mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t [f(s) - f^\Pi(s)] dB_s \right|^2 \right] \\
 & \leq 4 \int_0^T \mathbb{E} [|f(s) - f^\Pi(s)|^2] ds \\
 & = 4 \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \mathbb{E} [|f(s) - f(s_{j-1})|^2] ds \\
 & \leq 4 \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \underbrace{\sup_{u,v \in [s_{j-1}, s_j]} \mathbb{E} [|f(u) - f(v)|^2]}_{\rightarrow 0, |\Pi| \rightarrow 0} ds \xrightarrow{|\Pi| \rightarrow 0} 0.
 \end{aligned}$$

■ ■

Problem 15.19. Solution: To simplify notation, we drop the n in Π_n and write only $0 = t_0 < t_1 < \dots < t_k = T$ and

$$\theta_{n,j}^\alpha = \theta_j = \alpha t_j + (1 - \alpha)t_{j-1}.$$

We get

$$L_T(\alpha) := L^2(\mathbb{P})\text{-}\lim_{|\Pi| \rightarrow 0} \sum_{j=1}^k B_{\theta_j} (B_{t_j} - B_{t_{j-1}}) = \int_0^T B_s dB_s + \alpha T.$$

Indeed, we have

$$\begin{aligned}
 & \sum_{j=1}^k B_{\theta_j} (B_{t_j} - B_{t_{j-1}}) \\
 & = \sum_{j=1}^k B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) + \sum_{j=1}^k (B_{\theta_j} - B_{t_{j-1}}) (B_{t_j} - B_{t_{j-1}}) \\
 & = \sum_{j=1}^k B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) + \sum_{j=1}^k (B_{\theta_j} - B_{t_{j-1}})^2 + \sum_{j=1}^k (B_{t_j} - B_{\theta_j}) (B_{\theta_j} - B_{t_{j-1}}) \\
 & = X + Y + Z.
 \end{aligned}$$

We know already that $X \xrightarrow[|\Pi| \rightarrow 0]{L^2} \int_0^T B_s dB_s$. Moreover,

$$\begin{aligned}
 \mathbb{V} Z & = \mathbb{V} \left(\sum_{j=1}^k (B_{t_j} - B_{\theta_j}) (B_{\theta_j} - B_{t_{j-1}}) \right) \\
 & = \sum_{j=1}^k \mathbb{V} \left[(B_{t_j} - B_{\theta_j}) (B_{\theta_j} - B_{t_{j-1}}) \right] \\
 & = \sum_{j=1}^k \mathbb{E} \left[(B_{t_j} - B_{\theta_j})^2 (B_{\theta_j} - B_{t_{j-1}})^2 \right] \\
 & = \sum_{j=1}^k \mathbb{E} \left[(B_{t_j} - B_{\theta_j})^2 \right] \mathbb{E} \left[(B_{\theta_j} - B_{t_{j-1}})^2 \right] \\
 & = \sum_{j=1}^k (t_j - \theta_j) (\theta_j - t_{j-1})
 \end{aligned}$$

$$= \alpha(1 - \alpha) \sum_{j=1}^k (t_j - t_{j-1})(t_j - t_{j-1}) \xrightarrow{\text{as in Theorem 9.1}} 0.$$

Finally,

$$\begin{aligned} \mathbb{E} Y &= \mathbb{E} \left(\sum_{j=1}^k (B_{\theta_j} - B_{t_{j-1}})^2 \right) = \sum_{j=1}^k \mathbb{E} (B_{\theta_j} - B_{t_{j-1}})^2 \\ &= \sum_{j=1}^k (\theta_j - t_{j-1}) = \alpha \sum_{j=1}^k (t_j - t_{j-1}) = \alpha T. \end{aligned}$$

The L^2 -convergence follows now *literally* as in the proof of Theorem 9.1.

Consequence: $L_T(\alpha) = \frac{1}{2} (B_T^2 + (2\alpha - 1)T)$, and this stochastic integral is a martingale if, and only if, $\alpha = 0$, i.e. if $\theta_j = t_{j-1}$ is the left endpoint of the interval.

For $\alpha = \frac{1}{2}$ we get the so-called Stratonovich or mid-point stochastic integral. This will obey the usual calculus rules (instead of Itô's rule). A first sign is the fact that

$$L_T\left(\frac{1}{2}\right) = \frac{1}{2} B_T^2$$

and we usually write

$$L_T\left(\frac{1}{2}\right) = \int_0^T B_s \circ dB_s$$

with the *Stratonovich-circle* \circ to indicate the mid-point rule.

■ ■

Problem 15.20. Solution:

- a) Let τ_k be a sequence of stopping times with countably many, discrete values such that $\tau_k \downarrow \tau$. For example, $\tau_k := (\lfloor 2^k \tau \rfloor + 1)/2^k$, see Lemma A.15 in the appendix. Write $s_1 < \dots < s_K$ for the values of τ_k . In particular,

$$\mathbb{1}_{[0, T \wedge \tau_k)} = \sum_j \mathbb{1}_{\{T \wedge \tau_k = T \wedge s_j\}} \mathbb{1}_{[0, T \wedge s_j)}$$

And so

$$\{(s, \omega) : \mathbb{1}_{[0, T \wedge \tau_k(\omega))}(s) = 1\} = \bigcup_j [0, T \wedge s_j) \times \{T \wedge \tau_k = T \wedge s_j\}.$$

Since $\{T \wedge \tau_k = T \wedge s_j\} \in \mathcal{F}_{T \wedge s_j}$, it is clear that

$$\{(s, \omega) : \mathbb{1}_{[0, T \wedge \tau_k(\omega))}(s) = 1\} \cap ([0, t] \times \Omega) \in \mathcal{B}[0, t] \times \mathcal{F}_t \quad \text{for all } t \geq 0$$

and progressive measurability of $\mathbb{1}_{[0, T \wedge \tau_k)}$ follows.

- b) Since $T \wedge \tau_k \downarrow T \wedge \tau$ and $T \wedge \tau_k$ has only finitely many values, and we find

$$\lim_{k \rightarrow \infty} \mathbb{1}_{[0, T \wedge \tau_k)} = \mathbb{1}_{[0, T \wedge \tau)}$$

almost surely. Consequently, $\mathbb{1}_{[0, T \wedge \tau(\omega))}(s)$ is also \mathcal{P} -measurable.

In fact, we do not need to prove the progressive measurability of $\mathbb{1}_{[0, T \wedge \tau)}$ to evaluate the integral. If you want to show it nevertheless, have a look at Problem 16.2 below.

- c) Fix k and write $0 \leq s_1 < \dots < s_K$ for the values of $T \wedge \tau_k$. Following the proof of Theorem 15.11.c)

$$\begin{aligned}
 \int \mathbb{1}_{[0, T \wedge \tau_k)}(s) dB_s &= \int \sum_j \mathbb{1}_{[T \wedge s_{j-1}, T \wedge s_j)}(s) \mathbb{1}_{[0, T \wedge \tau_k)}(s) dB_s \\
 &= \sum_j \int \underbrace{\mathbb{1}_{[T \wedge s_{j-1}, T \wedge s_j \wedge \tau_k)}(s)}_{T \wedge s_j \wedge \tau_k = T \wedge s_j} \underbrace{\mathbb{1}_{\{T \wedge \tau_k > T \wedge s_{j-1}\}}}_{\mathcal{F}_{T \wedge s_{j-1}}\text{-mble}} dB_s \\
 &= \sum_j \mathbb{1}_{\{T \wedge \tau_k > T \wedge s_{j-1}\}} \int \mathbb{1}_{[T \wedge s_{j-1}, T \wedge s_j)}(s) dB_s \\
 &= \sum_j \mathbb{1}_{\{T \wedge \tau_k > T \wedge s_{j-1}\}} (B_{T \wedge s_j} - B_{T \wedge s_{j-1}}) \\
 &= B_{T \wedge \tau_k}.
 \end{aligned}$$

- d) $\underline{\mathbb{1}_{[0, T \wedge \tau)}} = L^2\text{-}\lim_k \mathbb{1}_{[0, T \wedge \tau_k)}$: This follows from

$$\begin{aligned}
 \mathbb{E} \int |\mathbb{1}_{[0, T \wedge \tau_k)}(s) - \mathbb{1}_{[0, T \wedge \tau)}(s)|^2 ds &= \mathbb{E} \int |\mathbb{1}_{[T \wedge \tau, T \wedge \tau_k)}(s)|^2 ds \\
 &= \mathbb{E} \int \mathbb{1}_{[T \wedge \tau, T \wedge \tau_k)}(s) ds \\
 &= \mathbb{E}(T \wedge \tau_k - T \wedge \tau) \xrightarrow[k \rightarrow \infty]{} 0
 \end{aligned}$$

by dominated convergence.

- e) By the very definition of the stochastic integral we find now

$$\int \mathbb{1}_{[0, T \wedge \tau)}(s) dB_s \stackrel{\text{d)}}{=} L^2\text{-}\lim_k \int \mathbb{1}_{[0, T \wedge \tau_k)}(s) dB_s \stackrel{\text{c)}}{=} L^2\text{-}\lim_k B_{T \wedge \tau_k} = B_{T \wedge \tau}$$

by the continuity of Brownian motion and dominated convergence: $\sup_{s \leq T} |B_s|$ is integrable.

- f) The result is, in the light of the localization principle of Theorem 15.15 not unexpected. ■ ■

Problem 15.21. Solution: Throughout the proof $t \geq 0$ is arbitrary but fixed.

- Clearly, $\emptyset, [0, T] \times \Omega \in \mathcal{P}$.
- Let $\Gamma \in \mathcal{P}$. Then

$$\Gamma^c \cap ([0, t] \times \Omega) = \underbrace{([0, t] \times \Omega)}_{\in \mathcal{B}[0, t] \otimes \mathcal{F}_t} \setminus \underbrace{(\Gamma \cap ([0, t] \times \Omega))}_{\in \mathcal{B}[0, t] \otimes \mathcal{F}_t} \in \mathcal{B}[0, t] \otimes \mathcal{F}_t,$$

thus $\Gamma^c \in \mathcal{P}$.

- Let $\Gamma_n \in \mathcal{P}$. By definition

$$\Gamma_n \cap ([0, t] \times \Omega) \in \mathcal{B}[0, t] \otimes \mathcal{F}_t$$

and we can take the union over n to get

$$\left(\bigcup_n \Gamma_n\right) \cap ([0, t] \times \Omega) = \bigcup_n (\Gamma_n \cap ([0, t] \times \Omega)) \in \mathcal{B}[0, t] \otimes \mathcal{F}_t$$

i.e. $\bigcup_n \Gamma_n \in \mathcal{P}$.

■ ■

Problem 15.22. Solution: Let $f(t, \omega)$ be right continuous on the interval $[0, T]$. (We consider only $T < \infty$ since the case of the infinite interval $[0, \infty)$ is actually easier.)

Set

$$f_n^T(s, \omega) := f\left(\frac{\lfloor 2^n s \rfloor + 1}{2^n} \wedge T, \omega\right)$$

then

$$f_n^T(s, \omega) = \sum_k f\left(\frac{k+1}{2^n} \wedge T, \omega\right) \mathbb{1}_{[k2^{-n}, (k+1)2^{-n})}(s) \quad (s \leq T)$$

and, since $(\lfloor 2^n s \rfloor + 1)/2^n \downarrow s$, we find by right continuity that $f_n \rightarrow f$ as $n \rightarrow \infty$. This means that it is enough to consider the \mathcal{P} -measurability of the step-function f_n .

Fix $n \geq 0$, write $t_j = j2^{-n}$. Then $t_0 = 0 < t_1 < \dots < t_N \leq T$ for some suitable N . Observe that for any $x \in \mathbb{R}$

$$\{(s, \omega) : f(s, \omega) \leq x\} = \{T\} \times \{\omega : f(T, \omega) \leq x\} \cup \bigcup_{j=1}^N [t_{j-1}, t_j) \times \{\omega : f(t_j, \omega) \leq x\}$$

and each set appearing in the union set on the right is in $\mathcal{B}[0, T] \otimes \mathcal{F}_T$.

This shows that f_n^T and f are $\mathcal{B}[0, T] \otimes \mathcal{F}_T$ measurable.

Now consider f_n^t and $f(t)\mathbb{1}_{[0, t]}$. We conclude, with the same reasoning, that both are $\mathcal{B}[0, t] \otimes \mathcal{F}_t$ measurable.

This shows that a right continuous f is progressive.

If f is left continuous, we use $\lfloor 2^n s \rfloor / 2^n \uparrow s$ and define the approximating function as

$$g_n^T(s, \omega) = \sum_k f\left(\frac{k}{2^n} \wedge T, \omega\right) \mathbb{1}_{[k2^{-n}, (k+1)2^{-n})}(s) \quad (s \leq T).$$

The rest of the proof is similar.

■ ■

Problem 15.23. Solution: Let τ_k be a sequence as constructed in Lemma A.16. We have $\tau_k \downarrow \tau$, thus we see with dominated convergence that $\mathbb{1}_{(0, \tau_k]} \rightarrow \mathbb{1}_{(0, \tau]}$ in $L^2(\lambda_T \otimes \mathbb{P})$ for any $T > 0$. So we can assume that τ has only countably many values in $[0, \infty)$ and only finitely many values in every compact interval $[0, t]$ where $t \in [0, T]$. Call the values of τ

$$0 = t_0 < t_1 < \dots < t_j < \dots$$

We have

$$\mathbb{1}_{(0, \tau]} = \sum_{j=0}^{\infty} \mathbb{1}_{(0, t_j]} \mathbb{1}_{\{\tau = t_j\}}.$$

The problem is that, relative to the time-interval, the set $\{\tau = t_j\}$ has the “wrong” measurability (namely, for the endpoint). The following trick helps (mind that all sums are actually finite sums!):

$$\mathbb{1}_{(0,\infty)\times\Omega} - \mathbb{1}_{((0,\tau])} = \sum_{j=0}^{\infty} (\mathbb{1}_{(0,\infty)} - \mathbb{1}_{(0,t_j]}) \mathbb{1}_{\{\tau=t_j\}} = \sum_{j=0}^{\infty} \mathbb{1}_{(t_j,\infty)} \mathbb{1}_{\{\tau=t_j\}}.$$

Now we can apply the definition of N_t and get

$$\begin{aligned} N_t(\mathbb{1}_{(0,\infty)\times\Omega} - \mathbb{1}_{((0,\tau])}) &= N(\mathbb{1}_{(0,t]\times\Omega} - \mathbb{1}_{((0,\tau\wedge t])}) \\ &= B_t - \sum_{j=0}^{\infty} (B_t - B_{t\wedge t_j}) \mathbb{1}_{\{\tau=t_j\}} = \sum_{j=0}^{\infty} B_{t\wedge t_j} \mathbb{1}_{\{\tau=t_j\}} = B_{t\wedge\tau}. \end{aligned}$$

■ ■

16 Stochastic integrals: Localization

Problem 16.1. Solution: Yes. In view of Lemma 16.3 we have to show that $\mathcal{L}_T^0 \supset \mathcal{L}_{T,\text{loc}}^2$. Let $f \in \mathcal{L}_{T,\text{loc}}^2$ and take some localizing sequence $(\sigma_n)_{n \geq 1}$ such that

$$\sigma_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \infty \quad \text{and} \quad \int_0^{T \wedge \sigma_n} |f(s, \cdot)|^2 ds < \infty$$

(the finiteness of the integral latter follows from the fact that $f \mathbb{1}_{[0, \sigma_n]}$ is in \mathcal{L}_T^2 for each n . Moreover, $f \mathbb{1}_{[0, \sigma_n]}$ is \mathcal{P} -measurable, hence $f \mathbb{1}_{[0, T \wedge \sigma_n]} \rightarrow f \mathbb{1}_{[0, T]}$ is \mathcal{P} -measurable. Note that the completeness of the filtration allows that $\sigma_n \rightarrow \infty$ holds only a.s.). Now observe that for every fixed ω there is some $n(T, \omega) \geq 1$ such that for all $n \geq n(T, \omega)$ we have $\sigma_n(\omega) \geq T$. Thus,

$$\int_0^T |f(s, \omega)|^2 ds = \int_0^{T \wedge \sigma_n(\omega)} |f(s, \omega)|^2 ds < \infty.$$

Problem 16.2. Solution: *Solution 1:* We have that the process $t \mapsto \mathbb{1}_{[0, \tau(\omega))}(t)$ is adapted

$$\{\omega : \mathbb{1}_{[0, \tau(\omega))}(t) = 0\} = \{\tau \leq t\} \in \mathcal{F}_t$$

since τ is a stopping time. By Problem 15.22 we conclude that $\mathbb{1}_{[0, \tau)}$ is progressive.

Solution 2: Set $t_j = j2^{-n}$ and define

$$I_n^t(s, \omega) := \mathbb{1}_{[0, \tau(\omega))}\left(\frac{\lfloor 2^n s \rfloor}{2^n} \wedge t\right) = \sum_j \mathbb{1}_{[0, \tau(\omega))}(t_{j+1} \wedge t) \mathbb{1}_{[t_j, t_{j+1})}(s \wedge t).$$

Since $\lfloor 2^n s \rfloor / 2^n \downarrow s$ we find, by right continuity, $I_n^t \rightarrow \mathbb{1}_{[0, \tau)}$. Therefore, it is enough to check that I_n^t is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$ -measurable. But this is obvious from the form of I_n^t .

Problem 16.3. Solution: Assume that σ_n are stopping times such that $(M_t^{\sigma_n} \mathbb{1}_{\{\sigma_n > 0\}})_t$ is a martingale. Clearly,

- $\tau_n := \sigma_n \wedge n \uparrow \infty$ almost surely as $n \rightarrow \infty$;
- $\{\sigma_n > 0\} = \{\sigma_n \wedge n > 0\} = \{\tau_n > 0\}$;
- by optional stopping, the following process is a martingale for each n :

$$M_{t \wedge n}^{\sigma_n} \mathbb{1}_{\{\sigma_n > 0\}} = M_t^{\sigma_n \wedge n} \mathbb{1}_{\{\sigma_n > 0\}} = M_t^{\sigma_n \wedge n} \mathbb{1}_{\{\sigma_n \wedge n > 0\}} = M_t^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}.$$

Remark: This has an interesting consequence:

$$\mathbb{E} \left[\sup_{s \leq T} |M(s \wedge \tau_n)|^2 \right] \stackrel{\text{Doob}}{\leq} 4 \mathbb{E} [|M(\tau_n)|^2] \leq 4 \mathbb{E} [|M(n)|^2].$$

Problem 16.4. Solution: “ \Leftarrow ” If $M_{t \wedge \sigma_n}$ is a martingale, then all rv are integrable, so $M_0 = M_{0 \wedge \sigma_n} \in L^1$. Moreover, we get for any $n \geq 1$, $F \in \mathcal{F}_s$ and $s \leq t$

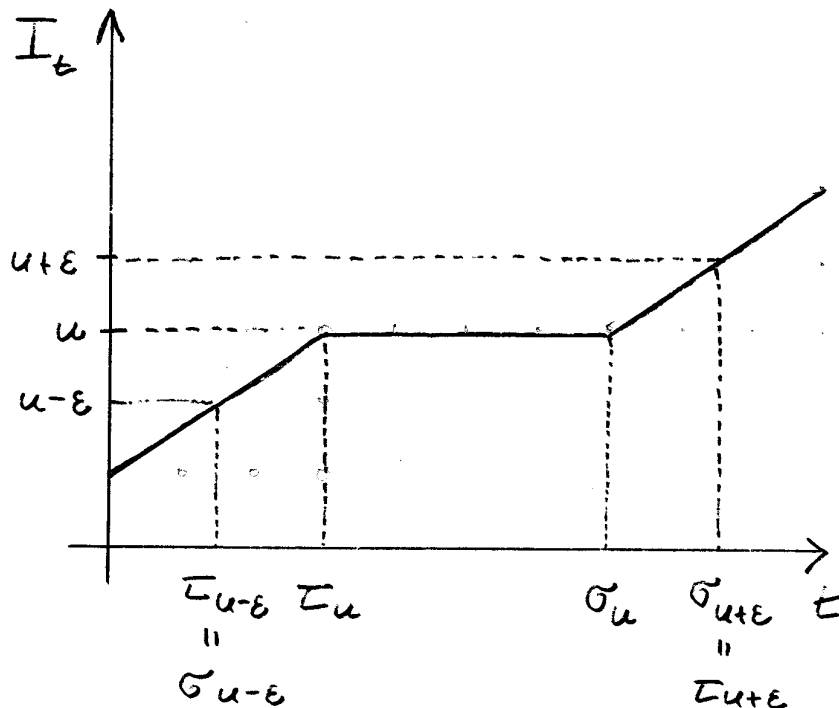
$$\begin{aligned} \int_F M_{t \wedge \sigma_n} \mathbb{1}_{\{\sigma_n > 0\}} d\mathbb{P} &= \int_F M_{t \wedge \sigma_n} d\mathbb{P} - \int_F M_{t \wedge \sigma_n} \mathbb{1}_{\{\sigma_n = 0\}} d\mathbb{P} \\ &\stackrel{\text{mg}}{=} \int_F M_{s \wedge \sigma_n} d\mathbb{P} - \int_F M_0 \mathbb{1}_{\{\sigma_n = 0\}} d\mathbb{P} \\ &= \int_F M_{s \wedge \sigma_n} d\mathbb{P} - \int_F M_{s \wedge \sigma_n} \mathbb{1}_{\{\sigma_n = 0\}} d\mathbb{P} \\ &= \int_F M_{s \wedge \sigma_n} \mathbb{1}_{\{\sigma_n > 0\}} d\mathbb{P} \end{aligned}$$

proving that $M_{t \wedge \sigma_n} \mathbb{1}_{\{\sigma_n > 0\}}$ is a martingale. Thus, M_t is a local martingale with $M_0 \in L^1$.

The reverse direction “ \Rightarrow ” follows from almost the same calculation.

Problem 16.5. Solution:

a) The picture below show that $I_{\sigma_u} = I_{\tau_u} = u$ since $t \mapsto I_t$ is continuous.



Thus,

$$\begin{aligned} \omega \in \{\sigma_u \geq t\} &\iff \sigma_u(\omega) \geq t \\ &\iff \inf\{s \geq 0 : I_s(\omega) > u\} \geq t \end{aligned}$$

$$\begin{aligned} & \stackrel{I_t \text{ cts.}}{\iff} I_t(\omega) \leq u \\ & \iff \omega \in \{I_t \leq u\} \end{aligned}$$

and

$$\begin{aligned} \omega \in \{\tau_u > t\} & \iff \tau_u(\omega) > t \\ & \iff \inf\{s \geq 0 : I_s(\omega) \geq u\} > t \\ & \stackrel{I_t \text{ cts.}}{\iff} I_t(\omega) < u \\ & \iff \omega \in \{I_t < u\}. \end{aligned}$$

(b) We have

$$\{\tau_u \leq t\} = \{\tau_u > t\}^c \stackrel{(a)}{=} \{I_t < u\} \in \mathcal{F}_t$$

and

$$\{\sigma_u \leq t\} = \bigcap_k \left\{ \sigma_u < t + \frac{1}{k} \right\} = \bigcap_k \left\{ \sigma_u \geq t + \frac{1}{k} \right\}^c = \bigcap_k \left\{ I_{t+\frac{1}{k}} \leq u \right\}^c \in \bigcap_k \mathcal{F}_{t+\frac{1}{k}} = \mathcal{F}_{t+}.$$

(c) Proof for σ : Clearly, $\sigma_u \leq \sigma_{u+\epsilon}$ for all $\epsilon \geq 0$. Thus, $\sigma_u \leq \lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon}$.

In order to show that $\sigma_u \geq \lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon}$, it is enough to check that

$$\lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon} \geq t \implies \sigma_u \geq t. \quad (*)$$

Indeed: if $\lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon} > \sigma_u$, then there is some q such that $\lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon} > q > \sigma_u$, and this contradicts (*).

Let us show (*):

$$\lim_{\epsilon \downarrow 0} \sigma_{u+\epsilon} \geq t \implies \forall \epsilon < \epsilon_0 : I_t \leq u + \epsilon \implies I_t \leq u \stackrel{(a)}{\implies} \sigma_u \geq t.$$

Proof for τ : Clearly, $\tau_{u-\epsilon} \leq \tau_u$ for all $\epsilon \geq 0$. Thus, $\tau_u \geq \lim_{\epsilon \downarrow 0} \tau_{u-\epsilon}$.

In order to show that $\tau_u \leq \lim_{\epsilon \downarrow 0} \tau_{u-\epsilon}$, it is enough to check that

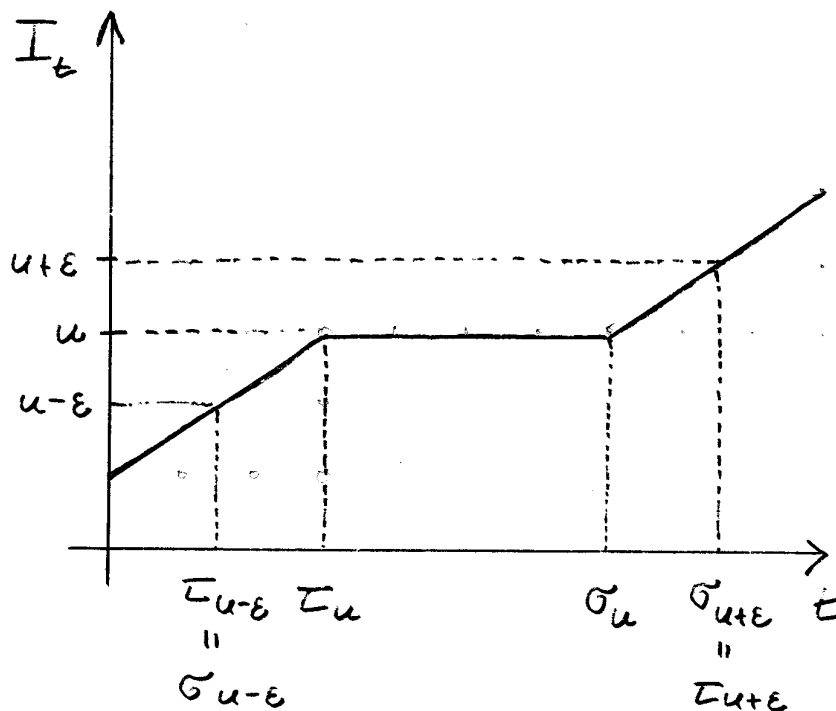
$$\lim_{\epsilon \downarrow 0} \tau_{u-\epsilon} \leq t \implies \tau_u \leq t. \quad (**)$$

Indeed: if $\lim_{\epsilon \downarrow 0} \tau_{u-\epsilon} < \tau_u$, then there is some q such that $\lim_{\epsilon \downarrow 0} \tau_{u-\epsilon} < q < \tau_u$ and this contradicts (**).

Let us show (**):

$$\lim_{\epsilon \downarrow 0} \tau_{u-\epsilon} \leq t \implies \forall \epsilon < \epsilon_0 : I_t \geq u - \epsilon \implies I_t \geq u \stackrel{(a)}{\implies} \tau_u \leq t.$$

The following picture motivates why we should expect that $\tau_u = \sigma_{u-}$:



(d) Clearly, $\sigma_{u-\epsilon} \leq \tau_u$ for all $\epsilon > 0$, i.e. $\sigma_{u-} \leq \tau_u$.

We show now $\sigma_{u-} \geq \tau_u$. For this it is enough to check that

$$\sigma_{u-} < t \implies \tau_u \leq t. \tag{***}$$

Indeed: if $\sigma_{u-} < \tau_u$, then there is some q with $\sigma_{u-} < q < \tau_u$ contradicting (***) .

Let us verify (***) . We have

$$\begin{aligned} \sigma_{u-} < t &\iff \lim_{\epsilon \downarrow 0} \sigma_{u-\epsilon} < t \\ &\implies \forall \epsilon < \epsilon_0 : \sigma_{u-\epsilon} < t \\ &\stackrel{(a)}{\implies} \forall \epsilon < \epsilon_0 : I_t > u - \epsilon \\ &\implies I_t \geq u \\ &\implies \tau_u \leq t. \end{aligned}$$

(e) Clear, since in this case I_t is continuously invertible and σ, τ are the left and right continuous inverses. ■ ■

17 Stochastic integrals: Martingale drivers

Problem 17.1. Solution: Proposition 17.2: Let X_t be a local martingale with continuous paths of bounded variation. Without loss of generality we assume that $X_0 = 0$. Let σ_n be a localizing sequence and set $X_t^n := X_{t \wedge \sigma_n}$. Clearly, X_t^n is a martingale with continuous and BV paths, so $X_t^n = X_0^n = X_0 = 0$ a.s.

Corollary 17.3: The same type of argument applies here, too we just make sure that all localizing sequences are the same. We can enforce this by using θ_n as the minimum of all localizing sequences appearing somewhere in the argument.

Problem 17.2. Solution: Let $M = (M_t)_{t \geq 0}$ be a local martingale. Without loss we assume that $M_0 = 0$. Let ρ_n be a localizing sequence and define stopping times $\sigma_n := \inf\{s \geq 0 : |M_s| > n\}$. Since $t \mapsto M_t$ is continuous, $\sigma_n \uparrow \infty$ as $n \rightarrow \infty$ and $|M_{s \wedge \sigma_n}| \leq n$, i.e. the stopped process M^{σ_n} is square integrable.

Thus, $M_t^n := M_t^{\rho_n \wedge \sigma_n}$ is a square integrable martingale satisfying the assumptions of Theorem 17.1. Set $\tau_n := \sigma_n \wedge \rho_n$. Thus, there is a unique process such that $M_{\tau_n \wedge t}^2 - \langle M^{\tau_n} \rangle_t$ is a martingale. Moreover, for any refining sequence of partitions $|\Pi_m| \rightarrow 0$ we get, in ucp-sense (uniform convergence in probability),

$$\langle M^n \rangle_t = \lim_m \sum_{t_j, t_{j-1} \in \Pi_m} (M_{t \wedge t_j \wedge \tau_n} - M_{t \wedge t_{j-1} \wedge \tau_n}).$$

This shows that $\langle M \rangle$ can be defined via $\langle M^{\tau_n} \rangle_t = \langle M \rangle_{t \wedge \tau_n} = \langle M \rangle_t^{\tau_n}$.

Of course, in the statement of Theorem 17.1 we have now to claim that the process $\langle M \rangle$ is unique, increasing, continuous and such that $M - \langle M \rangle$ is a **local** martingale.

Problem 17.3. Solution:

- a) Note that the bracket is defined for $M \pm N \in \mathcal{M}_{T, \text{loc}}^2$, so everything is well-defined. The cross-variation is BV since it is the difference of two increasing processes.
- b) By polarization we have

$$4(MN - \langle M, N \rangle) = \underbrace{[(M + N)^2 - \langle M + N \rangle]}_{\in \mathcal{M}_{T, \text{loc}}^c} - \underbrace{[(M - N)^2 - \langle M - N \rangle]}_{\in \mathcal{M}_{T, \text{loc}}^c}.$$

Uniqueness (up to constants) follows from (the localized version of) Corollary 17.3, see Problem 17.1.

c) Assume first that $M, N \in \mathcal{M}_T^{2,c}$. If we polarize the formula (17.1)

$$\langle M \rangle_t = \text{ucp-} \lim_{|\Pi| \rightarrow 0} \sum_{\Pi} (M_{t_i} - M_{t_{i-1}})^2$$

we get the claimed formula. Now use

$$\langle M, N \rangle^{\tau_k} = \langle M^{\tau_k}, N^{\tau_k} \rangle \quad \text{and Lemma 17.14.}$$

d) This follows immediately from part c) and the linearity of the limit.

e) This follows as in the case for scalar products:

$$\begin{aligned} 0 &\leq \langle \lambda M - \lambda^{-1} N, \lambda M - \lambda^{-1} N \rangle \\ &= \lambda^2 \langle M \rangle - 2 \langle M, N \rangle + \lambda^{-2} \langle N \rangle \end{aligned}$$

and choose $\lambda^2 = \sqrt{\langle N \rangle} / \sqrt{\langle M \rangle}$.

Problem 17.4. Solution:

a) Use the Chebyshev–Markov inequality.

b) Use that for any processes U, V we have

$$(U + V)_T^* \leq U_T^* + V_T^* \implies \{(U + V)_T^* > c\} \subset \{U_T^* > c/2\} \cup \{V_T^* > c/2\}.$$

c) We know that \mathcal{E}_T is $L^2(\mathbb{P})$ -dense in $\mathcal{L}_T^2(M)$. Fix $f \in \mathcal{L}_{T,\text{loc}}^2(M)$, pick some localizing sequence τ_j . Then

$$\begin{aligned} f \in \mathcal{L}_{T,\text{loc}}^2(M) &\implies f_{\bullet \wedge \tau_j} \in \mathcal{L}_T^2(M) \\ &\stackrel{\text{a)}}{\implies} \mathcal{E}_T \ni f_{\bullet \wedge \tau_j}^{\Pi(n,j)} \xrightarrow[n \rightarrow \infty]{\text{ucp}} f_{\bullet \wedge \tau_j} \\ &\stackrel{\S 17.14}{\implies} \mathcal{E}_T \ni f^{\Pi(n,n)} \xrightarrow[n \rightarrow \infty]{\text{ucp}} f. \end{aligned}$$

Problem 17.5. Solution: Throughout we assume that $f^n \rightarrow f$ in ucp-sense.

a) —part i) We have

$$\begin{aligned} \mathbb{P} \left(\sup_{t \leq T} (f - f^n)^2 \bullet \langle M \rangle_t > C \right) &= \mathbb{P} \left((f - f^n)^2 \bullet \langle M \rangle_T > C \right) \\ &\leq \mathbb{P} \left(\sup_{t \leq T} (f - f^n)_t^2 \langle M \rangle_T > C \right) \\ &= \mathbb{P} \left(\sup_{t \leq T} |f - f^n|_t > \sqrt{C / \langle M \rangle_T} \right) \\ &\xrightarrow[n \rightarrow \infty]{\text{ucp}} 0. \end{aligned}$$

This is enough to prove the next two steps. In the end we show how to use the other two steps to prove the claimed continuity.

b) By Theorem 17.13.d) we get

$$\begin{aligned} \mathbb{P}\left(\sup_{t \leq T} |(f - f^n) \bullet M_t| > \epsilon\right) &\leq \frac{4C}{\epsilon^2} + \mathbb{P}\left((f - f^n)^2 \bullet \langle M \rangle_T \geq C\right) \\ &\xrightarrow[n \rightarrow \infty]{\text{a), part i)}} \frac{4C}{\epsilon^2} \xrightarrow[C \rightarrow 0]{} 0. \end{aligned}$$

c) Let $M^n \rightarrow M$ in ucp. Define

$$\begin{aligned} \sigma_\epsilon^n &:= \inf\{t \in [0, T] : (M - M^n)_t^* > \epsilon\}, \\ \tau_k^n &:= \text{joint localizing sequence for } M^n, M. \end{aligned}$$

For all $C > 0$ we see

$$\begin{aligned} &\mathbb{P}(\langle M - M^n \rangle_T > C) \\ &\leq \mathbb{P}\left(\langle M - M^n \rangle_T > C, \underbrace{\sigma_\epsilon^n > T}_{=(M - M^n)_T^* \leq \epsilon}\right) + \mathbb{P}\left((M - M^n)_T^* > \epsilon\right) \\ &\leq \mathbb{P}\left(\langle M - M^n \rangle_T > C, \sigma_\epsilon^n > T, \tau_k^n \geq T\right) + \mathbb{P}\left(\tau_k^n < T\right) + \mathbb{P}\left((M - M^n)_T^* > \epsilon\right) \\ &\leq \mathbb{P}\left(\langle M - M^n \rangle_{T \wedge \sigma_\epsilon^n \wedge \tau_k^n} > C\right) + \mathbb{P}\left(\tau_k^n < T\right) + \mathbb{P}\left((M - M^n)_T^* > \epsilon\right) \\ &\leq \frac{1}{C^2} \mathbb{E}\left(\langle M - M^n \rangle_{T \wedge \sigma_\epsilon^n \wedge \tau_k^n}\right) + \mathbb{P}\left(\tau_k^n < T\right) + \mathbb{P}\left((M - M^n)_T^* > \epsilon\right) \\ &= \frac{1}{C^2} \mathbb{E}\left(|M - M^n|_{T \wedge \sigma_\epsilon^n \wedge \tau_k^n}^2\right) + \mathbb{P}\left(\tau_k^n < T\right) + \mathbb{P}\left((M - M^n)_T^* > \epsilon\right) \\ &\leq \underbrace{\frac{\epsilon^2}{C^2}}_{3^\circ: \epsilon \rightarrow 0} + \underbrace{\mathbb{P}\left(\tau_k^n < T\right)}_{1^\circ: k \rightarrow \infty} + \underbrace{\mathbb{P}\left((M - M^n)_T^* > \epsilon\right)}_{2^\circ: n \rightarrow \infty} \xrightarrow[k \rightarrow \infty, n \rightarrow \infty, \epsilon \rightarrow 0]{} 0 \end{aligned}$$

d) —part ii) Now we use that

$$f^n \xrightarrow{\text{ucp}} f \implies f^n \bullet M \xrightarrow{\S 17.13.b), \text{ucp}} f \bullet M \implies \langle f^n \bullet M \rangle \xrightarrow{\S 17.13.c), \text{ucp}} \langle \bullet M \rangle$$

and this is the claimed continuity of $f^2 \bullet \langle M \rangle = \langle f \bullet M \rangle$.

■ ■

Problem 17.6. Solution:

a) By definition, $|f(t, \omega)| < \infty$ for each $t \in [0, T]$ (finite left limits at $T!$). Moreover, $\sup_{t \in [0, T]} |f(t, \omega)| \leq c(\omega) < \infty$. Assume that this were not true. Then we'd find a sequence $t_n \in [0, T]$ such that $|f(t_n, \omega)| \rightarrow \infty$. Since $[0, T]$ is compact, we may assume that $t_n \rightarrow t$ (otherwise: extract convergent subsequence) and $t_n \uparrow t$ or $t_n \downarrow t$ (again: subsequence). Then we get $|f(t-, \omega)| = \infty$ or $|f(t+, \omega)| = |f(t, \omega)| = \infty$, contradicting the càdlàg property. Thus,

$$\left| \int_0^T f(t, \omega) d\langle M \rangle_t \right| \leq \int_0^T |f(t, \omega)| d\langle M \rangle_t \leq c(\omega) \langle M \rangle_T(\omega) < \infty$$

almost surely.

- b) Take a localizing sequence τ_k for M which also satisfies that $\int_0^{\tau_k} f(t) d\langle M \rangle_t < \infty$. Then we know from Theorem 17.9.c), d) that $(f^{\Pi} \bullet M)^{\tau_k} \rightarrow (f \bullet M)_k^T$ in $L^2(\mathbb{P})$, even uniformly in $t \in [0, T]$. Thus the claim follows from the localization principle for ucp-convergence, cf. Lemma 17.14.

■ ■

Problem 17.7. Solution: We have for any sufficiently large c

$$\begin{aligned} \mathbb{P} \left(\int_0^T f^2(s) d\langle M \rangle_s > c \right) &\leq \mathbb{P} \left(\int_0^T (f - f_n)^2(s) d\langle M \rangle_s > 1 \right) + \mathbb{P} \left(\int_0^T f_n^2(s) d\langle M \rangle_s > c - 1 \right) \\ &\leq \mathbb{P} \left(\sup_{s \leq T} (f - f_n)^2(s) > 1/\langle M \rangle_T \right) + \mathbb{P} \left(\int_0^T f_n^2(s) d\langle M \rangle_s > c - 1 \right) \\ &\xrightarrow[c \rightarrow \infty]{n \text{ fixed}} \mathbb{P} \left(\sup_{s \leq T} (f - f_n)^2(s) > 1/\langle M \rangle_T \right) \\ &\xrightarrow[n \rightarrow \infty]{\text{ucp}} 0. \end{aligned}$$

■ ■

Problem 17.8. Solution: By definition, there is a sequence f_n of elementary processes, i.e. of processes of the form

$$f_n(s, \omega) = \sum_j \phi_{j-1}(s) \mathbb{1}_{[t_{j-1}, t_j)}(s)$$

where ϕ_{j-1} is $\mathcal{F}_{t_{j-1}}$ measurable such that $f_n \rightarrow f$ in $L^2(\mu_T \otimes \mathbb{P})$. In particular, there is a subsequence such that

$$\lim_{k \rightarrow \infty} \int_0^t |f_{n(k)}(s)|^2 dA_s = \int_0^t |f(s)|^2 dA_s \quad \text{a.s.}$$

so that it is enough to check that the integrals $\int_0^t |f_{n(j)}(s)|^2 dA_s$ are adapted. By definition

$$\int_0^t |f_{n(j)}(s)|^2 dA_s = \sum_j \phi_{j-1}^2(A_{t_j \wedge t} - A_{t_{j-1} \wedge t})$$

and from this it is clear that the integral is \mathcal{F}_t measurable for each t .

■ ■

18 Itô's formula

Problem 18.1. Solution: We try to identify the bits and pieces as parts of Itô's formula. For $f(x) = e^x$ we get $f'(x) = f''(x) = e^x$ and so

$$e^{B_t} - 1 = \int_0^t e^{B_s} dB_s + \frac{1}{2} \int_0^t e^{B_s} ds.$$

Thus,

$$X_t = e^{B_t} - 1 - \frac{1}{2} \int_0^t e^{B_s} ds.$$

With the same trick we try to find $f(x)$ such that $f'(x) = xe^{x^2}$. A moment's thought reveals that $f(x) = \frac{1}{2} e^{x^2}$ will do. Moreover $f''(x) = e^{x^2} + 2x^2 e^{x^2}$. This then gives

$$\frac{1}{2} e^{B_t^2} - \frac{1}{2} = \int_0^t B_s e^{B_s^2} dB_s + \frac{1}{2} \int_0^t (e^{B_s^2} + 2B_s^2 e^{B_s^2}) ds$$

and we see that

$$Y_t = \frac{1}{2} \left(e^{B_t^2} - 1 - \int_0^t (e^{B_s^2} + 2B_s^2 e^{B_s^2}) ds \right).$$

Note: the integrand $B_s^2 e^{B_s^2}$ is not of class \mathcal{L}_T^2 , thus we have to use a stopping technique (as in step 4° of the proof of Itô's formula or as in Chapter 16).

Problem 18.2. Solution: For $\gamma = 1$ we get a telescoping sum

$$T = t_N - t_0 = \sum_{j=1}^N (t_j - t_{j-1}) = \sum_{j=1}^N (t_j - t_{j-1})^\gamma.$$

If $\gamma = 1 + \epsilon > 1$ we get

$$\sum_{j=1}^N (t_j - t_{j-1})^{1+\epsilon} \leq |\Pi|^\epsilon \sum_{j=1}^N (t_j - t_{j-1}) = |\Pi|^\epsilon T \xrightarrow{|\Pi| \rightarrow 0} 0,$$

and if $\gamma = 1 - \epsilon < 1$ we have

$$\sum_{j=1}^N (t_j - t_{j-1})^{1-\epsilon} \geq |\Pi|^{-\epsilon} \sum_{j=1}^N (t_j - t_{j-1}) = |\Pi|^{-\epsilon} T \xrightarrow{|\Pi| \rightarrow 0} \infty.$$

Problem 18.3. Solution: Let $0 = t_0 < t_1 < \dots < t_N = T$ be a generic partition of $[0, T]$ and write $\Delta_j = B_{t_j} - B_{t_{j-1}}$. Then we get

$$\left(\sum_j \Delta_j^2 \right)^4 = \sum_j \sum_k \sum_l \sum_m \Delta_j^2 \Delta_k^2 \Delta_l^2 \Delta_m^2$$

$$\begin{aligned}
&= c_{1,1,1,1} \sum_{j<k<l<m} \sum \sum \sum \sum \Delta_j^2 \Delta_k^2 \Delta_l^2 \Delta_m^2 \\
&\quad + c_{1,1,2} \sum_{j<k<l} \sum \sum \sum \Delta_j^2 \Delta_k^2 (\Delta_l^2)^2 + c_{1,2,1} \sum_{j<k<l} \sum \sum \sum \Delta_j^2 (\Delta_k^2)^2 \Delta_l^2 + c_{2,1,1} \sum_{j<k<l} \sum \sum \sum (\Delta_j^2)^2 \Delta_k^2 \Delta_l^2 \\
&\quad + c_{2,2} \sum_{j<k} \sum \sum (\Delta_j^2)^2 (\Delta_k^2)^2 + c_{1,3} \sum_{j<k} \sum \sum \Delta_j^2 (\Delta_k^2)^3 + c_{3,1} \sum_{j<k} \sum \sum (\Delta_j^2)^3 \Delta_k^2 \\
&\quad + c_4 \sum_j (\Delta_j^2)^4.
\end{aligned}$$

By the scaling property $\mathbb{E}(\Delta_j^2)^n = (t_j - t_{j-1})^n \mathbb{E} B_1^{2n} = \delta_j^n \mathbb{E} B_1^{2n}$ where $\delta_j = t_j - t_{j-1}$. Using the independent increments property we get

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_j \Delta_j^2 \right)^4 \right] &= \sum_j \sum_k \sum_l \sum_m \mathbb{E} \left(\Delta_j^2 \Delta_k^2 \Delta_l^2 \Delta_m^2 \right) \\
&= c'_{1,1,1,1} \sum_{j<k<l<m} \sum \sum \sum \sum \delta_j \delta_k \delta_l \delta_m \\
&\quad + c'_{1,1,2} \sum_{j<k<l} \sum \sum \sum \delta_j \delta_k \delta_l^2 + c'_{1,2,1} \sum_{j<k<l} \sum \sum \sum \delta_j \delta_k^2 \delta_l + c'_{2,1,1} \sum_{j<k<l} \sum \sum \sum \delta_j^2 \delta_k \delta_l \\
&\quad + c'_{2,2} \sum_{j<k} \sum \sum \delta_j^2 \delta_k^2 + c'_{1,3} \sum_{j<k} \sum \sum \delta_j \delta_k^3 + c'_{3,1} \sum_{j<k} \sum \sum \delta_j^3 \delta_k \\
&\quad + c'_4 \sum_j \delta_j^4 \\
&= c'_{1,1,1,1} \left(\sum_j \delta_j \right)^4 \\
&\quad + c''_{1,1,2} \sum_{j<k<l} \sum \sum \sum \delta_j \delta_k \delta_l^2 + c''_{1,2,1} \sum_{j<k<l} \sum \sum \sum \delta_j \delta_k^2 \delta_l + c''_{2,1,1} \sum_{j<k<l} \sum \sum \sum \delta_j^2 \delta_k \delta_l \\
&\quad + c''_{2,2} \sum_{j<k} \sum \sum \delta_j^2 \delta_k^2 + c''_{1,3} \sum_{j<k} \sum \sum \delta_j \delta_k^3 + c''_{3,1} \sum_{j<k} \sum \sum \delta_j^3 \delta_k \\
&\quad + c''_4 \sum_j \delta_j^4.
\end{aligned}$$

Since $\sum_j \delta_j = T$ and since we can estimate the terms containing powers of δ_j by, for example,

$$\sum_{j<k<l} \sum \sum \sum \delta_j \delta_k^2 \delta_l \leq |\Pi| \sum_{j<k<l} \sum \sum \sum \delta_j \delta_k \delta_l \leq |\Pi| \sum_j \sum_k \sum_l \delta_j \delta_k \delta_l = |\Pi| T^3 \xrightarrow{|\Pi| \rightarrow 0} 0$$

we get

$$\mathbb{E} \left[\left(\sum_j (B_{t_j} - B_{t_{j-1}})^2 \right)^4 \right] \xrightarrow{|\Pi| \rightarrow 0} c'_{1,1,1,1} T^4.$$

We will use this on page 301 (of *Brownian Motion*) when we estimate $|J_2|$:

$$|J_2|^2 \leq \max_{1 \leq l \leq N} |g(\xi_l) - g(B_{t_{l-1}})|^2 \left[\sum_{l=1}^N (B_{t_l} - B_{t_{l-1}})^2 \right]^2,$$

and taking now the Cauchy-Schwarz inequality gives

$$\mathbb{E} [J_2^2] \leq \sqrt{\mathbb{E} \left(\max_{1 \leq l \leq n} |g(\xi_l) - g(B_{t_{l-1}})|^4 \right)} \sqrt{\mathbb{E} [(S : 2^\Pi(B; [0, t]))^4]}.$$

The second factor is, however, bounded by CT^2 , see the considerations from above, and the L^2 -convergence follows.

Alternative Solution: Let $0 = t_0 < t_1 < \dots < t_n = T$ be a generic partition of $[0, T]$ and write $\Delta_j = B_{t_j} - B_{t_{j-1}}$. By the independence and stationarity of the increments, we have for any $\xi \in \mathbb{R}$

$$\mathbb{E} \exp \left(i \xi \sum_{j=1}^n \Delta_j^2 \right) = \prod_{j=1}^n \mathbb{E} \exp \left(i \xi (t_j - t_{j-1}) B_1^2 \right) = \prod_{j=1}^n \frac{1}{\sqrt{1 - 2i \xi (t_j - t_{j-1})}} =: \prod_{j=1}^n g_j(\xi)$$

using that B_1^2 is χ_1^2 -distributed. Obviously,

$$\frac{d^k}{d\xi^k} g_j(\xi) = c_k \frac{(t_j - t_{j-1})^k}{(1 - 2i \xi (t_j - t_{j-1}))^{1/2+k}}$$

for some constants c_k , $k \geq 1$, which do not depend on ξ , j . In particular,

$$\left| \frac{d^k}{d\xi^k} g_j(\xi) \right| \Big|_{\xi=0} = |c_k \cdot (t_j - t_{j-1})^k| \leq |c_k| \cdot |\Pi|^k.$$

From

$$\mathbb{E} \left[\left(\sum_{j=1}^n \Delta_j^2 \right)^4 \right] = \frac{d^4}{d\xi^4} \left[\mathbb{E} \exp \left(i \xi \sum_{j=1}^n \Delta_j^2 \right) \right] \Big|_{\xi=0}$$

we conclude, by applying Leibniz' product rule,

$$\mathbb{E} \left[\left(\sum_{j=1}^n \Delta_j^2 \right)^4 \right] = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=4}} \left(C_\alpha \prod_{j=1}^n \frac{d^{\alpha_j}}{d\xi^{\alpha_j}} g_j(\xi) \right) \Big|_{\xi=0} \leq C \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=4}} \left(C_\alpha \prod_{j=1}^n \underbrace{(t_j - t_{j-1})^{\alpha_j}}_{\leq |\Pi|^{\alpha_j}} \right) \leq C |\Pi|^4 n^4.$$

By the definition of the mesh size we have $n \leq T/|\Pi|$; thus,

$$\mathbb{E} \left[\left(\sum_{j=1}^n \Delta_j^2 \right)^4 \right] \leq CT^4.$$

Note that the constant C does not depend on the partition Π . The rest of the proof follows as in the preceding solution. ■ ■

Problem 18.4. Solution: Multiply out the outer square: $(\sum_l s_l)^2 = \sum_l \sum_m s_l s_m$. In the last equality only the pure squares survive when we multiply out the outer square. This is due to the fact that $(B_t^2 - t)_{t \geq 0}$ is a martingale: Indeed, if $l < m$, then $t_{l-1} < t_l \leq t_{m-1} < t_m$. For brevity we write

$$g_{l-1} := g(B_{t_{l-1}}), \quad \Delta_l B := B_{t_l} - B_{t_{l-1}}, \quad \Delta_l t := t_l - t_{l-1}.$$

By the tower property,

$$\mathbb{E} \left(g_{l-1} \{ (\Delta_l B)^2 - \Delta_l t \} g_{m-1} \{ (\Delta_m B)^2 - \Delta_m t \} \right)$$

$$\begin{aligned}
 &= \mathbb{E} \left(\mathbb{E} \left[g_{l-1} \{ (\Delta_l B)^2 - \Delta_l t \} g_{m-1} \{ (\Delta_m B)^2 - \Delta_m t \} \mid \mathcal{F}_{t_{m-1}} \right] \right) \\
 &= \mathbb{E} \left(\underbrace{g_{l-1} \{ (\Delta_l B)^2 - \Delta_l t \} g_{m-1}}_{\mathcal{F}_{t_{m-1}} \text{ measurable}} \cdot \underbrace{\mathbb{E} \left[\{ (\Delta_m B)^2 - \Delta_m t \} \mid \mathcal{F}_{t_{m-1}} \right]}_{=0 \text{ martingale, cf. 5.2.c}} \right) = 0,
 \end{aligned}$$

i.e. the mixed terms break away. ■ ■

Problem 18.5. Solution:

- a) Assume first that $f \in \mathcal{C}_b^1$. Let $\Pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ be any partition. We have

$$B_t f(B_t) = \sum_{l=1}^n (B_{t_l} f(B_{t_l}) - B_{t_{l-1}} f(B_{t_{l-1}})).$$

Using

$$\begin{aligned}
 &B_t f(B_t) - B_s f(B_s) \\
 &= B_s (f(B_t) - f(B_s)) + f(B_s) (B_t - B_s) + (f(B_t) - f(B_s)) (B_t - B_s) \\
 &= B_s (f(B_t) - f(B_s)) + f(B_s) (B_t - B_s) + f'(\xi) (B_t - B_s)^2
 \end{aligned}$$

with some intermediate value ξ between B_s and B_t , the identity follows.

Letting $|\Pi| \rightarrow 0$ in this identity, we see that the left-hand side converges (it is constant!) and the second and third term on the right converge, in probability, to

$$\int_0^t f(B_s) dB_s \quad \text{and} \quad \int_0^t f'(B_s) ds,$$

respectively, cf. Lemma 18.4. Therefore, the first term has to converge, i.e.

$$\int_0^t B_s df(B_s)$$

makes sense (and this is all we need!).

If f' is not bounded, we can use a stopping and cutting technique as in the proof of Theorem 18.1 (step 4°).

- (b) This follows from (a) after having taken the limit.
 (c) Applying (b) to $f(x) = x^{n-1}$ gives

$$dB_t^n = d(B_t^{n-1} B_t) = B_t dB_t^{n-1} + B_t^{n-1} dB_t + (n-1) B_t^{n-2} dt$$

and iterating this yields

$$dB_t^n = n B_t^{n-1} dB_t + \frac{1}{2} n(n-1) B_t^{n-2} dt = n p_{n-1}(B_t) dB_t + \frac{1}{2} p_n''(B_t) dt$$

where we use $p_n(x) = x^n$ for the monomial of order n . Since the Itô integral is linear, we get the claim for all polynomials of any order.

(d) This follows directly from Itô's isometry:

$$\mathbb{E} \left[\left| \int_0^T (g(s) - g_n(s)) dB_s \right|^2 \right] = \int_0^T \mathbb{E} [|g(s) - g_n(s)|^2] ds.$$

(e) We can assume that f has compact support, $\text{supp } f \subset [-K, K]$, say. Otherwise, we use the stopping and cutting technique from the proof (step 4°) of Theorem 18.1, to remove this assumption.

The classical version of Weierstraß' approximation theorem tells us that f can be uniformly approximated on $[-K, K]$ by a sequence of polynomials $(p_n^f)_n$, see e.g. [18, Theorem 28.6]. We apply this theorem to f'' and observe that f' and f are still uniformly approximated by the primitives $P_n^f := \int p_n^f$ and $Q_n^f := \int P_n^f = \iint p_n^f$.

The rest follows from the previous step (d) which allows us to interchange (stochastic) integration and limits. (The Riemann part in Itô's formula is clear, since we have uniform convergence!).

Problem 18.6. Solution:

a) Set $F(x, y) = xy$ and $G(t) = (f(t), g(t))$.

Then $f(t)g(t) = F \circ G(t)$. If we differentiate this using the chain rule we get

$$\frac{d}{dt} (F \circ G) = \partial_x F \circ G(t) \cdot f'(t) + \partial_y F \circ G(t) \cdot g'(t) = g(t) \cdot f'(t) + f(t) \cdot g'(t)$$

(surprised?) and if we integrate this up we see

$$\begin{aligned} F \circ G(t) - F \circ G(0) &= \int_0^t f(s)g'(s) ds + \int_0^t g(s)f'(s) ds \\ &= \int_0^t f(s) dg(s) + \int_0^t g(s) df(s). \end{aligned}$$

Note: For the first equality we have to assume that f', g' exist Lebesgue a.e. and that their primitives are f and g , respectively. This is tantamount to saying that f, g are absolutely continuous with respect to Lebesgue measure.

b) $f(x, y) = xy$. Then $\partial_x f(x, y) = y, \partial_y f(x, y) = x$ and $\partial_x \partial_y f(x, y) = \partial_y \partial_x f(x, y) = 1$ and $\partial_x^2 f(x, y) = \partial_y^2 f(x, y) = 0$. Thus, the two-dimensional Itô formula yields

$$\begin{aligned} b_t \beta_t &= \int_0^t b_s d\beta_s + \int_0^t \beta_s db_s + \\ &\quad + \frac{1}{2} \int_0^t \partial_x^2 f(b_s, \beta_s) ds + \frac{1}{2} \int_0^t \partial_y^2 f(b_s, \beta_s) ds + \int_0^t \partial_x \partial_y f(b_s, \beta_s) d\langle b, \beta \rangle_s \\ &= \int_0^t b_s d\beta_s + \int_0^t \beta_s db_s + \langle b, \beta \rangle_t. \end{aligned}$$

If $b \perp \beta$ we have $\langle b, \beta \rangle \equiv 0$ (note our Itô formula has no mixed second derivatives!) and we get the formula as in the statement. Otherwise we have to take care of

$\langle b, \beta \rangle$. This is not so easy to calculate since we need more information on the joint distribution. In general, we have

$$\langle b, \beta \rangle_t = \lim_{|\Pi| \rightarrow 0} \sum_{t_j, t_{j-1}} (b(t_j) - b(t_{j-1}))(\beta(t_j) - \beta(t_{j-1})).$$

Where Π stands for a partition of the interval $[0, t]$.

Problem 18.7. Solution: The following proof works without changes if $f \in \mathcal{C}^{2,2}$. **Formally** it works in $\mathcal{C}^{1,2}$, too, but for this we need some justification. Here it is:

- *üheither you go through the proof of the Itô formula and you see that, whenever we deal with the t -coordinate, we only need derivatives up to order one.*
- *or your use that $\mathcal{C}^{2,2}$ is dense in $\mathcal{C}^{1,2}$, you work first in $\mathcal{C}^{2,2}$ and then approximate. This will work since the final result holds for $\mathcal{C}^{1,2}$*

A bit more details are given in the sketch of the proof of Theorem 18.11.

The main point of this exercise is that you learn that the extended process (t, B_t) is sometimes a good choice to play with.

Consider the two-dimensional Itô process $X_t = (t, B_t)$ with parameters

$$\sigma \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad b \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Applying the Itô formula (18.14) we get

$$\begin{aligned} f(t, B_t) - f(0, 0) &= f(X_t) - f(X_0) \\ &= \int_0^t \left(\partial_1 f(X_s) \sigma_{11} + \partial_2 f(X_s) \sigma_{21} \right) dB_s \\ &\quad + \int_0^t \left(\partial_1 f(X_s) b_1 + \partial_2 f(X_s) b_2 + \frac{1}{2} \partial_2 \partial_2 f(X_s) \sigma_{21}^2 \right) ds \\ &= \int_0^t \partial_2 f(X_s) dB_s + \int_0^t \left(\partial_1 f(X_s) b_1 + \frac{1}{2} \partial_2 \partial_2 f(X_s) \right) ds \\ &= \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds. \end{aligned}$$

In the same way we obtain the d -dimensional counterpart:

Let $(B_t^1, \dots, B_t^d)_{t \geq 0}$ be a BM^d and $f : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. Consider the $(d+1)$ -dimensional Itô process $X_t = (t, B_t^1, \dots, B_t^d)$ with parameters

$$\sigma \in \mathbb{R}^{(d+1) \times d}, \quad \sigma_{ik} = \begin{cases} 1, & \text{if } i = k + 1; \\ 0, & \text{else;} \end{cases} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The multidimensional Itô formula (18.14) yields

$$\begin{aligned}
 & f(t, B_t^1, \dots, B_t^d) - f(0, 0, \dots, 0) \\
 &= f(X_t) - f(X_0) \\
 &= \sum_{k=1}^d \int_0^t \left[\sum_{j=1}^{d+1} \partial_j f(X_s) \sigma_{jk} \right] dB_s^k + \sum_{j=1}^{d+1} \int_0^t \partial_j f(X_s) b_j ds + \frac{1}{2} \sum_{i,j=1}^{d+1} \int_0^t \partial_i \partial_j f(X_s) \sum_{k=1}^d \sigma_{ik} \sigma_{jk} ds \\
 &= \sum_{k=1}^d \int_0^t \partial_{k+1} f(X_s) dB_s^k + \int_0^t \partial_1 f(X_s) ds + \frac{1}{2} \sum_{j=2}^{d+1} \int_0^t \partial_j \partial_j f(X_s) ds \\
 &= \sum_{k=1}^d \int_0^t \frac{\partial f}{\partial x_k}(s, B_s^1, \dots, B_s^d) dB_s^k + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s^1, \dots, B_s^d) + \frac{1}{2} \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(s, B_s^1, \dots, B_s^d) \right) ds.
 \end{aligned}$$

■ ■

Problem 18.8. Solution: Let $B_t = (B_t^1, \dots, B_t^d)$ be a BM^d and $f \in \mathcal{C}^{1,2}((0, \infty) \times \mathbb{R}^d, \mathbb{R})$ as in Theorem 5.6. Then the multidimensional time-dependent Itô's formula shown in Problem 18.7 yields

$$\begin{aligned}
 M_t^f &= f(t, B_t) - f(0, B_0) - \int_0^t Lf(s, B_s) ds \\
 &= f(t, B_t) - f(0, B_0) - \int_0^t \left(\frac{\partial}{\partial t} f(s, B_s) + \frac{1}{2} \Delta_x f(s, B_s) \right) ds \\
 &= \sum_{k=1}^d \int_0^t \frac{\partial f}{\partial x_k}(s, B_s^1, \dots, B_s^d) dB_s^k.
 \end{aligned}$$

By Theorem 15.15 it follows that M_t^f is a martingale (note that the assumption (5.5) guarantees that the integrand is of class \mathcal{L}_T^2 !)

■ ■

Problem 18.9. Solution: First we show that $X_t = e^{t/2} \cos B_t$ is a martingale. We use the time-dependent Itô's formula from Problem 18.7. Therefore, we set $f(t, x) = e^{t/2} \cos x$. Then

$$\frac{\partial f}{\partial t}(t, x) = \frac{1}{2} e^{t/2} \cos x, \quad \frac{\partial f}{\partial x}(t, x) = -e^{t/2} \sin x, \quad \frac{\partial^2 f}{\partial x^2}(t, x) = -e^{t/2} \cos x.$$

Hence we obtain

$$\begin{aligned}
 X_t &= e^{t/2} \cos B_t = f(t, B_t) - f(0, 0) + 1 \\
 &= \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + 1 \\
 &= - \int_0^t e^{s/2} \sin B_s dB_s + \int_0^t \left(\frac{1}{2} e^{s/2} \cos B_s - \frac{1}{2} e^{s/2} \cos B_s \right) ds + 1 \\
 &= - \int_0^t e^{s/2} \sin B_s dB_s + 1,
 \end{aligned}$$

and the claim follows from Theorem 15.15.

Analogously, we show that $Y_t = (B_t + t)e^{-B_t - t/2}$ is a martingale. We set $f(t, x) = (x + t)e^{-x - t/2}$. Then

$$\begin{aligned}\frac{\partial f}{\partial t}(t, x) &= e^{-x - t/2} - \frac{1}{2}(x + t)e^{-x - t/2}, \\ \frac{\partial f}{\partial x}(t, x) &= e^{-x - t/2} - (x + t)e^{-x - t/2}, \\ \frac{\partial f}{\partial x^2}(t, x) &= -2e^{-x - t/2} + (x + t)e^{-x - t/2}.\end{aligned}$$

By the time-dependent Itô's formula we have

$$\begin{aligned}Y_t &= (B_t + t)e^{-B_t - t/2} \\ &= f(t, B_t) - f(0, 0) \\ &= \int_0^t \left(e^{-B_s - s/2} - (B_s + s)e^{-B_s - s/2} \right) dB_s + \\ &\quad + \int_0^t \left(e^{-B_s - s/2} - \frac{1}{2}(B_s + s)e^{-B_s - s/2} + \frac{1}{2}(-2e^{-B_s - s/2} + (B_s + s)e^{-B_s - s/2}) \right) ds \\ &= \int_0^t \left(e^{-B_s - s/2} - (B_s + s)e^{-B_s - s/2} \right) dB_s.\end{aligned}$$

Again, from Theorem 15.15 we deduce that Y_t is a martingale. ■ ■

Problem 18.10. Solution:

- a) The stochastic integrals exist if b_s/r_s and β_s/r_s are in \mathcal{L}_T^2 . As $|b_s/r_s| \leq 1$ we get

$$\|b/r\|_{L^2(\lambda_T \otimes \mathbb{P})}^2 = \int_0^T [\mathbb{E}(|b_s/r_s|^2)] ds \leq \int_0^T 1 ds = T < \infty.$$

Since b_s/r_s is adapted and has continuous sample paths, it is progressive and so an element of \mathcal{L}_T^2 . Analogously, $|\beta_s/r_s| \leq 1$ implies $\beta_s/r_s \in \mathcal{L}_T^2$.

- b) We use Lévy's characterization of a BM¹, Theorem 9.13 or 19.5. From Theorem 15.15 it follows that

- $t \mapsto \int_0^t b_s/r_s db_s, t \mapsto \int_0^t \beta_s/r_s d\beta_s$ are continuous; thus $t \mapsto W_t$ is a continuous process.
- $\int_0^t b_s/r_s db_s, \int_0^t \beta_s/r_s d\beta_s$ are square integrable martingales, and so is W_t .
- the quadratic variation is given by

$$\begin{aligned}\langle W \rangle_t &= \langle b/r \bullet b \rangle_t + \langle \beta/r \bullet \beta \rangle_t \\ &= \int_0^t b_s^2/r_s^2 ds + \int_0^t \beta_s^2/r_s^2 ds \\ &= \int_0^t \frac{b_s^2 + \beta_s^2}{r_s^2} ds \\ &= \int_0^t ds = t,\end{aligned}$$

i.e. $(W_t^2 - t)_{t \geq 0}$ is a martingale.

Therefore, W_t is a BM¹.

Note, that the above processes can be used to calculate Lévy's stochastic area formula, see Protter [13, Chapter II, Theorem 43]

Problem 18.11. Solution: The function $f = u + iv$ is analytic, and as such it satisfies the Cauchy–Riemann equations, see e.g. Rudin [16, Theorem 11.2],

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

First, we show that $u(b_t, \beta_t)$ is a BM¹. Therefore we apply Itô's formula

$$\begin{aligned} & u(b_t, \beta_t) - u(b_0, \beta_0) \\ &= \int_0^t u_x(b_s, \beta_s) db_s + \int_0^t u_y(b_s, \beta_s) d\beta_s + \frac{1}{2} \int_0^t (u_{xx}(b_s, \beta_s) + u_{yy}(b_s, \beta_s)) ds \\ &= \int_0^t u_x(b_s, \beta_s) db_s + \int_0^t u_y(b_s, \beta_s) d\beta_s, \end{aligned}$$

where the last term cancels as $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Theorem 15.15 implies

- $t \mapsto u(b_t, \beta_t) = \int_0^t u_x(b_s, \beta_s) db_s + \int_0^t u_y(b_s, \beta_s) d\beta_s$ is a continuous process.
- $\int_0^t u_x(b_s, \beta_s) db_s, \int_0^t u_y(b_s, \beta_s) d\beta_s$ are square integrable martingales, and so $u(b_t, \beta_t)$ is a square integrable martingale.
- the quadratic variation is given by

$$\begin{aligned} \langle u(b, \beta) \rangle_t &= \langle u_x(b, \beta) \bullet b \rangle_t + \langle u_y(b, \beta) \bullet \beta \rangle_t \\ &= \int_0^t u_x^2(b_s, \beta_s) ds + \int_0^t u_y^2(b_s, \beta_s) ds = \int_0^t 1 ds = t, \end{aligned}$$

i.e. $(u^2(b_t, \beta_t) - t)_{t \geq 0}$ is a martingale.

Due to Lévy's characterization of a BM¹, Theorem 9.13 or 19.5, we know that $u(b_t, \beta_t)$ is a BM¹. Analogously, we see that $v(b_t, \beta_t)$ is also a BM¹. Just note that, due to the Cauchy–Riemann equations we get from $u_x^2 + u_y^2 = 1$ also $v_y^2 + v_x^2 = 1$.

The quadratic covariation is (we drop the arguments, for brevity):

$$\begin{aligned} \langle u, v \rangle_t &= \frac{1}{4} (\langle u + v \rangle_t - \langle u - v \rangle_t) \\ &= \frac{1}{4} \left(\int_0^t (u_x + v_x)^2 ds + \int_0^t (u_y + v_y)^2 ds - \int_0^t (u_x - v_x)^2 ds - \int_0^t (u_y - v_y)^2 ds \right) \\ &= \int_0^t (u_x v_x + u_y v_y) ds \\ &= \int_0^t (-v_y u_y + u_y v_y) ds = 0. \end{aligned}$$

As an abbreviation we write $u_t = u(b_t, \beta_t)$ and $v_t = v(b_t, \beta_t)$. Applying Itô's formula to the function $g(u_t, v_t) = e^{i(\xi u_t + \eta v_t)}$ and $s < t$ yields

$$g(u_t, v_t) - g(u_s, v_s) = i\xi \int_s^t g(u_r, v_r) du_r + i\eta \int_s^t g(u_r, v_r) dv_r - \frac{1}{2} (\xi^2 + \eta^2) \int_s^t g(u_r, v_r) dr,$$

as the quadratic covariation $\langle u, v \rangle_t = 0$. Since $|g| \leq 1$ and since $g(u_t, v_t)$ is progressive, the integrand is in \mathcal{L}_T^2 and the above stochastic integrals exist. From Theorem 15.15 we deduce that

$$\mathbb{E} \left(\int_s^t g(u_r, v_r) du_r \mathbb{1}_F \right) = 0 \quad \text{and} \quad \mathbb{E} \left(\int_s^t g(u_r, v_r) dv_r \mathbb{1}_F \right) = 0.$$

for all $F \in \sigma(u_r, v_r : r \leq s) =: \mathcal{F}_s$. If we multiply the above equality by $e^{-i(\xi u_s + \eta v_s)} \mathbb{1}_F$ and take expectations, we get

$$\underbrace{\mathbb{E} \left(g(u_t - u_s, v_t - v_s) \mathbb{1}_F \right)}_{=\Phi(t)} = \mathbb{P}(F) - \frac{1}{2}(\xi^2 + \eta^2) \int_0^t \underbrace{\mathbb{E} \left(g(u_r - u_s, v_r - v_s) \mathbb{1}_F \right)}_{=\Phi(r)} dr.$$

Since this integral equation has a unique solution (use Gronwall's lemma, Theorem A.42), we get

$$\begin{aligned} \mathbb{E} \left(e^{i(\xi(u_t - u_s) + \eta(v_t - v_s))} \mathbb{1}_F \right) &= \mathbb{P}(F) e^{-\frac{1}{2}(t-s)(\xi^2 + \eta^2)} \\ &= \mathbb{P}(F) e^{-\frac{1}{2}(t-s)\xi^2} e^{-\frac{1}{2}(t-s)\eta^2} \\ &= \mathbb{P}(F) \mathbb{E} \left(e^{i\xi(u_t - u_s)} \right) \mathbb{E} \left(e^{i\eta(v_t - v_s)} \right). \end{aligned}$$

From this we deduce with Lemma 5.4 that $(u(b_t, \beta_t), v(b_t, \beta_t))$ is a BM².

Note that the above calculation is essentially the proof of Lévy's characterization theorem. Only a few modifications are necessary for the proof of the multidimensional version, see e.g. Karatzas, Shreve [10, Theorem 3.3.16].

■ ■

Problem 18.12. Solution: Let $X_t = \int_0^t \sigma(s) dB_s + \int_0^t b(s) ds$ be an d -dimensional Itô process. Assuming that $f = u + iv$ and thus $u = \operatorname{Re} f = \frac{1}{2} f + \frac{1}{2} \bar{f}$ and $v = \operatorname{Im} f = \frac{1}{2i} f - \frac{1}{2i} \bar{f}$ are \mathcal{C}^2 -functions, we may apply the real d -dimensional Itô formula (18.14) to the functions $u, v : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\begin{aligned} f(X_t) - f(X_0) &= u(X_t) - u(X_0) + i(v(X_t) - v(X_0)) \\ &= \int_0^t \nabla u(X_s)^\top \sigma(s) dB_s + \int_0^t \nabla u(X_s)^\top b(s) ds + \frac{1}{2} \int_0^t \operatorname{trace}(\sigma(s)^\top D^2 u(X_s) \sigma(s)) ds \\ &\quad + i \left(\int_0^t \nabla v(X_s)^\top \sigma(s) dB_s + \int_0^t \nabla v(X_s)^\top b(s) ds + \frac{1}{2} \int_0^t \operatorname{trace}(\sigma(s)^\top D^2 v(X_s) \sigma(s)) ds \right) \\ &= \int_0^t \nabla f(X_s)^\top \sigma(s) dB_s + \int_0^t \nabla f(X_s)^\top b(s) ds + \frac{1}{2} \int_0^t \operatorname{trace}(\sigma(s)^\top D^2 f(X_s) \sigma(s)) ds, \end{aligned}$$

by the linearity of the differential operators and the (stochastic) integral.

■ ■

Problem 18.13. Solution:

- a) By definition we have $\text{supp } \chi \subset [-1, 1]$ hence it is obvious that for $\chi_n(x) := n\chi(nx)$ we have $\text{supp } \chi_n \subset [-1/n, 1/n]$. Substituting $y = nx$ we get

$$\int_{-1/n}^{1/n} \chi_n(x) dx = \int_{-1/n}^{1/n} n\chi(nx) dx = \int_{-1}^1 \chi(y) dy = 1$$

- b) For derivatives of convolutions we know that $\partial(f \star \chi_n) = f \star (\partial\chi_n)$. Hence we obtain

$$\begin{aligned} |\partial^k f_n(x)| &= |f \star (\partial^k \chi_n)(x)| \\ &= \left| \int_{\mathbb{B}(x, 1/n)} f(y) \partial^k \chi_n(x-y) dy \right| \\ &\leq \sup_{y \in \mathbb{B}(x, 1/n)} |f(y)| \int_{\mathbb{R}} n |\partial^k \chi(n(x-y))| dy \\ &= \sup_{y \in \mathbb{B}(x, 1/n)} |f(y)| \int_{\mathbb{R}} n^k |\partial^k \chi(z)| dz \\ &= \sup_{y \in \mathbb{B}(x, 1/n)} |f(y)| n^k \|\partial^k \chi\|_{L^1}, \end{aligned}$$

where we substituted $z = n(y-x)$ in the penultimate step.

- c) For $x \in \mathbb{R}$ we have

$$\begin{aligned} |f \star \chi_n(x) - f(x)| &= \left| \int_{\mathbb{R}} (f(y) - f(x)) \chi_n(x-y) dy \right| \\ &\leq \sup_{y \in \mathbb{B}(x, 1/n)} |f(y) - f(x)| \cdot \|\chi\|_{L^1} \\ &= \sup_{y \in \mathbb{B}(x, 1/n)} |f(y) - f(x)|. \end{aligned}$$

This shows that $\lim_{n \rightarrow \infty} |f \star \chi_n(x) - f(x)| = 0$, i.e. $\lim_{n \rightarrow \infty} f \star \chi_n(x) = f(x)$, at all x where f is continuous.

- d) Using the above result and taking the supremum over all $x \in \mathbb{R}$ we get

$$\sup_{x \in \mathbb{R}} |f \star \chi_n(x) - f(x)| \leq \sup_{x \in \mathbb{R}} \sup_{y \in \mathbb{B}(x, 1/n)} |f(y) - f(x)|.$$

Thus $\lim_{n \rightarrow \infty} \|f \star \chi_n - f\|_{\infty} = 0$ whenever the function f is uniformly continuous. ■ ■

Problem 18.14. Solution: We follow the hint and use Lévy's characterization of a BM¹, Theorem 9.13 or 19.5.

- $t \mapsto \beta_t$ is a continuous process.
- the integrand $\text{sgn } B_s$ is bounded, hence it is in \mathcal{L}_T^2 for any $T > 0$.
- by Theorem 15.15 β_t is a square integrable martingale
- by Theorem 15.15 the quadratic variation is given by

$$\langle \beta \rangle_t = \left\langle \int_0^\bullet \text{sgn}(B_s) dB_s \right\rangle_t = \int_0^t (\text{sgn}(B_s))^2 ds = \int_0^t ds = t,$$

i.e. $(\beta_t^2 - t)_{t \geq 0}$ is also a martingale.

Thus, β is a BM¹.

Problem 18.15. Solution: 1° — Consider the Itô processes

$$dX_j(t) = \sigma_j(t) dB_t + b_j(t) dt, \quad X_j(0) = 0, \quad (j = 1, 2)$$

where $\sigma_j \in \mathcal{L}_T^2$ and b_j is bounded. Then we get from the two-dimensional Itô's formula

$$X_1(t)X_2(t) = \int_0^t \sigma_1(s)\sigma_2(s) ds + \int_0^t X_1(s) dX_2(s) + \int_0^t X_2(s) dX_1(s).$$

Taking expectations, the martingale parts containing $dB(s)$ vanish, so

$$\mathbb{E}(X_1(t)X_2(t)) = \mathbb{E} \int_0^t \sigma_1(s)\sigma_2(s) ds + \mathbb{E} \int_0^t X_1(s) b_2(s) ds + \mathbb{E} \int_0^t X_2(s) b_1(s) ds.$$

2° — Now let $X_1 = f \bullet B$ and $X_2 = \Phi(g \bullet B)$ with $\Phi \in \mathcal{C}_b^2(\mathbb{R})$. Then, by Itô's formula (18.1),

$$\begin{aligned} dX_1(t) &= f(t) dB_t, \\ dX_2(t) &= d\Phi(g \bullet B_t) = \Phi'(g \bullet B_t) g(t) dB_t + \frac{1}{2} \Phi''(g \bullet B_t) g^2(t) dt. \end{aligned}$$

3° — Combining steps 1° and 2° gives

$$\begin{aligned} \mathbb{E} \left(\int_0^t f(r) dB_r \cdot \Phi \left(\int_0^t g(r) dB_r \right) \right) &= \mathbb{E} \left(\int_0^t f(s)g(s) \Phi' \left(\int_0^s g(r) dB_r \right) ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^t \int_0^s f(r) dB_r \Phi'' \left(\int_0^s g(r) dB_r \right) g^2(s) ds \right) \end{aligned}$$

Problem 18.16. Solution: Let $\sigma_\Pi, b_\Pi \in \mathcal{E}_T$ such that $\sigma_\Pi \xrightarrow[|\Pi| \rightarrow 0]{L^2(\lambda_T \otimes \mathbb{P})} \sigma$, $b_\Pi \xrightarrow[|\Pi| \rightarrow 0]{L^2(\lambda_T \otimes \mathbb{P})} b$. By the Chebyshev inequality, Doob's maximal inequality, and Itô's isometry, we have

$$\begin{aligned} \mathbb{P} \left(\sup_{t \leq T} \left| \int_0^t g(X_s^\Pi) \sigma_\Pi(s) dB_s - \int_0^t g(X_s) \sigma(s) dB_s \right| > \epsilon \right) \\ \leq \frac{1}{\epsilon^2} \mathbb{E} \left(\sup_{t \leq T} \left| \int_0^t (g(X_s^\Pi) \sigma_\Pi(s) - g(X_s) \sigma(s)) dB_s \right|^2 \right) \\ \leq \frac{4}{\epsilon^2} \mathbb{E} \left(\left| \int_0^T (g(X_s^\Pi) \sigma_\Pi(s) - g(X_s) \sigma(s)) dB_s \right|^2 \right) \\ = \frac{4}{\epsilon^2} \mathbb{E} \left(\int_0^T |g(X_s^\Pi) \sigma_\Pi(s) - g(X_s) \sigma(s)|^2 ds \right). \end{aligned}$$

From

$$g(X_s^\Pi) \sigma_\Pi(s) - g(X_s) \sigma(s) = g(X_s^\Pi) (\sigma_\Pi(s) - \sigma(s)) - \sigma(s) (g(X_s) - g(X_s^\Pi))$$

and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we conclude

$$\mathbb{P} \left(\sup_{t \leq T} \left| \int_0^t g(X_s^\Pi) \sigma_\Pi(s) dB_s - \int_0^t g(X_s) \sigma(s) dB_s \right| > \epsilon \right)$$

$$\begin{aligned} &\leq \frac{8}{\epsilon^2} \mathbb{E} \left(\int_0^T g(X_s^\Pi)^2 |\sigma_\Pi(s) - \sigma(s)|^2 ds \right) + \frac{8}{\epsilon^2} \mathbb{E} \left(\int_0^T \sigma(s)^2 |g(X_s^\Pi) - g(X_s)|^2 ds \right) \\ &=: I_1 + I_2. \end{aligned}$$

Since g is bounded and $\sigma_\Pi \xrightarrow[|\Pi| \rightarrow 0]{L^2(\lambda_T \otimes \mathbb{P})} \sigma$, it follows that $I_1 \rightarrow 0$ as $|\Pi| \rightarrow 0$. For the second term we note that, by Lemma 18.5,

$$\sup_{s \leq T} |g(X_s^\Pi) - g(X_s)| \xrightarrow[|\Pi| \rightarrow 0]{\mathbb{P}} 0.$$

Hence, by Vitali's convergence theorem, $I_2 \rightarrow 0$ as $|\Pi| \rightarrow 0$.

A similar, but simpler, calculation shows

$$\mathbb{P} \left(\sup_{t \leq T} \left| \int_0^t g(X_s^\Pi) b_\Pi(s) ds - \int_0^t g(X_s) b(s) ds \right| > \epsilon \right) \xrightarrow[|\Pi| \rightarrow 0]{} 0.$$

Consequently,

$$\mathbb{P} \left(\sup_{t \leq T} \left| \int_0^t g(X_s^\Pi) dX_s^\Pi - \int_0^t g(X_s) dX_s \right| > \epsilon \right) \xrightarrow[|\Pi| \rightarrow 0]{} 0.$$

Problem 18.17. Solution: Use Corollary 17.3 and localize it, choosing a common localization sequence for all objects appearing in the statement.

19 Applications of Itô's formula

Problem 19.1. Solution: Lemma. Let $(B_t, \mathcal{F}_t)_{t \geq 0}$ be a BM^d , $f = (f_1, \dots, f_d)$, $f_j \in L^2_{\mathbb{P}}(\lambda_T \otimes \mathbb{P})$ for all $T > 0$, and assume that $|f_j(s, \omega)| \leq C$ for some $C > 0$ and all $s \geq 0, 1 \leq j \leq d$, and $\omega \in \Omega$. Then

$$\exp\left(\sum_{j=1}^d \int_0^t f_j(s) dB_s^j - \frac{1}{2} \sum_{j=1}^d \int_0^t f_j^2(s) ds\right), \quad t \geq 0, \quad (19.1)$$

is a martingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Proof. Set $X_t = \sum_{j=1}^d \int_0^t f_j(s) dB_s^j - \frac{1}{2} \sum_{j=1}^d \int_0^t f_j^2(s) ds$. Itô's formula, Theorem 18.7, yields

$$\begin{aligned} e^{X_t} - 1 &= \sum_{j=1}^d \int_0^t e^{X_s} f_j(s) dB_s^j - \frac{1}{2} \sum_{j=1}^d \int_0^t e^{X_s} f_j^2(s) ds + \frac{1}{2} \sum_{j=1}^d \int_0^t e^{X_s} f_j^2(s) ds \\ &= \sum_{j=1}^d \int_0^t \exp\left(\sum_{k=1}^d \int_0^s f_k(r) dB_r^k - \frac{1}{2} \sum_{k=1}^d \int_0^s f_k^2(r) dr\right) f_j(s) dB_s^j \\ &= \sum_{j=1}^d \int_0^t \prod_{k=1}^d \exp\left(\int_0^s f_k(r) dB_r^k - \frac{1}{2} \int_0^s f_k^2(r) dr\right) f_j(s) dB_s^j. \end{aligned}$$

If we can show that the integrand is in $L^2_{\mathbb{P}}(\lambda_T \otimes \mathbb{P})$ for every $T > 0$, then Theorem 15.15 applies and shows that the stochastic integral, hence e^{X_t} , is a martingale.

We will see that we can reduce the d -dimensional setting to a one-dimensional setting. The essential step in the proof is the analogue of the estimate on page 250, line 6 from above. In the d -dimensional setting we have for each $k = 1, \dots, d$

$$\begin{aligned} \mathbb{E} \left[\left| e^{\sum_{j=1}^d \int_0^T f_j(r) dB_r^j - \frac{1}{2} \sum_{j=1}^d \int_0^T f_j^2(r) dr} f_k(T) \right|^2 \right] &\leq C^2 \mathbb{E} \left[e^{2 \sum_{j=1}^d \int_0^T f_j(r) dB_r^j} \right] \\ &= C^2 \mathbb{E} \left[\prod_{j=1}^d e^{2 \int_0^T f_j(r) dB_r^j} \right] \\ &\leq C^2 \prod_{j=1}^d \left(\mathbb{E} \left[e^{2d \int_0^T f_j(r) dB_r^j} \right] \right)^{1/d}. \end{aligned}$$

In the last step we used the generalized Hölder inequality

$$\int \prod_{k=1}^n \phi_k d\mu \leq \prod_{k=1}^n \left(\int |\phi_k|^{p_k} d\mu \right)^{1/p_k} \quad \forall (p_1, \dots, p_n) \in [1, \infty)^n : \sum_{k=1}^n \frac{1}{p_k} = 1$$

with $n = d$ and $p_1 = \dots = p_d = d$. Now the one-dimensional argument with df_j playing the role of f shows (cf. page 250, line 9 from above)

$$\begin{aligned} \mathbb{E} \left[\left| e^{\sum_{j=1}^d \int_0^T f_j(r) dB_r^j - \frac{1}{2} \sum_{j=1}^d \int_0^T f_j^2(r) dr} f_k(T) \right|^2 \right] &\leq C^2 \prod_{j=1}^d \left(\mathbb{E} \left[e^{2d \int_0^T f_j(r) dB_r^j} \right] \right)^{1/d} \\ &\leq C^2 e^{2dC^2T} < \infty. \quad \square \end{aligned}$$



Problem 19.2. Solution: As for a Brownian motion one can see that the independent increments property of a Poisson process is equivalent to saying that $N_t - N_s \perp \mathcal{F}_s^N$ for all $s \leq t$, cf. Lemma 2.14 or Section 5.1. Thus, we have for $s \leq t$

$$\begin{aligned} \mathbb{E}(N_t - t \mid \mathcal{F}_s^N) &= \mathbb{E}(N_t - N_s - (t - s) \mid \mathcal{F}_s^N) + \mathbb{E}(N_s - s \mid \mathcal{F}_s^N) \\ &\stackrel{N_t - N_s \perp \mathcal{F}_s^N}{=} \mathbb{E}(N_t - N_s - (t - s)) + N_s - s \\ &\stackrel{\text{pull out}}{=} \mathbb{E}(N_t - N_s - (t - s)) + N_s - s \\ &\stackrel{N_t - N_s \sim N_{t-s}}{=} \mathbb{E}(N_t - N_s) - (t - s) + N_s - s \\ &= \mathbb{E}(N_{t-s}) - (t - s) + N_s - s \\ &= N_s - s. \end{aligned}$$

Observe that

$$\begin{aligned} (N_t - t)^2 - t &= (N_t - N_s - (t - s) + (N_s - s))^2 - t \\ &= (N_t - N_s - (t - s))^2 + (N_s - s)^2 + 2(N_s - s)(N_t - N_s - t + s) - t. \end{aligned}$$

Thus,

$$\begin{aligned} ((N_t - t)^2 - t) - ((N_s - s)^2 - s) &= (N_t - N_s - (t - s))^2 + 2(N_s - s)(N_t - N_s - t + s) - (t - s). \end{aligned}$$

Now take $\mathbb{E}(\dots \mid \mathcal{F}_s^N)$ in the last equality and observe that $N_t - N_s \perp \mathcal{F}_s$. Then

$$\begin{aligned} &\mathbb{E} \left[((N_t - t)^2 - t) - ((N_s - s)^2 - s) \mid \mathcal{F}_s^N \right] \\ &\stackrel{N_t - N_s \perp \mathcal{F}_s^N}{=} \mathbb{E} \left[(N_t - N_s - (t - s))^2 \right] + 2 \mathbb{E} \left[(N_s - s)(N_t - N_s - t + s) \mid \mathcal{F}_s^N \right] - (t - s) \\ &\stackrel{N_t - N_s \sim N_{t-s}}{\stackrel{\text{pull out}}{=}} \mathbb{E} \left[(N_{t-s} - (t - s))^2 \right] + 2(N_s - s) \mathbb{E} \left[(N_t - N_s - t + s) \mid \mathcal{F}_s^N \right] - (t - s) \\ &\stackrel{N_t - N_s \perp \mathcal{F}_s^N}{=} \mathbb{V} N_{t-s} + 2(N_s - s) \mathbb{E}(N_t - N_s - t + s) - (t - s) \\ &= t - s + 2(N_s - s) \cdot 0 - (t - s) = 0. \end{aligned}$$

Since $t \mapsto N_t$ is not continuous, this does not contradict Theorem 19.5.



Problem 19.3. Solution: We want to use Lévy's characterization, Theorem 19.5. Clearly, $t \mapsto W_t$ is continuous and $W_0 = 0$. Set $\mathcal{F}_t^b = \sigma(b_r : r \leq t)$, $\mathcal{F}_t^\beta = \sigma(\beta_r : r \leq t)$ and $\mathcal{F}_t^W = \sigma(b_r, \beta_r : r \leq t) = \sigma(\mathcal{F}_t^b, \mathcal{F}_t^\beta)$, and

$$\begin{aligned} \lambda &= \sigma_1 / \sqrt{\sigma_1^2 + \sigma_2^2}, \\ \mu &= \sigma_2 / \sqrt{\sigma_1^2 + \sigma_2^2}. \end{aligned}$$

We have

$$\mathbb{E}(W_t \mid \mathcal{F}_s^W) = \mathbb{E}(\lambda b_t \mid \mathcal{F}_s^W) + \mathbb{E}(\mu \beta_t \mid \mathcal{F}_s^W)$$

$$\mathcal{F}_t^b \perp \mathcal{F}_t^\beta \quad \lambda \mathbb{E}(b_t \mid \mathcal{F}_s^b) + \mu \mathbb{E}(\beta_t \mid \mathcal{F}_s^\beta) = \lambda b_s + \mu \beta_s = W_s.$$

proving that (W_t, \mathcal{F}_t^W) is a martingale. Similarly one shows that $(W_t^2 - t, \mathcal{F}_t^W)_{t \geq 0}$ is a martingale. Now Theorem 19.5 applies. ■ ■

Problem 19.4. Solution: *Solution 1:* Note that

$$\begin{aligned} \mathbb{Q}(W(t_j) \in A_j, \forall j = 1, \dots, n) &= \int \prod_{j=1}^n \mathbb{1}_{A_j}(W(t_j)) d\mathbb{Q} \\ &= \int \prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(T) - \frac{1}{2}\xi^2 T} d\mathbb{P}. \end{aligned}$$

By the tower property and the fact that $e^{\xi B(t) - \frac{1}{2}\xi^2 t}$ is a martingale we get

$$\begin{aligned} &\int \prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(T) - \frac{1}{2}\xi^2 T} d\mathbb{P} \\ &= \mathbb{E} \left[\mathbb{E} \left(\prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(T) - \frac{1}{2}\xi^2 T} \mid \mathcal{F}_{t_n} \right) \right] \\ &= \mathbb{E} \left[\prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) \mathbb{E} \left(e^{\xi B(T) - \frac{1}{2}\xi^2 T} \mid \mathcal{F}_{t_n} \right) \right] \\ &= \mathbb{E} \left[\prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(t_n) - \frac{1}{2}\xi^2 t_n} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(\prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(t_n) - \frac{1}{2}\xi^2 t_n} \mid \mathcal{F}_{t_{n-1}} \right) \right] \\ &= \mathbb{E} \left[\prod_{j=1}^{n-1} \mathbb{1}_{A_j}(B(t_j) - \xi t_j) e^{\xi B(t_{n-1}) - \frac{1}{2}\xi^2 t_{n-1}} \times \right. \\ &\quad \left. \times \mathbb{E} \left(\mathbb{1}_{A_n}(B(t_n) - \xi t_n) e^{\xi(B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2(t_n - t_{n-1})} \mid \mathcal{F}_{t_{n-1}} \right) \right] \end{aligned}$$

Now, since $B(t_n) - B(t_{n-1}) \perp \mathcal{F}_{t_{n-1}}$ we get

$$\begin{aligned} &\mathbb{E} \left(\mathbb{1}_{A_n}(B(t_n) - \xi t_n) e^{\xi(B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2(t_n - t_{n-1})} \mid \mathcal{F}_{t_{n-1}} \right) \\ &= \mathbb{E} \left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + B(t_{n-1}) - \xi t_{n-1}) \times \right. \\ &\quad \left. \times e^{\xi(B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2(t_n - t_{n-1})} \mid \mathcal{F}_{t_{n-1}} \right) \\ &= \mathbb{E} \left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + y) \times \right. \\ &\quad \left. \times e^{\xi(B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2(t_n - t_{n-1})} \right) \Bigg|_{y=B(t_{n-1}) - \xi t_{n-1}} \end{aligned}$$

A direct calculation now gives

$$\mathbb{E} \left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + y) e^{\xi(B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2(t_n - t_{n-1})} \right)$$

$$\begin{aligned}
 &= \mathbb{E} \left(\mathbb{1}_{A_n} (B(t_n - t_{n-1}) - \xi(t_n - t_{n-1}) + y) e^{\xi B(t_n - t_{n-1}) - \frac{1}{2} \xi^2 (t_n - t_{n-1})} \right) \\
 &= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n} (x - \xi(t_n - t_{n-1}) + y) e^{\xi x - \frac{1}{2} \xi^2 (t_n - t_{n-1})} e^{-\frac{1}{2(t_n - t_{n-1})} x^2} dx \\
 &= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n} (x - \xi(t_n - t_{n-1}) + y) e^{-\frac{1}{2(t_n - t_{n-1})} (x - \xi(t_n - t_{n-1}))^2} dx \\
 &= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n} (z + y) e^{-\frac{1}{2(t_n - t_{n-1})} z^2} dz \\
 &= \mathbb{E} \mathbb{1}_{A_n} (B(t_n) - B(t_{n-1}) + y)
 \end{aligned}$$

In the next iteration we get

$$\begin{aligned}
 &\mathbb{E} \mathbb{1}_{A_n} \left((B(t_n) - B(t_{n-1})) + (B(t_{n-1}) - B(t_{n-2}) + y) \right) \mathbb{1}_{A_{n-1}} \left((B(t_{n-1}) - B(t_{n-2}) + y) \right) \\
 &= \mathbb{E} \mathbb{1}_{A_n} \left((B(t_n) - B(t_{n-2}) + y) \right) \mathbb{1}_{A_{n-1}} \left((B(t_{n-1}) - B(t_{n-2}) + y) \right)
 \end{aligned}$$

etc. and we finally arrive at

$$\mathbb{Q}(W(t_j) \in A_j, \forall j = 1, \dots, n) = \mathbb{E} \prod_{j=1}^n \mathbb{1}_{A_j} \left(\sum_{k=1}^j (B(t_k) - B(t_{k-1})) \right).$$

Solution 2: As in the first part of Solution 1 we see that we can assume that $T = t_n$. Since we know the joint distribution of $(B(t_1), \dots, B(t_n))$, cf. (2.10b), we get (using $x_0 = t_0 = 0$)

$$\begin{aligned}
 &\mathbb{Q}(W(t_1) \in A_1, \dots, W(t_n) \in A_n) \\
 &= \int \prod_{j=1}^n \mathbb{1}_{A_j} (B(t_j) - \xi t_j) e^{\xi B(t_n) - \frac{1}{2} \xi^2 t_n} d\mathbb{P} \\
 &= \int \dots \int \prod_{j=1}^n \mathbb{1}_{A_j} (x_j - \xi t_j) e^{\xi x_n - \frac{1}{2} \xi^2 t_n} e^{-\frac{1}{2} \sum_{j=1}^n \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}} \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \\
 &= \int \dots \int \left(\prod_{j=1}^n \left[\mathbb{1}_{A_j} (x_j - \xi t_j) e^{-\frac{1}{2} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}} \right] \right) e^{\sum_{j=1}^n (\xi(x_j - x_{j-1}) - \frac{1}{2} \xi^2 (t_j - t_{j-1}))} \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \\
 &= \int \dots \int \prod_{j=1}^n \left[\mathbb{1}_{A_j} (x_j - \xi t_j) e^{-\frac{1}{2} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \xi(x_j - x_{j-1}) - \frac{1}{2} \xi^2 (t_j - t_{j-1})} \right] \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \\
 &= \int \dots \int \prod_{j=1}^n \left[\mathbb{1}_{A_j} (x_j - \xi t_j) e^{-\frac{1}{2(t_j - t_{j-1})} ((x_j - x_{j-1}) + \xi(t_j - t_{j-1}))^2} \right] \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \\
 &= \int \dots \int \prod_{j=1}^n \left[\mathbb{1}_{A_j} (z_j) e^{-\frac{1}{2(t_j - t_{j-1})} (z_j - z_{j-1})^2} \right] \frac{dz_1 \dots dz_n}{(2\pi)^{n/2} \prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \\
 &= \mathbb{P}(B(t_1) \in A_1, \dots, B(t_n) \in A_n).
 \end{aligned}$$

Problem 19.5. Solution: We have

$$\mathbb{P} \left(B_t + \alpha t \leq x, \sup_{s \leq t} (B_s + \alpha s) \leq y \right)$$

$$\begin{aligned}
 &= \int \mathbf{1}_{(-\infty, x]}(B_t + \alpha t) \mathbf{1}_{(-\infty, y]}(\sup_{s \leq t} (B_s + \alpha s)) d\mathbb{P} \\
 &= \int \mathbf{1}_{(-\infty, x]}(B_t + \alpha t) \mathbf{1}_{(-\infty, y]}(\sup_{s \leq t} (B_s + \alpha s)) \frac{1}{\beta_t} d\mathbb{Q}
 \end{aligned}$$

where $\mathbb{Q} = \beta_t \cdot \mathbb{P}$ with $\beta_t = \exp(-\alpha B_t - \frac{1}{2} \alpha^2 t)$

$$\begin{aligned}
 &= \int \mathbf{1}_{(-\infty, x]}(B_t + \alpha t) \mathbf{1}_{(-\infty, y]}(\sup_{s \leq t} (B_s + \alpha s)) e^{\alpha B_t + \frac{1}{2} \alpha^2 t} d\mathbb{Q} \\
 &= \int \mathbf{1}_{(-\infty, x]}(B_t + \alpha t) \mathbf{1}_{(-\infty, y]}(\sup_{s \leq t} (B_s + \alpha s)) e^{\alpha(B_t + \alpha t)} e^{-\frac{1}{2} \alpha^2 t} d\mathbb{Q} \\
 &\stackrel{\text{Girsanov}}{=} e^{-\frac{1}{2} \alpha^2 t} \int \mathbf{1}_{(-\infty, x]}(W_t) \mathbf{1}_{(-\infty, y]}(\sup_{s \leq t} W_s) e^{\alpha W_t} d\mathbb{Q} \\
 &= e^{-\frac{1}{2} \alpha^2 t} \int_{\mathbb{R}^d} \mathbf{1}_{(-\infty, x]}(\xi) e^{\alpha \xi} \mathbb{Q}(W_t \in dxi, \sup_{s \leq t} W_s \leq y).
 \end{aligned}$$

where $(W_s)_{s \leq t}$ is a Brownian motion for the probability measure \mathbb{Q} .

From Solution 2 of Problem 6.12 (or with Theorem 6.19) we have

$$\begin{aligned}
 \mathbb{Q}(\sup_{s \leq t} W_t < y, W_t \in dxi) &= \lim_{a \rightarrow -\infty} \mathbb{Q}(\inf_{s \leq t} W_s > a, \sup_{s \leq t} W_t < y, W_t \in dxi) \\
 &\stackrel{(6.20)}{=} \frac{dxi}{\sqrt{2\pi t}} \left[e^{-\frac{\xi^2}{2t}} - e^{-\frac{(\xi-2y)^2}{2t}} \right]
 \end{aligned}$$

and we get the same result for $\mathbb{Q}(\sup_{s \leq t} W_t \leq y, W_t \in dxi)$. Thus,

$$\begin{aligned}
 &\mathbb{P}(B_t + \alpha t \leq x, \sup_{s \leq t} (B_s + \alpha s) \leq y) \\
 &= \int_{-\infty}^x e^{\alpha \xi} e^{-\frac{1}{2} t \alpha^2} \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{\xi^2}{2t}} - e^{-\frac{(\xi-2y)^2}{2t}} \right) dxi \\
 &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^x \left(e^{-\frac{(\xi-\alpha t)^2}{2t}} - e^{2\alpha y} e^{-\frac{(\xi-2y-\alpha t)^2}{2t}} \right) dxi \\
 &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{x-\alpha t}{\sqrt{t}}} e^{-\frac{z^2}{2}} dz - \frac{e^{2\alpha y}}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{x-2y-\alpha t}{\sqrt{t}}} e^{-\frac{z^2}{2}} dz \\
 &= \Phi\left(\frac{x-\alpha t}{\sqrt{t}}\right) - e^{2\alpha y} \Phi\left(\frac{x-2y-\alpha t}{\sqrt{t}}\right).
 \end{aligned}$$

■ ■

Problem 19.6. Solution:

a) Since X_t has continuous sample paths we find that

$$\widehat{\tau}_b = \inf \{t \geq 0 : X_t \geq b\}.$$

Moreover, we have

$$\{\widehat{\tau}_b \leq t\} = \{\sup_{s \leq t} X_s \geq b\}.$$

Indeed,

$$\begin{aligned}
 \omega \in \{\sup_{s \leq t} X_s \geq b\} &\implies \exists s \leq t : X_s(\omega) \geq b \quad (\text{continuous paths!}) \\
 &\implies \widehat{\tau}_b(\omega) \leq t \\
 &\implies \omega \in \{\widehat{\tau}_b \leq t\},
 \end{aligned}$$

and so $\{\widehat{\tau}_b \leq t\} \supset \{\sup_{s \leq t} X_s \geq b\}$. Conversely,

$$\begin{aligned} \omega \in \{\widehat{\tau}_b \leq t\} &\implies \widehat{\tau}_b(\omega) \leq t \\ &\implies X_{\widehat{\tau}_b(\omega)}(\omega) \geq b, \widehat{\tau}_b(\omega) \leq t \\ &\implies \sup_{s \leq t} X_s(\omega) \geq b \\ &\implies \omega \in \{\sup_{s \leq t} X_s \geq b\}, \end{aligned}$$

and so $\{\widehat{\tau}_b \leq t\} \subset \{\sup_{s \leq t} X_s \geq b\}$.

By the previous problem, Problem 19.5, $\mathbb{P}(\sup_{s \leq t} X_s = b) = 0$. This means that

$$\begin{aligned} \mathbb{P}(\widehat{\tau}_b > t) &= \mathbb{P}\left(\sup_{s \leq t} X_s < b\right) \\ &= \mathbb{P}\left(\sup_{s \leq t} X_s \leq b\right) \\ &= \mathbb{P}\left(X_t \leq b, \sup_{s \leq t} X_s \leq b\right) \\ &\stackrel{\text{Prob. 5}}{=} \Phi\left(\frac{b-\alpha t}{\sqrt{t}}\right) - e^{2\alpha b} \Phi\left(\frac{-b-\alpha t}{\sqrt{t}}\right) \\ &= \Phi\left(\frac{b}{\sqrt{t}} - \alpha\sqrt{t}\right) - e^{2\alpha b} \Phi\left(-\frac{b}{\sqrt{t}} - \alpha\sqrt{t}\right). \end{aligned}$$

Differentiating in t yields

$$\begin{aligned} -\frac{d}{dt} \mathbb{P}(\widehat{\tau}_b > t) &= e^{2\alpha b} \left(\frac{b}{2t\sqrt{t}} - \frac{\alpha}{2\sqrt{t}}\right) \Phi'\left(-\frac{b}{\sqrt{t}} - \alpha\sqrt{t}\right) + \left(\frac{b}{2t\sqrt{t}} + \frac{\alpha}{2\sqrt{t}}\right) \Phi'\left(\frac{b}{\sqrt{t}} - \alpha\sqrt{t}\right) \\ &= \frac{1}{\sqrt{2\pi}} \left(e^{2\alpha b} \left(\frac{b}{2t\sqrt{t}} - \frac{\alpha}{2\sqrt{t}}\right) e^{-\frac{(b+\alpha t)^2}{2t}} + \left(\frac{b}{2t\sqrt{t}} + \frac{\alpha}{2\sqrt{t}}\right) e^{-\frac{(b-\alpha t)^2}{2t}} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\left(\frac{b}{2t\sqrt{t}} - \frac{\alpha}{2\sqrt{t}}\right) e^{-\frac{(b-\alpha t)^2}{2t}} + \left(\frac{b}{2t\sqrt{t}} + \frac{\alpha}{2\sqrt{t}}\right) e^{-\frac{(b-\alpha t)^2}{2t}} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2b}{2t\sqrt{t}} e^{-\frac{(b-\alpha t)^2}{2t}} \\ &= \frac{b}{t\sqrt{2\pi t}} e^{-\frac{(b-\alpha t)^2}{2t}} \end{aligned}$$

b) We have seen in part a) that

$$\begin{aligned} \mathbb{P}(\widehat{\tau}_b > t) &= \Phi\left(\frac{b-\alpha t}{\sqrt{t}}\right) - e^{2\alpha b} \Phi\left(\frac{-b-\alpha t}{\sqrt{t}}\right) \\ &\xrightarrow{t \rightarrow \infty} \begin{cases} \Phi(-\infty) - e^{2\alpha b} \Phi(-\infty) = 0 & \text{if } \alpha > 0 \\ \Phi(0) - e^0 \Phi(0) = 0 & \text{if } \alpha = 0 \\ \Phi(\infty) - e^{2\alpha b} \Phi(\infty) = 1 - e^{2\alpha b} & \text{if } \alpha < 0 \end{cases} \end{aligned}$$

Therefore, we get

$$\mathbb{P}(\widehat{\tau}_b < \infty) = \begin{cases} 1 & \text{if } \alpha \geq 0 \\ e^{2\alpha b} & \text{if } \alpha < 0. \end{cases}$$

Problem 19.7. Solution: Basically, the claim follows from Lemma 19.11. Indeed, if we set

$$g(s) := \sum_{j=1}^n (\xi_j + \dots + \xi_n) \mathbb{1}_{[t_{j-1}, t_j)}(s),$$

then

$$\begin{aligned} \int_0^T g(s) dB_s &= \sum_{j=1}^n (\xi_j + \dots + \xi_n) (B_{t_j} - B_{t_{j-1}}) \\ &= \sum_{j=1}^n ((\xi_j + \dots + \xi_n) - (\xi_{j+1} + \dots + \xi_n)) B_{t_j} \\ &= \sum_{j=1}^n \xi_j B_{t_j}. \end{aligned}$$

Lemma 19.11 shows that $e^{i \int_0^T g(s) dB_s} = e^{i \sum_{j=1}^n \xi_j B_{t_j}}$ is in $\mathcal{H}_T^2 \oplus i\mathcal{H}_T^2$.

If you want to be a bit more careful, you should treat the real and imaginary parts of $\exp\left(i \int_0^T g(s) dB_s\right) = \cos\left(\int_0^T g(s) dB_s\right) + i \sin\left(\int_0^T g(s) dB_s\right)$ separately. Let us do this for the real part.

We apply the two-dimensional Itô-formula (18.14) to $f(x, y) = \cos(x)e^{y/2}$ and the process $(X_t, Y_t) = \left(\int_0^t g(s) dB_s, \int_0^t g^2(s) ds\right)$: Since

$$\begin{aligned} \partial_x f(x, y) &= -\sin(x)e^{y/2} \\ \partial_x^2 f(x, y) &= -\cos(x)e^{y/2} \\ \partial_y f(x, y) &= \frac{1}{2} \cos(x)e^{y/2} \end{aligned}$$

we get

$$\begin{aligned} &\cos(X_T)e^{Y_T/2} - 1 \\ &= -\int_0^T \sin(X_s)e^{Y_s/2} dX_s + \frac{1}{2} \int_0^T \cos(X_s)e^{Y_s/2} dY_s - \frac{1}{2} \int_0^T \cos(X_s)e^{Y_s/2} g^2(s) ds \\ &= -\int_0^T \sin(X_s)e^{Y_s/2} g(s) dB_s. \end{aligned}$$

Thus, by the definition of g , X and Y ,

$$\begin{aligned} \cos\left(\sum_{j=1}^n \xi_j B_{t_j}\right) &= \cos\left(\int_0^T g(s) dB_s\right) \\ &= e^{-\int_0^T g^2(s) ds} - \int_0^T \sin\left(\int_0^s g(r) dB_r\right) e^{-\frac{1}{2} \int_s^T g^2(r) dr} g(s) dB_s. \end{aligned}$$

Since the integrand of the stochastic integral is continuous and bounded, it is clear that it is in $L_{\mathcal{F}}^2(\lambda_T \otimes \mathbb{P})$. Hence $\cos\left(\sum_{j=1}^n \xi_j B_{t_j}\right) \in \mathcal{H}_T^2$.

The imaginary part can be treated in a similar way. ■ ■

Problem 19.8. Solution: Denote by $\overline{\mathcal{H}}$ the closure of \mathcal{H} in \mathcal{L} . Then we have the unique orthogonal decomposition $\mathcal{L} = \overline{\mathcal{H}} \oplus \mathcal{H}^\perp$ where $\mathcal{H}^\perp = \{y \in \mathcal{L} : y \perp \mathcal{H}\}$. (Note that \mathcal{H}^\perp is automatically a closed subspace.)

If \mathcal{H} is dense, then $\overline{\mathcal{H}} = \mathcal{L}$ and we get $\mathcal{H}^\perp = \{0\}$, so that the property holds.

Conversely, the property says that $\mathcal{H}^\perp = \{0\}$, so $\overline{\mathcal{H}} = \mathcal{L}$, i.e. \mathcal{H} is dense in \mathcal{L} .

■ ■

Problem 19.9. Solution: Set $\Sigma_{t_1, \dots, t_n} := \sigma(B_{t_1}, \dots, B_{t_n})$. There are several possibilities to prove this result.

Possibility 1: Set $\mathbf{t}_n = (t_1, \dots, t_n)$ and $\Sigma(\mathbf{t}_n) = \Sigma_{t_1, \dots, t_n}$. Then the family of σ -algebras $\Sigma(\mathbf{t}_n)$ is upwards filtering, i.e. whenever we have \mathbf{t}_n and \mathbf{s}_m there is some \mathbf{u}_{n+m} such that $\Sigma(\mathbf{s}_m) \cup \Sigma(\mathbf{t}_n) \subset \Sigma(\mathbf{u}_{n+m})$. Therefore we can use Lévy's (upwards) martingale convergence theorem and conclude that

$$\mathbb{E}(Y \mid \mathcal{F}_T^B) = L^1\text{-}\lim \mathbb{E}(Y \mid \Sigma(\mathbf{t}_n)) = 0.$$

Since \mathcal{F}_T^B and $\tilde{\mathcal{F}}_T^B$ differ only by trivial sets (with probability zero or one), we get a.s. $Y = \mathbb{E}(Y \mid \tilde{\mathcal{F}}_T^B) = \mathbb{E}(Y \mid \mathcal{F}_T^B) = 0$.

Possibility 2: Set $\Sigma_T := \bigcup_{n \geq 1} \bigcup_{0 \leq t_1 < \dots < t_n = T} \Sigma_{t_1, \dots, t_n}$. Then $\sigma(\Sigma_T) = \mathcal{F}_T^B$ and Σ_T is stable under intersections. Consider the measures

$$\mu^\pm(F) := \int_F Y^\pm d\mathbb{P} \quad \forall F \in \Sigma_T.$$

By assumption, $\mu^+(F) = \mu^-(F)$ on Σ_T , and by the uniqueness theorem for measures we get $\mu^+ = \mu^-$ on \mathcal{F}_T . But then we get $\int_F Y d\mathbb{P} = 0$ for all $F \in \mathcal{F}_T^B$.

If we add to Σ_T all \mathbb{P} null set, the above considerations remain valid (without changes!) and we get $\int_F Y d\mathbb{P} = 0$ for all $F \in \tilde{\mathcal{F}}_T^B$, hence $Y = 0$ as Y is $\tilde{\mathcal{F}}_T^B$ measurable.

■ ■

Problem 19.10. Solution: Because of the properties of conditional expectations we have for $s \leq t$

$$\mathbb{E}(M_t \mid \mathcal{H}_s) = \mathbb{E}(M_t \mid \sigma(\mathcal{F}_s, \mathcal{G}_s)) \stackrel{M_{\mathbb{1}^{\mathcal{G}_\infty}}}{=} \mathbb{E}(M_t \mid \mathcal{F}_s) = M_s.$$

Thus, $(M_t, \mathcal{H}_t)_{t \geq 0}$ is still a martingale; $(B_t, \mathcal{H}_t)_{t \geq 0}$ is treated in a similar way.

■ ■

Problem 19.11. Solution: Recall that

$$\tau(s) = \inf\{t \geq 0 : a(t) > s\}.$$

Since for any $\epsilon > 0$

$$\{t : a(t) \geq s\} \subset \{t : a(t) > s - \epsilon\} \subset \{t : a(t) \geq s - \epsilon\}$$

we get

$$\inf\{t : a(t) \geq s\} \geq \inf\{t : a(t) > s - \epsilon\} \geq \inf\{t : a(t) \geq s - \epsilon\}$$

and

$$\inf\{t : a(t) \geq s\} \geq \liminf_{\epsilon \uparrow 0} \inf\{t : a(t) > s - \epsilon\} \geq \liminf_{\epsilon \uparrow 0} \inf\{t : a(t) \geq s - \epsilon\}.$$

$$\underbrace{\liminf_{\epsilon \uparrow 0} \inf\{t : a(t) > s - \epsilon\}}_{=\lim_{\epsilon \uparrow 0} \tau(s-\epsilon)=\tau(s-)}$$

Thus, $\inf\{t : a(t) \geq s\} \geq \tau(s-)$. Assume that $\inf\{t : a(t) \geq s\} > \tau(s-)$. Then

$$a(\tau(s-)) < s.$$

On the other hand, by Lemma 19.16 b)

$$s - \epsilon \leq a(\tau(s - \epsilon)) \leq a(\tau(s-)) < s \quad \forall \epsilon > 0.$$

This leads to a contradiction, and so $\inf\{t : a(t) \geq s\} \leq \tau(s-)$.

The proof for $a(s-)$ is similar.

Assume that $\tau(s) \geq t$. Then $a(t-) = \inf\{s \geq 0 : \tau(s) \geq t\} \leq s$. On the other hand,

$$a(t-) \leq s \implies \forall \epsilon > 0 : a(t - \epsilon) \leq s \stackrel{19.16 \text{ d)}}{\implies} \forall \epsilon > 0 : \tau(s) > t - \epsilon \implies \tau(s) \geq t.$$

■ ■

Problem 19.12. Solution: We have

$$\begin{aligned} \{\langle M \rangle_t \leq s\} &= \bigcap_{n \geq 1} \{\langle M \rangle_t < s + 1/n\} = \bigcap_{n \geq 1} \{\langle M \rangle_t \geq s + 1/n\}^c \\ &\stackrel{19.16}{=} \bigcap_{n \geq 1} \{\tau_{s+1/n-} \leq s\}^c \in \bigcap_{n \geq 1} \mathcal{F}_{\tau_{s+1/n}} \stackrel{A.15}{=} \mathcal{F}_{\tau_{s+}}. \end{aligned}$$

As \mathcal{F}_t is right continuous, $\mathcal{F}_{\tau_{s+}} = \mathcal{F}_{\tau_s} = \mathcal{G}_s$ and we conclude that $\langle M \rangle_t$ is a \mathcal{G}_t stopping time.

■ ■

Problem 19.13. Solution: Solution 1: Assume that $f \in \mathcal{C}^2$. Then we can apply Itô's formula. Use Itô's formula for the deterministic process $X_t = f(t)$ and apply it to the function x^a (we assume that $f \geq 0$ to make sure that f^a is defined for all $a > 0$):

$$f^a(t) - f^a(0) = \int_0^t \left[\frac{d}{dx} x^a \right]_{x=f(s)} df(s) = \int_0^t a f^{a-1}(s) df(s).$$

This proves that the primitive $\int f^{a-1} df = f^a/a$. The rest is an approximation argument ($f \in \mathcal{C}^1$ is pretty immediate).

Solution 2: Any absolutely continuous function has an Lebesgue a.e. defined derivative f' and $f = \int f' ds$. Thus,

$$\int_0^t f^{a-1}(s) df(s) = \int_0^t f^{a-1}(s) f'(s) ds = \int_0^t \frac{1}{a} \frac{d}{ds} f^a(s) ds = \left[\frac{f^a(s)}{a} \right]_0^t = \frac{f^a(t) - f^a(0)}{a}.$$

■ ■

Problem 19.14. Solution: Theorem. Let $B_t = (B_t^1, \dots, B_t^d)$ be a d -dimensional Brownian motion and $f_1, \dots, f_d \in L^2_{\mathbb{P}}(\lambda_T \otimes \mathbb{P})$ for all $T > 0$. Then, we have for $2 \leq p < \infty$

$$\mathbb{E} \left[\left(\int_0^T \sum_{k=1}^d |f_k(s)|^2 ds \right)^{p/2} \right] \asymp \mathbb{E} \left[\sup_{t \leq T} \left| \sum_k \int_0^t f_k(s) dB_s^k \right|^p \right] \quad (19.2)$$

with finite comparison constants which depend only on p .

Proof. Let $X_t = \sum_k \int_0^t f_k(s) dB_s^k$. Then we have

$$\begin{aligned} \langle X \rangle_t &= \left\langle \sum_k \int_0^t f_k(s) dB_s^k, \sum_l \int_0^t f_l(s) dB_s^l \right\rangle \\ &= \sum_{k,l} \left\langle \int_0^t f_k(s) dB_s^k, \int_0^t f_l(s) dB_s^l \right\rangle \\ &= \sum_{k,l} \int_0^t f_k(s) f_l(s) d\langle B^k, B^l \rangle_s \\ &= \sum_k \int_0^t f_k^2(s) ds \end{aligned}$$

since $dB_s^k dB_s^l = d\langle B^k, B^l \rangle_s = \delta_{kl} ds$.

With these notations, the proof of Theorem 17.16 goes through almost unchanged and we get the inequalities for $p \geq 2$. \square

Remark: Often one needs only one direction (as we do later in the book) and one can use 19.21 directly, without going through the proof again. Note that

$$\begin{aligned} \left| \sum_{k=1}^d \int_0^t f_k(s) dB_s^k \right|^p &\leq \left(\sum_{k=1}^d \left| \int_0^t f_k(s) dB_s^k \right| \right)^p \\ &\leq c_{d,p} \sum_{k=1}^d \left| \int_0^t f_k(s) dB_s^k \right|^p. \end{aligned}$$

Thus, by (19.21)

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} \left| \sum_{k=1}^d \int_0^t f_k(s) dB_s^k \right|^p \right] &\leq c_{d,p} \sum_{k=1}^d \mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t f_k(s) dB_s^k \right|^p \right] \\ &\asymp c_{d,p} \sum_{k=1}^d \mathbb{E} \left[\left(\int_0^T |f_k(s)|^2 ds \right)^{p/2} \right] \\ &\asymp c_{d,p} \mathbb{E} \left[\left(\int_0^T \sum_{k=1}^d |f_k(s)|^2 ds \right)^{p/2} \right]. \end{aligned}$$

Problem 19.15. Solution: The proof that $\text{supp}[dL_t^a(\omega)] \subset \{t \geq 0 : B_t(\omega) = a\}$ holds for all $\omega \notin N_a$, $\mathbb{P}(N_a) = 0$, for each $a \in \mathbb{R}$, works with the same argument that we have used for the lemma, if we replace B_t by $B_t - a$. If $a \in \mathbb{Q}$ we can use the null set $N = \bigcup_{a \in \mathbb{Q}} N_a$. If $a \in \mathbb{R}$, we use the continuity of the local time. Enlarge, if necessary, N such that $(t, a) \mapsto L_t^a(\omega)$

is continuous for all $\omega \notin N$. Suppose that for $\omega \notin N$, some $a \in \mathbb{R}$ and $0 \leq s < t$ we have $L_t^a(\omega) > L_s^a(\omega)$ and $B_r(\omega) \neq a$ for every $r \in [s, t]$. Then there is some rational $q \in \mathbb{Q} \subset \mathbb{R}$, sufficiently close to a , such that $L_t^q(\omega) > L_s^q(\omega)$ and still $B_r(\omega) \neq a$ for every $r \in [s, t]$ – but this is impossible since we already have $\text{supp}[dL_t^q(\omega)] \subset \{t \geq 0 : B_t(\omega) = a\}$ for all $\omega \notin N$.

Problem 19.16. Solution: *Existence:* The function

$$a(t) = 0 \vee \sup_{s \leq t} (-b(s)) = - \left[0 \wedge \inf_{s \leq t} b(s) \right]$$

is obviously positive, continuous and increasing. Moreover, $a(t)$ increases strictly whenever $b(t)$ reaches a new running infimum. Setting $p(t) := b(t) + a(t)$ shows that the pair $\{p(t), a(t)\}$ satisfies i)–iii).

Uniqueness: Let $\{\bar{p}(t), \bar{a}(t)\}$ be another decomposition such that i)–iii) hold. From $p(t) - a(t) = b(t) = \bar{p}(t) - \bar{a}(t)$ we see that $p(t) - \bar{p}(t) = a(t) - \bar{a}(t)$. Assume that there is some $t_1 > 0$ such that $p(t_1) > \bar{p}(t_1)$, and define $t_0 := \inf\{t < t_1 : p(t) = \bar{p}(t)\}$. Since $(p - \bar{p})|_{(t_0, t_1]} > 0$, we know that $p|_{(t_0, t_1]} > \bar{p}|_{(t_0, t_1]} \geq 0$, i.e. $a(t_0) = a(t_1)$ because of property iii). Since \bar{a} is increasing, we see

$$0 < p(t_1) - \bar{p}(t_1) = a(t_1) - \bar{a}(t_1) \stackrel{\text{iii)}}{=} a(t_0) - \bar{a}(t_1) \stackrel{\text{ii)}}{\leq} a(t_0) - \bar{a}(t_0) = p(t_0) - \bar{p}(t_0) = 0.$$

This is a contradiction, and so $p(t) \leq \bar{p}(t)$ for all $t \geq 0$. A similar argument shows $\bar{p}(t) \leq p(t)$, i.e. $p = \bar{p}$ and $a = \bar{a}$.

Problem 19.17. Solution: As in the third step of the proof of Theorem 19.28 we can use Tanaka's formula to get

$$\begin{aligned} L_{t+s}^0 - L_t^0 &= |B_{t+s}| - |B_t| - \int_t^{t+s} \text{sgn}(B_r) dB_r \\ &= |B_s \circ \theta_t| - |B_0 \circ \theta_t| - \int_0^s \text{sgn}(B_r \circ \theta_t) d(B_r \circ \theta_t) \end{aligned}$$

and the latter is $L_s^0 \circ \theta_t$ – it is the local time for the Brownian motion $W_s = B_s \circ \theta_t - B_0 \circ \theta_t$, if we condition on $B_t = a$; observe that $(W_s)_{s \geq 0}$ is independent of $(B_r)_{r \leq t}$, cf. Theorem 6.1.

Problem 19.18. Solution: Note that the probability space on which τ_v is defined is governed by \mathbb{P} which derives from the underlying Brownian motion. Denote by \mathbb{P}^x the probability measure for B_t such that $B_0 = x$ a.s. \mathbb{P} is then a short-hand for \mathbb{P}^0 . We use the strong Markov property for $(B_t)_{t \geq 0}$ and observe that $B_{\tau_u} = 0$ (by definition of τ_u). Thus,

$$\begin{aligned} &\mathbb{P}^0(\tau_{u_i} - \tau_{u_{i-1}} \in A_i, i = 1, \dots, n) \\ &= \mathbb{P}^0(\tau_{u_n - u_{n-1}} \circ \theta_{\tau_{u_{n-1}}} \in A_n, \tau_{u_i} - \tau_{u_{i-1}} \in A_i, i = 1, \dots, n-1) \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_{u_n - u_{n-1}} \circ \theta_{\tau_{u_{n-1}}} \in A_n, \tau_{u_i} - \tau_{u_{i-1}} \in A_i, i=1, \dots, n-1\}} \right) \\
 &= \mathbb{E}^0 \left(\mathbb{P}^{B(\tau_{t_{n-1}})} [\tau_{u_n - u_{n-1}} \in A_n] \mathbb{1}_{\{\tau_{u_i} - \tau_{u_{i-1}} \in A_i, i=1, \dots, n-1\}} \right) \\
 &= \mathbb{E}^0 \left(\mathbb{P}^0 [\tau_{u_n - u_{n-1}} \in A_n] \mathbb{1}_{\{\tau_{u_i} - \tau_{u_{i-1}} \in A_i, i=1, \dots, n-1\}} \right) \\
 &= \mathbb{P}^0 [\tau_{u_n - u_{n-1}} \in A_n] \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_{u_i} - \tau_{u_{i-1}} \in A_i, i=1, \dots, n-1\}} \right)
 \end{aligned}$$

and we can now proceed recursively. ■■

Problem 19.19. Solution: From our assumption we get immediately $\mathbb{E} [e^{i\xi(X_t - X_s)} | \mathcal{G}_s] = \mathbb{E} [e^{i\xi X_{t-s}}]$. Taking expectations on both sides, yields $\mathbb{E} [e^{i\xi(X_t - X_s)}] = \mathbb{E} [e^{i\xi X_{t-s}}]$, hence $X_t - X_s \sim X_{t-s}$. Denote by $\phi(t-s) := \mathbb{E} [e^{i\xi X_{t-s}}]$. We can now use the tower property just as in the proof of Lemma 5.4 to see that consecutive increments $(X_{t_i} - X_{t_{i-1}})$, $0 = t_0 < t_1 < \dots < t_n < \infty$ are independent. See also [21, Lemma 49.1]. ■■

20 Wiener Chaos and iterated Wiener–Itô integrals

Problem 20.1. Solution: Let

$$\phi(s, t) = \sum_{i \neq k} a_{ik} \mathbb{1}_{[t_{i-1}, t_i] \times [t_{k-1}, t_k]}(s, t) = \sum_{m \neq n} b_{ik} \mathbb{1}_{[s_{m-1}, s_m] \times [s_{n-1}, s_n]}(s, t)$$

and take the common refinement of the underlying partitions. This gives a third representation and we show that each of the above representations and the common-refinement representation leads to the same result. WLOG we can assume that the partition with the t_i 's is already finer. Therefore, $[s_{m-1}, s_m] = \cup_{i: [s_{m-1}, s_m] \supset [t_{i-1}, t_i]} [t_{i-1}, t_i]$. Since everything is linear, we can consider a single step of the s_m -Partition:

$$\mathbb{1}_{[s_{m-1}, s_m] \times [s_{n-1}, s_n]}(s, t) = \sum_{[t_{i-1}, t_i] \subset [s_{m-1}, s_m]} \sum_{[t_{k-1}, t_k] \subset [s_{n-1}, s_n]} \mathbb{1}_{[t_{i-1}, t_i] \times [t_{k-1}, t_k]}(s, t)$$

Since $m \neq n$, the t_i -and- t_k -intervals are also disjoint, i.e. off-diagonal. Now observe that

$$B_{s_m} - B_{s_{m-1}} = \sum_{[t_{i-1}, t_i] \subset [s_{m-1}, s_m]} (B_{t_i} - B_{t_{i-1}})$$

and so we get that the above two representations have the same double Itô integral. ■■

Problem 20.2. Solution: Recall that $\int_a^b = \int_{[a,b]}$. Since we do not know that one-point sets have zero μ -measure, we have to take care about sets being open or closed. Using Fubini and the symmetry of the integrand $f(s, t) = f(t, s)$ we have

$$\begin{aligned} & \int_a^b \int_a^b f(s, t) \mu(ds) \mu(dt) \\ &= \int_{[a,b]} \int_{[a,b]} f(s, t) \mu(ds) \mu(dt) \\ &= \int_{[a,b]} \int_{[a,t]} f(s, t) \mu(ds) \mu(dt) + \int_{[a,b]} \int_{[t,b]} f(s, t) \mu(ds) \mu(dt) \\ &= \int_{[a,b]} \int_{[a,t]} f(s, t) \mu(ds) \mu(dt) + \int_{[a,b]} \int_{[a,s]} f(s, t) \mu(dt) \mu(ds) \\ &= \int_{[a,b]} \int_{[a,t]} f(s, t) \mu(ds) \mu(dt) + \int_{[a,b]} \int_{[a,s]} f(s, t) \mu(dt) \mu(ds) + \int_{[a,b]} \int_{\{s\}} f(s, t) \mu(dt) \mu(ds) \\ &= \int_{[a,b]} \int_{[a,t]} f(s, t) \mu(ds) \mu(dt) + \int_{[a,b]} \int_{[a,t]} f(t, s) \mu(ds) \mu(dt) + \int_{[a,b]} \mu\{s\} f(s, s) \mu(ds) \\ &= 2 \int_{[a,b]} \int_{[a,t]} f(s, t) \mu(ds) \mu(dt) + \int_{[a,b]} \mu\{s\} f(s, s) \mu(ds) \end{aligned}$$

and this shows that the diagonal term does have an influence. This also explains why it is important to consider off-diagonal simple functions.

Problem 20.3. Solution: This is a standard argument (in many areas of analysis and probability). Let $\phi_n \rightarrow f$ and $\psi_n \rightarrow f$ be two sequences of step-functions converging to f in $L^2(\mathbb{R}_+^2)$. Then we have

$$\|\phi_n - \psi_n\|_{L^2} \leq \|\phi_n - f\|_{L^2} + \|f - \psi_n\|_{L^2} \rightarrow 0$$

and so

$$\begin{aligned} \|I_2(\phi_n) - I_2(\psi_n)\|_{L^2(\Omega)} &= \|I_2(\phi_n - \psi_n)\|_{L^2(\Omega)} = 2\|\widehat{\phi}_n - \widehat{\psi}_n\|_{L^2(dt, ds)} \\ &\leq 2\|\phi_n - \psi_n\|_{L^2(dt, ds)} \rightarrow 0 \end{aligned}$$

i.e. both sequences lead to the same $L^2(\Omega)$ -limit.

Problem 20.4. Solution: We have to show that

$$\sum_{k=1}^N \lambda_k H_{n(k)}(x) = 0 \quad \forall x \implies \lambda_1 = \dots = \lambda_N = 0.$$

The idea behind the following proof is that the Hermite polynomials H_n increase in order as n goes up.

WLOG $n(1) < \dots < n(N)$. If we re-arrange the above sum we get

$$\sum_{k=1}^N \lambda_k H_{n(k)}(x) = \lambda_N x^{n(N)} + \sum_{i=0}^{n(N)-1} c_i x^i = 0 \quad \forall x$$

Now divide by $x^{n(N)}$ and let $x \rightarrow \infty$. This shows

$$\lambda_N = 0.$$

This reduces our problem to

$$\sum_{k=1}^{N-1} \lambda_k H_{n(k)}(x) = 0 \quad \forall x \implies \lambda_1 = \dots = \lambda_{N-1} = 0.$$

and we can repeat our reasoning. Finally we get that all $\lambda_i = 0$.

Problem 20.5. Solution: Fix $\xi, \eta \in \mathbb{R}$. We have

$$\begin{aligned} \mathbb{E} e^{i\xi I_1(f) + i\eta I_1(g)} &= \mathbb{E} e^{iI_1(\xi f + \eta g)} \\ &= e^{-\frac{1}{2} \int_0^\infty \xi^2 f^2(t) + \eta^2 g^2(t) + 2\xi\eta f(t)g(t) dt} \\ &= e^{-\frac{1}{2} \xi^2 \int_0^\infty f^2(t) dt} e^{-\frac{1}{2} \eta^2 \int_0^\infty g^2(t) dt} \end{aligned}$$

$$\begin{aligned}
 &= e^{-\frac{1}{2}\xi^2 \int_0^\infty f^2(t) dt} e^{-\frac{1}{2}\eta^2 \int_0^\infty g^2(t) dt} \\
 &= \mathbb{E} e^{i\xi I_1(f)} \mathbb{E} e^{i\eta I_1(g)}
 \end{aligned}$$

Since ξ, η are arbitrary, we conclude that $I_1(f), I_1(g)$ are independent (Theorem of M. Kac).

Problem 20.6. Solution: We can make a direct calculation, but the following trick is much more effective: X, Y and $X \pm Y$ are one-dimensional mean zero Gaussian and therefore $\mathbb{E}(X \pm Y)^3 = \mathbb{E} X^3 \pm \mathbb{E} Y^3 = 0$. Thus,

$$0 = \mathbb{E}(X \pm Y)^3 = \mathbb{E} X^3 \pm \mathbb{E} Y^3 \pm 3 \mathbb{E} X^2 Y + 3 \mathbb{E} X Y^2 \implies \mathbb{E} X Y^2 = \pm \mathbb{E} X^2 Y$$

and this is only possible if $\mathbb{E} X Y^2 = \mathbb{E} Y X^2 = 0$.

We have to show that for any $g \in L^2(\mathbb{R}_+)$

$$\mathbb{E} \left[(I_1^2(f) - \|f\|_{L^2}^2) I_1(g) \right] = 0.$$

Note that $\mathbb{E}(\|f\|_{L^2}^2) I_1(g) = \|f\|_{L^2}^2 \mathbb{E} I_1(g) = 0$. Moreover,

$$\mathbb{E}(I_1^2(f) I_1(g)) = 0$$

since $(X, Y) = (I_1(f), I_1(g))$ is mean zero Gaussian — and now we can use the first part of the problem.

Problem 20.7. Solution:

- a) When calculating the product $e_\alpha(t) e_\beta(t)$ we match the t_k -coordinates, which we arrange in blocks:

$$t = (t_1, t_2, \dots, t_{\alpha_1})(t_{\alpha_1+1}, t_{\alpha_1+2}, \dots, t_{\alpha_1+\alpha_2})(t_{\alpha_1+\alpha_2+1}, t_{\alpha_1+\alpha_2+2}, \dots, t_{\alpha_1+\alpha_2+\alpha_3}) \dots$$

Thus, we get (note that all products are finite since our multiindices have only finitely many entries $\neq 0$)

$$\prod_{k=1}^{\infty} \prod_{j=1}^{\alpha_k} e_k(t_{j+\alpha_1+\dots+\alpha_{k-1}}) \prod_{k=1}^{\infty} \prod_{j=1}^{\beta_k} e_k(t_{j+\beta_1+\dots+\beta_{k-1}})$$

If $\alpha = \beta$, this means that we have “perfect matches” of the form $\langle e_k, e_k \rangle_{L^2(\mathbb{R}_+)} = 1$ only and these scalar products produce always a 1.

If $\alpha \neq \beta$, there is at least one (well, necessarily two) mismatch in one coordinate i.e. $\langle e_k, e_j \rangle_{L^2(\mathbb{R}_+)} = 0$ since $k \neq j$. This means that the whole product becomes zero.

- b) We continue the notation from above but we also write $\sigma, \tau : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ for permutations and we assume that $|\alpha| = |\beta| = n$. Moreover, if $t = (t_1, \dots, t_n)$, then $\sigma(t) = (t_{\sigma(1)}, \dots, t_{\sigma(n)})$ – and $\tau(t)$ is understood in a similar way. By definition

$$\widehat{e}_\alpha(t) = \frac{1}{n!} \sum_{\sigma} e_\alpha(\sigma(t))$$

and we get

$$\begin{aligned} \|\widehat{e}_\alpha\|_{L^2(\mathbb{R}_+^n)}^2 &= \int_{\mathbb{R}_+^n} \left(\frac{1}{n!} \sum_{\sigma} e_\alpha(\sigma(t)) \right)^2 dt \\ &= \left(\frac{1}{n!} \right)^2 \sum_{\sigma} \sum_{\tau} \int_{\mathbb{R}_+^n} e_\alpha(\sigma(t)) e_\alpha(\tau(t)) dt. \end{aligned}$$

All we have to do is to multiply this out in an orderly manner. Observe the following

$\sigma = \tau$: In this case $\int_{\mathbb{R}_+^n} e_\alpha(\sigma(t)) e_\alpha(\tau(t)) dt = 1$. This gives us $n!$ -many “1” in the double sum $\sum_{\sigma} \sum_{\tau}$.

$\sigma \neq \tau$ but σ, τ differ only in the coordinate block t_1, \dots, t_{α_1} : In this case we have $\alpha_1!$ additional possibilities to get a “perfect match” leading to $\int_{\mathbb{R}_+^n} e_\alpha(\sigma(t)) e_\alpha(\tau(t)) dt = 1$.

...

$\sigma \neq \tau$ but σ, τ differ only in the α_k -coordinate block: this gives us $\alpha_k!$ new possibilities.

By the counting principle, we thus get

$$n! \cdot \prod_{k=1}^{\infty} \alpha_k! \quad (0! = 1)$$

“ones”, i.e.

$$\begin{aligned} \|\widehat{e}_\alpha\|_{L^2(\mathbb{R}_+^n)}^2 &= \int_{\mathbb{R}_+^n} \left(\frac{1}{n!} \sum_{\sigma} e_\alpha(\sigma(t)) \right)^2 dt \\ &= \left(\frac{1}{n!} \right)^2 n! \cdot \prod_{k=1}^{\infty} \alpha_k! \\ &= \frac{\prod_{k=1}^{\infty} \alpha_k!}{n!} \end{aligned}$$

as claimed. ■ ■

Problem 20.8. Solution:

- a) We have $F = f(I_1(e_1), \dots, I_1(e_n))$ and $G = g(I_1(e_1), \dots, I_1(e_n))$ (wlog F, G depend on the same number of variables $I_1(e_k)$). We have for any e_l

$$\langle D(FG), e_l \rangle_H = D_{e_l}(FG) = \partial_l(FG)$$

and we know already that ∂_l satisfies the product rule:

$$\begin{aligned}\partial_l(FG) &= G\partial_l F + F\partial_l G \\ &= GD_{e_l}F + FD_{e_l}G \\ &= G\langle DF, e_l \rangle + F\langle DG, e_l \rangle \\ &= \langle GDF, e_l \rangle + \langle FDG, e_l \rangle \\ &= \langle GDF + FDG, e_l \rangle\end{aligned}$$

We can also make a direct calculation in the following way:

$$D(FG) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (fg)(I_1(e_1), \dots, I_1(e_n))$$

and now use the product rule for $\frac{\partial}{\partial x_i} \dots$

b) Similar to the previous calculation, just use the classical Leibniz rule.

Problem 20.9. Solution: A solution using induction based on $n = m + 1$ can be found in Nualart's book [11, Proposition 1.1.3, p. 12]. We will now give a combinatorial argument.

In what follows, $\alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}_0^k$ are multiindices and we use the standard notation for multiindices: $\xi^\alpha = \prod_1^k \xi_i^{\alpha_i}$ for vectors $\xi = (\xi_1, \dots, \xi_k)$ and $|\alpha| = \alpha_1 + \dots + \alpha_k$. Moreover, we agree that $\binom{p}{q} = 0$ if $q > p$ (also a standard convention)

We have the following combinatorial identity: for any r, m, n and $|\alpha| = m$ and $|\beta| = n$ we have

$$\sum_{\substack{r_1, \dots, r_k \geq 0 \\ r_1 + \dots + r_k = r}} r_1! \binom{\alpha_1}{r_1} \binom{\beta_1}{r_1} \cdot \dots \cdot r_k! \binom{\alpha_k}{r_k} \binom{\beta_k}{r_k} = r! \binom{m}{r} \binom{n}{r}. \quad (*)$$

In order to see this, rewrite this identity as

$$\sum_{\substack{r_1, \dots, r_k \geq 0 \\ r_1 + \dots + r_k = r}} \frac{r!}{r_1! \cdot \dots \cdot r_k!} r_1! \binom{\alpha_1}{r_1} \binom{\beta_1}{r_1} \cdot \dots \cdot r_k! \binom{\alpha_k}{r_k} \binom{\beta_k}{r_k} = r! \binom{m}{r} r! \binom{n}{r}.$$

and this can be interpreted in the following way: We draw r distinguishable balls from an urn M containing m balls and from urn N with n balls. The right-hand side gives all possible drawings.

Now imagine that the urns have k sub-compartments M_1, \dots, M_k and N_1, \dots, N_k containing $\alpha_i \geq 0$ and $\beta_i \geq 0$ balls. The right-hand side gives all possibilities of drawings from these urns, and the additional multinomial factor gives the number of possible drawings such that we take r_i balls from each of M_i and N_i .

Since both drawing schemes give the same outcomes, the claimed identity holds.

By a density argument, it is clearly enough to show the formula of the problem for $f = \sum_{i=1}^k \xi_i e_i$ and $g = \sum_{i=1}^k \eta_i e_i$ where $\xi, \eta \in \mathbb{R}^k$ and $(e_i)_{i \geq 1}$ is a some ONB of $L^2(0, \infty)$. We may

also assume that $\|f\|_{L^2} = \|g\|_{L^2} = 1$, which amounts to saying that $\sum_{i=1}^k \xi_i^2 = \sum_{i=1}^k \eta_i^2 = 1$ (use Parvseval's identity). Finally we set $e = (e_1, \dots, e_k)$ and use the notation $e_\alpha = e^{\otimes \alpha} = \otimes_1^k e_i^{\otimes \alpha_i}$.

We have

$$\begin{aligned}
 & I_m(f)I_n(g) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta I_m(e^{\otimes \alpha})I_n(e^{\otimes \beta}) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta \prod_{i=1}^k H_{\alpha_i}(I_1(e_i)) \prod_{j=1}^k H_{\beta_j}(I_1(e_j)) \quad (\text{use Corollary 20.21}) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta \prod_{i=1}^k H_{\alpha_i}(I_1(e_i))H_{\beta_i}(I_1(e_i)) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta \prod_{i=1}^k \sum_{r_i=1}^{\alpha_i \wedge \beta_i} r_i! \binom{\alpha_i}{r_i} \binom{\beta_i}{r_i} H_{\alpha_i + \beta_i - 2r_i}(I_1(e_i)) \quad (\text{Theorem 20.13.i}) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta \sum_{r_1, \dots, r_k \geq 0} r_1! \binom{\alpha_1}{r_1} \binom{\beta_1}{r_1} \cdots r_k! \binom{\alpha_k}{r_k} \binom{\beta_k}{r_k} \prod_{i=1}^k H_{\alpha_i + \beta_i - 2r_i}(I_1(e_i)) \\
 &= \sum_{\substack{|\alpha|=m \\ |\beta|=n}} \xi^\alpha \eta^\beta \sum_{r=0}^{m \wedge n} \sum_{\substack{r_1, \dots, r_k \geq 0 \\ r_1 + \dots + r_k = r}} r_1! \binom{\alpha_1}{r_1} \binom{\beta_1}{r_1} \cdots r_k! \binom{\alpha_k}{r_k} \binom{\beta_k}{r_k} I_{m+n-2r}(e^{\otimes(\alpha'+\beta')})
 \end{aligned}$$

here we use Corollary 20.21, $\alpha = \alpha' + \gamma$, $\beta = \beta' + \gamma$ and $|\gamma| = r$)

$$\begin{aligned}
 &= \sum_{r=0}^{m \wedge n} \sum_{\substack{r_1, \dots, r_k \geq 0 \\ r_1 + \dots + r_k = r}} r_1! \binom{\alpha_1}{r_1} \binom{\beta_1}{r_1} \cdots r_k! \binom{\alpha_k}{r_k} \binom{\beta_k}{r_k} \sum_{\substack{\alpha = \alpha' + \gamma, |\alpha|=m, |\gamma|=r \\ \beta = \beta' + \gamma, |\beta|=n}} \xi^\alpha \eta^\beta I_{m+n-2r}(e^{\otimes(\alpha'+\beta')}) \\
 &= \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} \sum_{\substack{\alpha = \alpha' + \gamma, |\alpha|=m, |\gamma|=r \\ \beta = \beta' + \gamma, |\beta|=n}} \xi^\alpha \eta^\beta I_{m+n-2r}(e^{\otimes(\alpha'+\beta')}) \quad (\text{use the identity } (*)) \\
 &= \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r} \left(\left(\sum_{i=1}^k \xi_i e_i \right)^{\otimes \alpha} \otimes_r \left(\sum_{i=1}^k \eta_i e_i \right)^{\otimes \beta} \right) \\
 &= \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f \otimes_r g).
 \end{aligned}$$

The penultimate equality follows from the definition of the contraction operation \otimes_k . Note that, b/o orthogonality, we have to contract the same e_j 's, otherwise we get zeros. This explains the structure of $\alpha = \alpha' + \gamma$ and $\beta = \beta' + \gamma$.

Problem 20.10. Solution: The idea is to use the following Integration-by-Parts formula which is a consequence of the Itô formula:

$$M_T N_T - M_0 N_0 = \int_0^T M_t dN_t + \int_0^T N_t dM_t + \langle M, N \rangle_T. \quad (*)$$

We will use this several times. Moreover, we use that

$$\frac{1}{2!}I_2(f) = \int_0^\infty dB_{t_2} \int_0^{t_1} dB_{t_1} f(t_1, t_2)$$

(we use physics notation where the integrator accompanies the integral and the integrand comes at the end). Thus

$$\begin{aligned} \frac{1}{4}I_2(f)I_2(g) &= \left(\int_0^\infty dB_{t_2} \int_0^{t_2} dB_{t_1} f(t_1, t_2) \right) \left(\int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} g(t_3, t_4) \right) \\ &\stackrel{(*)}{=} \int_0^\infty dB_{t_4} \left(\int_0^{t_4} dB_{t_2} \int_0^{t_2} dB_{t_1} f(t_1, t_2) \right) \left(\int_0^{t_4} dB_{t_3} g(t_3, t_4) \right) \\ &\quad + \boxed{\text{similar term with } f, g \text{ interchanged}} \\ &\quad + \langle I_2(f), I_2(g) \rangle. \end{aligned}$$

Now we observe that (cf. Theorem 20.15 and its proof)

$$\langle I_2(f), I_2(g) \rangle = \frac{1}{4} \int_0^\infty \int_0^\infty f(t_1, t_2)g(t_1, t_2) dt_1 dt_2 = \frac{1}{4}I_0(f \otimes_2 g)$$

which means that the $r = 2$ term appears and has the right multiplicity (factor 1).

We will now deal with the two other terms. For symmetry reasons it is enough to consider the term which is spelled out explicitly (and we should keep in mind to multiply every factor by 2, in the end...). We have, using integration by parts again for the product appearing under the first integral in the formula below,

$$\begin{aligned} &\int_0^\infty dB_{t_4} \left(\int_0^{t_4} dB_{t_2} \int_0^{t_2} dB_{t_1} f(t_1, t_2) \right) \left(\int_0^{t_4} dB_{t_3} g(t_3, t_4) \right) \\ &= \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} dB_{t_2} \int_0^{t_2} dB_{t_1} f(t_1, t_2)g(t_3, t_4) \\ &\quad + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \left(\int_0^{t_3} dB_{t_2} g(t_4, t_2) \right) \left(\int_0^{t_3} dB_{t_1} f(t_1, t_3) \right) \end{aligned}$$

There is no bracket term since the iterated integrals appearing in the bracket have different length (2 vs. 1)

$$\begin{aligned} &= \frac{1}{4!}I_4(f \otimes g) \\ &\quad + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \left(\int_0^{t_3} dB_{t_2} g(t_4, t_2) \right) \left(\int_0^{t_3} dB_{t_1} f(t_1, t_3) \right) \end{aligned}$$

And yet again using integration by parts for the product in the second integral yields

$$\begin{aligned} &= \frac{1}{4!}I_4(f \otimes g) \\ &\quad + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \left(\int_0^{t_3} dB_{t_1} f(t_1, t_3) \int_0^{t_1} dB_{t_2} g(t_4, t_2) \right) \\ &\quad + \boxed{\text{similar term with } f, g \text{ interchanged}} \\ &\quad + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \left\langle \int_0^\bullet g(t_4, t_2) dB_{t_2}, \int_0^\bullet f(t_2, t_3) dB_{t_2} \right\rangle_{t_3} \\ &= \frac{1}{4!}I_4(f \otimes g) \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} dB_{t_1} \int_0^{t_1} dB_{t_2} f(t_1, t_3) g(t_4, t_2) \\
 & + \boxed{\text{similar term with } f, g \text{ interchanged}} \\
 & + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} g(t_4, t_2) f(t_2, t_3) dt_2 \\
 = & \frac{1}{4!} I_4(f \otimes g) + \frac{1}{4!} I_4(f \otimes g) + \frac{1}{4!} I_4(f \otimes g) \\
 & + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} g(t_4, t_2) f(t_2, t_3) dt_2 \\
 = & \frac{1}{4} \frac{1}{2} I_4(f \otimes g) \\
 & + \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} g(t_4, t_2) f(t_2, t_3) dt_2
 \end{aligned}$$

This shows that the $r = 0$ term $I_4(f \otimes g)$ appears once (don't forget that we have to multiply by 2, b/o the "similar terms" in the very first calculation and that our left-hand side was $\frac{1}{4} I_2(f) I_2(g)$).

So, finally for the $r = 1$ -term $I_2(f \otimes_1 g)$ which has multiplicity 4. Set $\phi^s(a, b) := \phi(a, b) \mathbf{1}_{[0, s]^2}(a, b)$ and observe that

$$\begin{aligned}
 \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} \int_0^{t_3} g(t_4, t_2) f(t_2, t_3) dt_2 & = \int_0^\infty dB_{t_4} \int_0^{t_4} dB_{t_3} g^{t_3} \otimes_1 f^{t_3}(t_3, t_4) \\
 & = \frac{1}{2!} I_2(g \otimes_1 f).
 \end{aligned}$$

Again we have two of these terms (mind the factor 2 for the very first long calculation), and the lhs is $\frac{1}{4} I_2(f) I_2(g)$ which brings us to $4 I_2(g \otimes_1 f)$.

Problem 20.11. Solution: Since $B_T \sim \sqrt{T} B_1$, we have

$$\begin{aligned}
 \mathbb{E} \left[B_T^{2n} e^{-B_T^2} \right] & = \mathbb{E} \left[T^n B_1^{2n} e^{-T B_1^2} \right] \\
 & = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} T^n x^{2n} e^{-T x^2} e^{-x^2/2} dx \\
 & = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} T^n \frac{y^{2n}}{(2T+1)^n} e^{-y^2/2} \frac{dy}{\sqrt{2T+1}} \\
 & = \left(\frac{T}{2T+1} \right)^n \frac{1}{\sqrt{2T+1}} \mathbb{E} B_1^{2n}.
 \end{aligned}$$

Now we can use $\mathbb{E} B_1^{2n} = (2n-1)!!$, see § 2.3.

Problem 20.12. Solution: This is essentially Example 20.20, but we do it once again. The idea is to use the identity (20.11)

$$I_n(g^{\otimes n}) = H_n(I_1(g), \|g\|_{L^2}^2)$$

and a concrete Hermite expansion for $e^{rx - \frac{1}{2} r^2 t}$

$$F = \exp \left[I_1(g) - \frac{1}{2} \|g\|_{L^2}^2 \right] = \exp \left[rx - \frac{1}{2} r^2 t \right] \Big|_{r=1, t=\|g\|_{L^2}^2, x=I_1(g)}.$$

Since $e^{rx - \frac{1}{2}r^2t}$ is the generating function of the (extended) Hermite polynomials, we see (Theorem 20.13.c) that

$$e^{x-t/2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, t)$$

and from this we get

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(I_1(g), \|g\|_{L^2}^2) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(g^{\otimes n}) = 1 + \sum_{n=1}^{\infty} I_n\left(\frac{1}{n!} g^{\otimes n}\right)$$

which is the desired expansion. In view of the Taylor–Stroock theorem (Theorem 20.25) we can read off from this the values $\mathbb{E} D_{t_1, \dots, t_n}^n F$, but the problem asks for the derivatives without expected value. To find those we observe that F is essentially a cylindrical function

$$F = \exp\left[I_1(g) - \frac{1}{2}\|g\|_{L^2}^2\right] = \phi(I_1(g))$$

with $\phi(x) = \exp[x - \frac{1}{2}c]$ and $c = \|g\|_{L^2}^2$. So,

$$\begin{aligned} D_t F &= (\partial\phi)(I_1(g))g(t), & D_{st}^2 F &= (\partial^2\phi)(I_1(g))g(t)g(s), & \dots, \\ D_{t_1 \dots t_n}^n F &= (\partial^n\phi)(I_1(g))g(t_1) \dots g(t_n) = (\partial^n\phi)(I_1(g))g^{\otimes n}(t_1, \dots, t_n), \end{aligned}$$

and we note that $\partial^n\phi(x) = \partial_x^n e^{x-c/2} = e^{x-c/2}$, so

$$D_t F = Fg(t), \quad D_{st}^2 F = Fg(t)g(s), \quad \dots, \quad D_{t_1 \dots t_n}^n F = Fg(t_1) \dots g(t_n).$$

Since $\mathbb{E} F = 1$, this result is in-line with the Taylor–Stroock formula. ■ ■

Problem 20.13. Solution:

- a) This is a special case of the previous exercise, Problem 12. Taking $g(t) = \mathbb{1}_{[0,T]}(t)$, we see that $B_T = I_1(\mathbb{1}_{[0,T]})$ and $\|g\|_{L^2}^2 = \int_0^T \mathbb{1}_{[0,T]}(t) dt = T$. Therefore

$$e^{B_T - \frac{1}{2}T} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(\mathbb{1}_{[0,T]}^{\otimes n}).$$

- b) We use here the Taylor–Stroock formula. Set $F = B_T^5 = (I_1(\mathbb{1}_{[0,T]}))^5$ and $\phi(x) = x^5$. Then

$$B_T^5 = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E} D_{t_1, \dots, t_n}^n F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E} \partial^n \phi(x)|_{x=B_T} \mathbb{1}_{[0,T]}^{\otimes n}).$$

Since odd powers of Brownian motion have zero expectation, we get

$$\begin{aligned} B_T^5 &= I_1(5 \mathbb{E} B_T^4 \mathbb{1}_{[0,T]}) + \frac{1}{6} I_3(60 \mathbb{E} B_T^2 \mathbb{1}_{[0,T]}^{\otimes 3}) + \frac{1}{120} I_5(120 \mathbb{1}_{[0,T]}^{\otimes 5}) \\ &= 15T^2 I_1(\mathbb{1}_{[0,T]}) + 10T I_3(\mathbb{1}_{[0,T]}^{\otimes 3}) + I_5(\mathbb{1}_{[0,T]}^{\otimes 5}). \end{aligned}$$

- c) The idea is the following: $\int_0^1 F_t dB_t$ is just $I_1(F_t)$. So, if we work out the representation of F_t as

$$F_t = \sum_{n=0}^{\infty} I_n(f_n(t, \cdot)) \implies I_1(F_t) = \sum_{n=0}^{\infty} I_1(I_n(f_n(t, \cdot))) = \sum_{n=0}^{\infty} \frac{1}{n+1} I_{n+1}(\widetilde{f_n(t, \cdot)})$$

where $\widetilde{f_n(t, \cdot)}$ means the symmetrization of $(t, t_n, \dots, t_1) \mapsto f_n(t, t_n, \dots, t_1)$. The second equality in the display formula from above comes from the representation of I_n as iterated integral, see also the previous part of the problem.

Now let $F_t = r^3 B_t^3 + 2tB_t^2 = r^3 I_1(\mathbb{1}_{[0,t]})^3 + 2tI_1(\mathbb{1}_{[0,t]})^2 = [r^3 x^3 + 2tx^2]_{x=B_t}$. Applying the Taylor-Stroock formula we get

$$\begin{aligned} \mathbb{E} D_s F_t &= \mathbb{E} \left[3r^3 x^2 + 4tx \right]_{x=B_t} \mathbb{1}_{[0,t]}(s) = 3r^3 t \mathbb{1}_{[0,t]}(s) \\ \mathbb{E} D_{su}^2 F_t &= \mathbb{E} \left[6r^3 x + 4t \right]_{x=B_t} \mathbb{1}_{[0,t]}(s) \mathbb{1}_{[0,t]}(u) = 4t \mathbb{1}_{[0,t]}(s) \mathbb{1}_{[0,t]}(u) \\ \mathbb{E} D_{suw}^3 F_t &= \mathbb{E} \left[6r^3 \right]_{x=B_t} \mathbb{1}_{[0,t]}(s) \mathbb{1}_{[0,t]}(u) \mathbb{1}_{[0,t]}(w) = 6r^3 \mathbb{1}_{[0,t]}(s) \mathbb{1}_{[0,t]}(u) \mathbb{1}_{[0,t]}(w) \end{aligned}$$

and, thus,

$$\begin{aligned} F_t &= \mathbb{E} F_t + \frac{6r^3 t}{1!} I_1(\mathbb{1}_{[0,t]}) + \frac{4t}{2!} I_2(\mathbb{1}_{[0,t]}^{\otimes 2}) + \frac{6r^3}{3!} I_3(\mathbb{1}_{[0,t]}^{\otimes 3}) \\ &= 2t^2 + 6r^3 t I_1(\mathbb{1}_{[0,t]}) + 2t I_2(\mathbb{1}_{[0,t]}^{\otimes 2}) + r^3 I_3(\mathbb{1}_{[0,t]}^{\otimes 3}). \end{aligned}$$

Finally, with the observation made at the beginning, we get

$$F = 2tI_1(t^2 \mathbb{1}_{[0,1]}(t)) + 3r^3 I_2(\mathbb{1}_{[0,1]}^{\otimes 2}(t, u)) + \frac{2}{3} I_3(t \mathbb{1}_{[0,1]}^{\otimes 3}(t, u, v)) + \frac{1}{4} I_4(\mathbb{1}_{[0,1]}^{\otimes 4}(t, u, v, w)).$$

- d) Here we use the same strategy as for the previous problem. Write $F_t = te^{B_t}$ for the integrand, set $\phi(x) = te^x$ and use the Taylor–Stroock formula to get

$$D^k F_t = (\partial_x^k \phi)(B_t) \mathbb{1}_{[0,t]}^{\otimes k} \implies \mathbb{E} D^k F_t = t \mathbb{E} e^{B_t} \mathbb{1}_{[0,t]}^{\otimes k} = te^{t/2} \mathbb{1}_{[0,t]}^{\otimes k}.$$

Thus,

$$F_t = \sum_{k=0}^{\infty} \frac{1}{k!} te^{t/2} I_k(\mathbb{1}_{[0,t]}^{\otimes k})$$

and

$$F = I_1(F_\bullet) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{I_1 \left(te^{t/2} I_k(\mathbb{1}_{[0,t]}^{\otimes k}) \right)}_{I_1 \text{ acts on } t} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} I_{k+1} \left(te^{t/2} \mathbb{1}_{[0,1]}^{\otimes(k+1)}(t, t_k, \dots, t_1) \right);$$

mind the abuse of notation in the latter integral: t, t_k, \dots, t_1 are all integration variables for the $(k+1)$ -fold integral I_{k+1} .

Problem 20.14. Solution: Observe that $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Since $\mathbb{E} e^{B_T} = \mathbb{E} e^{-B_T} = e^{T/2}$, we see that $\mathbb{E} \sinh B_T = 0$ and $\mathbb{E} \cosh B_T = e^{T/2}$. ■■

a) Using the Taylor–Stroock formula we get

$$\begin{aligned} D^{2k+1}(\sinh B_T) &= \partial_x^{2k+1} \sinh x \Big|_{x=B_T} \mathbf{1}_{[0,T]}^{\otimes(2k+1)} = \cosh B_T \mathbf{1}_{[0,T]}^{\otimes(2k+1)} \\ &\implies \mathbb{E}[\cosh B_T] \mathbf{1}_{[0,T]}^{\otimes(2k+1)} = e^{T/2} \mathbf{1}_{[0,T]}^{\otimes(2k+1)}. \end{aligned}$$

and

$$\begin{aligned} D^{2k}(\sinh B_T) &= \partial_x^{2k} \sinh x \Big|_{x=B_T} \mathbf{1}_{[0,T]}^{\otimes 2k} = \sinh B_T \mathbf{1}_{[0,T]}^{\otimes 2k} \\ &\implies \mathbb{E}[\sinh B_T] \mathbf{1}_{[0,T]}^{\otimes 2k} = 0. \end{aligned}$$

Therefore,

$$\sinh B_T = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} e^{T/2} I_{2k+1} \left(\mathbf{1}_{[0,T]}^{\otimes(2k+1)} \right).$$

b) The calculation for the cosh is completely similar, only we get an even expansion:

$$\cosh B_T = \sum_{k=0}^{\infty} \frac{1}{(2k)!} e^{T/2} I_{2k} \left(\mathbf{1}_{[0,T]}^{\otimes 2k} \right).$$

Remark. An alternative solution can be based on the series representations for $\exp[\pm B_T]$, using Problem 12 or Problem 13(a).

Problem 20.15. Solution: If Σ is dense in $L^2(\mu)$, then $u = \sum_n \lambda_n \phi_n$ for some suitable $\phi_n \in \mathcal{D}$. Thus,

$$\|u\|^2 = \sum_n \lambda_n \langle u, \phi_n \rangle = \sum_n \lambda_n \cdot 0 = 0$$

and we get $u = 0$ in $L^2(\mu)$.

Conversely, since Σ is a linear subspace of $L^2(\mu)$, we get $L^2(\mu) = \overline{\Sigma} \oplus \overline{\Sigma}^\perp$. Thus, the condition $\langle u, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$ entails that $u \perp \mathcal{D}$, hence $u \perp \Sigma$, hence $u \perp \overline{\Sigma}$. If this entails $u = 0$, we get $\overline{\Sigma}^\perp = \{0\}$ or $\overline{\Sigma} = L^2(\mu)$.

Alternatively, we could use Hahn-Banach: $\langle u, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$ entails that $\langle u, \psi \rangle = 0$ for all $\psi \in \overline{\Sigma}$. If $\overline{\Sigma} \neq L^2(\mu)$, we can extend u , by Hahn-Banach, in such a way that $u \neq 0$ but $u|_{\overline{\Sigma}} = 0$. This gives a contradiction to $u = 0$, hence $\overline{\Sigma} = L^2(\mu)$.

21 Stochastic differential equations

Problem 21.1. Solution: We have

$$dX_t = b(t) dt + \sigma(t) dB_t$$

where b, σ are non-random coefficients such that the corresponding (stochastic) integrals exist. Without loss of generality we assume that $X_0 = x = 0$. Obviously,

$$(dX_t)^2 = \sigma^2(t) (dB_t)^2 = \sigma^2(t) dt$$

and we get for $0 \leq s \leq t < \infty$, using Itô's formula,

$$\begin{aligned} e^{i\xi X_t} - e^{i\xi X_s} &= \int_s^t i\xi e^{i\xi X_r} b(r) dr + \int_s^t i\xi e^{i\xi X_r} \sigma(r) dB_r \\ &\quad - \frac{1}{2} \int_s^t \xi^2 e^{i\xi X_r} \sigma^2(r) dr. \end{aligned}$$

Now take any $F \in \mathcal{F}_s$ and multiply both sides of the above formula by $e^{-\xi X_s} \mathbf{1}_F$. We get

$$\begin{aligned} e^{i\xi(X_t - X_s)} \mathbf{1}_F - \mathbf{1}_F &= \int_s^t i\xi e^{i\xi(X_r - X_s)} \mathbf{1}_F b(r) dr + \int_s^t i\xi e^{i\xi(X_r - X_s)} \mathbf{1}_F \sigma(r) dB_r \\ &\quad - \frac{1}{2} \int_s^t \xi^2 e^{i\xi(X_r - X_s)} \mathbf{1}_F \sigma^2(r) dr. \end{aligned}$$

Taking expectations gives

$$\begin{aligned} \mathbb{E} \left(e^{i\xi(X_t - X_s)} \mathbf{1}_F \right) &= \mathbb{P}(F) + \int_s^t i\xi \mathbb{E} \left(e^{i\xi(X_r - X_s)} \mathbf{1}_F \right) b(r) dr \\ &\quad - \frac{1}{2} \int_s^t \xi^2 \mathbb{E} \left(e^{i\xi(X_r - X_s)} \mathbf{1}_F \right) \sigma^2(r) dr \\ &= \mathbb{P}(F) + \int_s^t \left(i\xi b(r) - \frac{1}{2} \xi^2 \sigma^2(r) \right) \mathbb{E} \left(e^{i\xi(X_r - X_s)} \mathbf{1}_F \right) dr. \end{aligned}$$

Define $\phi_{s,t}(\xi) := \mathbb{E} \left(e^{i\xi(X_t - X_s)} \mathbf{1}_F \right)$. Then the integral equation

$$\phi_{s,t}(\xi) = \mathbb{P}(F) + \int_s^t \left(i\xi b(r) - \frac{1}{2} \xi^2 \sigma^2(r) \right) \phi_{r,s}(\xi) dr$$

has the unique solution (use Gronwall's lemma, cf. also the proof of Theorem 19.5)

$$\phi_{s,t}(\xi) = \mathbb{P}(F) e^{i\xi \int_s^t b(s) ds - \frac{1}{2} \xi^2 \int_s^t \sigma^2(r) dr}$$

and so

$$\mathbb{E} \left(e^{i\xi(X_t - X_s)} \mathbf{1}_F \right) = \mathbb{P}(F) e^{i\xi \int_s^t b(r) dr - \frac{1}{2} \xi^2 \int_s^t \sigma^2(r) dr}. \quad (*)$$

If we take in (*) $F = \Omega$ and $s = 0$, we see that

$$X_t \sim \mathbf{N}(\mu_t, \sigma_t^2), \quad \mu_t = \int_0^t b(r) dr, \quad \sigma_t^2 = \int_0^t \sigma^2(r) dr.$$

If we take in (*) $F = \Omega$ then the increment satisfies $X_t - X_s \sim \mathbf{N}(\mu_t - \mu_s, \sigma_t^2 - \sigma_s^2)$. If F is arbitrary, (*) shows that

$$X_t - X_s \perp\!\!\!\perp \mathcal{F}_s,$$

see the Lemma at the end of this section.

The above considerations show that

$$\mathbb{E} e^{\sum_{j=1}^n \xi_j (X_{t_j} - X_{t_{j-1}})} = \prod_{j=1}^n \exp\left(i\xi \int_{t_{j-1}}^{t_j} b(r) dr - \frac{1}{2}\xi^2 \int_{t_{j-1}}^{t_j} \sigma^2(r) dr\right),$$

i.e. $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ is a Gaussian random vector with independent components. Since $X_{t_k} = \sum_{j=1}^k (X_{t_j} - X_{t_{j-1}})$ we see that $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian random variable.

Let us, finally, compute $\mathbb{E}(X_s X_t)$. By independence, we have

$$\begin{aligned} \mathbb{E}(X_s X_t) &= \mathbb{E}(X_s^2) + \mathbb{E} X_s (X_t - X_s) \\ &= \mathbb{E}(X_s^2) + \mathbb{E} X_s \mathbb{E}(X_t - X_s) \\ &= \mathbb{E}(X_s^2) + \mathbb{E} X_s \mathbb{E} X_t - (\mathbb{E} X_s)^2 \\ &= \mathbb{V} X_s + \mathbb{E} X_s \mathbb{E} X_t \\ &= \int_0^s \sigma^2(r) dr + \int_0^s b(r) dr \int_0^t b(r) dr. \end{aligned}$$

In fact, since the mean is not zero, it would have been more elegant to compute the covariance

$$\text{Cov}(X_s, X_t) = \mathbb{E}(X_s - \mu_s)(X_t - \mu_t) = \mathbb{E}(X_s X_t) - \mathbb{E} X_s \mathbb{E} X_t = \mathbb{V} X_s = \int_0^s \sigma^2(r) dr.$$

Lemma. *Let X be a random variable and \mathcal{F} a σ field. Then*

$$\mathbb{E}(e^{i\xi X} \mathbf{1}_F) = \mathbb{E} e^{i\xi X} \cdot \mathbb{P}(F) \quad \forall \xi \in \mathbb{R} \implies X \perp\!\!\!\perp \mathcal{F}.$$

Proof. Note that $e^{i\eta \mathbf{1}_F} = e^{i\eta} \mathbf{1}_F + \mathbf{1}_{F^c}$. Thus,

$$\begin{aligned} \mathbb{E}(e^{i\xi X} \mathbf{1}_{F^c}) &= \mathbb{E}(e^{i\xi X}) - \mathbb{E}(e^{i\xi X} \mathbf{1}_F) \\ &= \mathbb{E}(e^{i\xi X}) - \mathbb{E}(e^{i\xi X}) \mathbb{P}(F) \\ &= \mathbb{E}(e^{i\xi X}) \mathbb{P}(F^c) \end{aligned}$$

and this implies

$$\mathbb{E}(e^{i\xi X} e^{i\eta \mathbf{1}_F}) = \mathbb{E}(e^{i\xi X}) \mathbb{E}(e^{i\eta \mathbf{1}_F}) \quad \forall \xi, \eta \in \mathbb{R}.$$

This shows that $X \perp\!\!\!\perp \mathbf{1}_F$ and $X \perp\!\!\!\perp F$ for all $F \in \mathcal{F}$. □

Problem 21.2. Solution:

a) We have $\Delta t = 2^{-n}$ and

$$\Delta X_n(t_{k-1}) = X_n(t_k) - X_n(t_{k-1}) = -\frac{1}{2} X_n(t_{k-1}) 2^{-n} + B(t_k) - B(t_{k-1})$$

and this shows

$$\begin{aligned} X_n(t_k) &= X_n(t_{k-1}) - \frac{1}{2} X_n(t_{k-1}) 2^{-n} + B(t_k) - B(t_{k-1}) \\ &= (1 - 2^{-n-1}) X_n(t_{k-1}) + B(t_k) - B(t_{k-1}) \\ &= (1 - 2^{-n-1}) [(1 - 2^{-n-1}) X_n(t_{k-2}) + B(t_{k-1}) - B(t_{k-2})] + [B(t_k) - B(t_{k-1})] \\ &\vdots \\ &= (1 - 2^{-n-1})^k X_n(t_0) + (1 - 2^{-n-1})^{k-1} [B(t_1) - B(t_0)] + \dots + \\ &\quad + (1 - 2^{-n-1}) [B(t_{k-1}) - B(t_{k-2})] + [B(t_k) - B(t_{k-1})] \\ &= (1 - 2^{-n-1})^k A + \sum_{j=1}^{k-1} (1 - 2^{-n-1})^j [B(t_{k-j}) - B(t_{k-j-1})] \end{aligned}$$

Observe that $B(t_j) - B(t_{j-1}) \sim \mathbf{N}(0, 2^{-n})$ for all j and $A \sim \mathbf{N}(0, 1)$. Because of the independence we get

$$X_n(t_n) = X_n(k2^{-n}) \sim \mathbf{N}\left(0, (1 - 2^{-n-1})^{2k} + \sum_{j=1}^{k-1} (1 - 2^{-n-1})^{2j} \cdot 2^{-n}\right)$$

For $k = 2^{n-1}$ we get $t_k = \frac{1}{2}$ and so

$$X_n\left(\frac{1}{2}\right) \sim \mathbf{N}\left(0, (1 - 2^{-n-1})^{2^n} + \sum_{j=1}^{2^n-1} (1 - 2^{-n-1})^{2j} \cdot 2^{-n}\right).$$

Using

$$\lim_{n \rightarrow \infty} (1 - 2^{-n-1})^{2^n} = e^{-\frac{1}{2}}$$

and

$$\sum_{j=1}^{2^n-1} (1 - 2^{-n-1})^{2j} \cdot 2^{-n} = \frac{1 - (1 - 2^{-n-1})^{2^n}}{1 - (1 - 2^{-n-1})^2} \cdot 2^{-n} = \frac{1 - (1 - 2^{-n-1})^{2^n}}{1 - 2^{-n-2}} \xrightarrow{n \rightarrow \infty} 1 - e^{-\frac{1}{2}}$$

finally shows that $X_n\left(\frac{1}{2}\right) \xrightarrow[n \rightarrow \infty]{d} X \sim \mathbf{N}(0, 1)$.

b) The solution of this SDE follows along the lines of Example 21.7 where $\alpha(t) \equiv 0$, $\beta(t) \equiv -\frac{1}{2}$, $\delta(t) \equiv 0$ and $\gamma(t) \equiv 1$:

$$\begin{aligned} dX_t^\circ &= \frac{1}{2} X_t^\circ dt \implies X_t^\circ = e^{t/2} \\ Z_t &= e^{t/2} X_t, \quad Z_0 = X_0 \\ dZ_t &= e^{t/2} dB_t \implies Z_t = Z_0 + \int_0^t e^{s/2} dB_s \\ X_t &= e^{-t/2} A + e^{-t/2} \int_0^t e^{s/2} dB_s. \end{aligned}$$

For $t = \frac{1}{2}$ we get

$$\begin{aligned} X_{1/2} &= A e^{-1/4} + e^{-1/4} \int_0^{1/2} e^{s/2} dB_s \\ \implies X_{1/2} &\sim \mathbf{N}\left(0, e^{-1/2} + e^{-1/2} \int_0^{1/2} e^s ds\right) = \mathbf{N}(0, 1). \end{aligned}$$

So, we find for all $s \leq t$

$$\begin{aligned} C(s, t) &= \mathbb{E} X_s X_t = e^{-s/2} e^{-t/2} \mathbb{E} A^2 + e^{-s/2} e^{-t/2} \mathbb{E} \left(\int_0^s e^{r/2} dB_r \int_0^t e^{u/2} dB_u \right) \\ &= e^{-(s+t)/2} + e^{-(s+t)/2} \int_0^s e^r dr \\ &= e^{-(t-s)/2}. \end{aligned}$$

This finally shows that $C(s, t) = e^{-|t-s|/2}$.

Problem 21.3. Solution: We can rewrite the SDE as

$$\begin{aligned} X_t &= x + b \int_0^t X_s ds + \int_0^t X_s d(\sigma_1 b_s + \sigma_2 \beta_s) \\ &= x + b \int_0^t X_s ds + \sqrt{\sigma_1^2 + \sigma_2^2} \int_0^t X_s dW_s \end{aligned}$$

where

$$W_s := \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} b_t + \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \beta_t$$

is, by Problem 19.3, a BM¹. This reduces the problem to a geometric Brownian motion as in Example a:

$$\begin{aligned} X_t &= x \exp \left(\left[b - \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \right] t + \sqrt{\sigma_1^2 + \sigma_2^2} W_t \right) \\ &= x \exp \left(\left[b - \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \right] t + \sigma_1 b_t + \sigma_2 \beta_t \right). \end{aligned}$$

Alternative Solution: As in Example 21.6, we assume that the initial condition $X_0 = x$ is positive and apply Itô's formula (18.15) to $Z_t := \log X_t$:

$$\begin{aligned} Z_t - Z_0 &= \int_0^t \frac{1}{X_s} dX_s + \frac{1}{2} \int_0^t \left(-\frac{1}{X_s^2} \right) (dX_s)^2 \\ &= \int_0^t b ds + \int_0^t \sigma_1 db_s + \int_0^t \sigma_2 d\beta_s - \frac{1}{2} \int_0^t (\sigma_1^2 + \sigma_2^2) ds \\ &= \left(b - \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \right) \cdot t + \sigma_1 b_t + \sigma_2 \beta_t. \end{aligned}$$

Since, by assumption,

$$\begin{aligned} dX_s &= b X_s ds + \sigma_1 X_s d\beta_s + \sigma_2 X_s db_s \\ \implies (dX_s)^2 &= (\sigma_1 X_s d\beta_s)^2 + (\sigma_2 X_s db_s)^2 = (\sigma_1^2 + \sigma_2^2) X_s^2 ds. \end{aligned}$$

Consequently,

$$X_t = x \exp\left(\sigma_1 b_t + \sigma_2 \beta_t + \left(b - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\right)t\right).$$

A direct calculation shows that X_t is indeed a solution of the given SDE. ■ ■

Problem 21.4. Solution: Since X_t° is such that $1/X_t^\circ$ solves the homogeneous SDE from Example 21.6, we see that

$$X_t^\circ = \exp\left(-\int_0^t (\beta(s) - \frac{1}{2}\delta^2(s)) ds\right) \exp\left(-\int_0^t \delta(s) dB_s\right)$$

(mind that the ‘minus’ sign comes from $1/X_t^\circ$).

Observe that $X_t^\circ = f(I_t^1, I_t^2)$ where I_t is an Itô process with

$$\begin{aligned} I_t^1 &= -\int_0^t (\beta(s) - \frac{1}{2}\delta^2(s)) ds \\ I_t^2 &= -\int_0^t \delta(s) dB_s. \end{aligned}$$

Now we get from Itô’s multiplication table

$$dI_t^1 dI_t^1 = dI_t^1 dI_t^2 = 0 \quad \text{and} \quad dI_t^2 dI_t^2 = \delta^2(t) dt$$

and, by Itô’s formula

$$\begin{aligned} dX_t^\circ &= \partial_1 f(I_t^1, I_t^2) dI_t^1 + \partial_2 f(I_t^1, I_t^2) dI_t^2 + \frac{1}{2} \sum_{j,k=1}^2 \partial_j \partial_k dI_t^j dI_t^k \\ &= X_t^\circ (dI_t^1 + dI_t^2 + \frac{1}{2} dI_t^2 dI_t^2) \\ &= X_t^\circ (-\beta(t) dt + \frac{1}{2} \delta^2(t) dt - \delta(t) dB_t + \frac{1}{2} \delta^2(t) dt) \\ &= X_t^\circ (-\beta(t) + \delta^2(t)) dt - X_t^\circ \delta(t) dB_t. \end{aligned}$$

Remark:

1. We used here the two-dimensional Itô formula (18.14) but we could have equally well used the one-dimensional version (18.13) with the Itô process $I_t^1 + I_t^2$.
2. Observe that Itô’s multiplication table gives us exactly the second-order term in (18.14).

Since

$$dZ_t = (\alpha(t) - \gamma(t)\delta(t))X_t^\circ dt + \gamma(t)X_t^\circ dB_t \quad \text{and} \quad X_t = Z_t/X_t^\circ$$

we get

$$X_t = \frac{1}{X_t^\circ} \left(X_0 + \int_0^t (\alpha(s) - \gamma(s)\delta(s))X_s^\circ ds + \int_0^t \gamma(s)X_s^\circ dB_s \right).$$

Problem 21.5. Solution:

a) We have $X_t = e^{-\beta t} X_0 + \int_0^t \sigma e^{-\beta(t-s)} dB_s$. This can be shown in four ways:

Solution 1: you guess the right result and use Itô's formula (18.7) to verify that the above X_t is indeed a solution to the SDE. For this rewrite the above solution as

$$e^{\beta t} X_t = X_0 + \int_0^t \sigma e^{\beta s} dB_s \implies d(e^{\beta t} X_t) = \sigma e^{\beta t} dB_t.$$

Now with the two-dimensional Itô formula for $f(x, y) = xy$ and the two-dimensional Itô-process $(e^{\beta t}, X_t)$ we get

$$d(e^{\beta t} X_t) = \beta X_t e^{\beta t} dt + e^{\beta t} dX_t$$

so that

$$\beta X_t e^{\beta t} dt + e^{\beta t} dX_t = \sigma e^{\beta t} dB_t \iff dX_t = -\beta X_t dt + \sigma dB_t.$$

Admittedly, this is unfair as one has to know the solution beforehand. On the other hand, this is exactly the way one *verifies* that the solution one has found is the correct one.

Solution 2: you apply the time-dependent Itô formula from Problem 18.7 or the two-dimensional Itô formula, Theorem 18.8 to

$$X_t = u(t, I_t) \quad \text{and} \quad I_t = \int_0^t e^{\beta s} dB_s \quad \text{and} \quad u(t, x) = e^{\beta t} X_0 + \sigma e^{\beta t} x$$

to get—as $dt dB_t = 0$ —

$$dX_t = \partial_t u(t, I_t) dt + \partial_x u(t, I_t) dI_t + \frac{1}{2} \partial_x^2 u(t, I_t) dt.$$

Again, this is best for the verification of the solution since you need to know its form beforehand.

Solution 3: you use Example 21.7 with $\alpha(t) \equiv 0$, $\beta(t) \equiv -\beta$, $\gamma(t) \equiv \sigma$ and $\delta(t) \equiv 0$. But, honestly, you will have to look up the formula in the book. We get

$$\begin{aligned} dX_t^\circ &= \beta X_t^\circ dt, \quad X_0^\circ = 1 \implies X_t^\circ = e^{\beta t}; \\ Z_t &= e^{\beta t} X_t, \quad Z_0 = X_0 = \xi = \text{const.}; \\ dZ_t &= \sigma e^{\beta t} dB_t; \\ Z_t &= \sigma \int_0^t e^{\beta s} dB_s + Z_0; \\ X_t &= e^{-\beta t} \xi + e^{-\beta t} \sigma \int_0^t e^{\beta s} dB_s, \quad t \geq 0. \end{aligned}$$

Solution 4: by bare hands and with Itô's formula! Consider first the *deterministic* ODE

$$x_t = x_0 - \beta \int_0^t x_s ds$$

which has the solution $x_t = x_0 e^{-\beta t}$, i.e. $e^{\beta t} x_t = x_0 = \text{const}$. This indicates that the transformation

$$Y_t := e^{\beta t} X_t$$

might be sensible. Thus, $Y_t = f(t, X_t)$ where $f(t, x) = e^{\beta t} x$. Thus,

$$\partial_t f(t, x) = \beta f(t, x) = \beta x e^{\beta t}, \quad \partial_x f(t, x) = e^{\beta t}, \quad \partial_x^2 f_{xx}(t, x) = 0.$$

By assumption,

$$dX_t = -\beta X_t dt + \sigma dB_t \implies (dX_t)^2 = \sigma^2 (dB_t)^2 = \sigma^2 dt,$$

and by Itô's formula (18.8) we get

$$\begin{aligned} Y_t - Y_0 &= \int_0^t \left(\underbrace{f_t(s, X_s) - \beta X_s f_x(s, X_s)}_{=0} + \underbrace{\frac{1}{2} \sigma^2 f_{xx}(s, X_s)}_{=0} \right) ds + \int_0^t \sigma f_x(s, X_s) dB_s \\ &= \int_0^t \sigma f_x(s, X_s) dB_s. \end{aligned}$$

So we have the solution, but we still have to go through the procedure in Solution 1 or 2 in order to verify our result.

- b) Since X_t is the limit of normally distributed random variables, it is itself Gaussian (see also part d))—if ξ is non-random or itself Gaussian and independent of everything else. In particular, if $X_0 = \xi = \text{const}$.

$$X_t \sim \mathbf{N}\left(e^{-\beta t} \xi, \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta s} ds\right) = \mathbf{N}\left(e^{-\beta t} \xi, \frac{\sigma^2}{2\beta} (1 - e^{-2\beta t})\right).$$

Now

$$C(s, t) = \mathbb{E} X_s X_t = e^{-\beta(t+s)} \xi^2 + \frac{\sigma^2}{2\beta} e^{-\beta(t+s)} (e^{2\beta s} - 1), \quad t \geq s \geq 0,$$

and, therefore

$$C(s, t) = e^{-\beta(t+s)} \xi^2 + \frac{\sigma^2}{2\beta} (e^{-\beta|t-s|} - e^{-\beta(t+s)}) \quad \text{for all } s, t \geq 0.$$

- c) The asymptotic distribution, as $t \rightarrow \infty$, is $X_\infty \sim \mathbf{N}(0, \sigma^2(2\beta)^{-1})$.
d) We have

$$\begin{aligned} &\mathbb{E} \left(\exp \left[i \sum_{j=1}^n \lambda_j X_{t_j} \right] \right) \\ &= \mathbb{E} \left(\exp \left[i \sum_{j=1}^n \lambda_j e^{-\beta t_j} \xi + i \sigma \sum_{j=1}^n \lambda_j e^{-\beta t_j} \int_0^{t_j} e^{\beta s} dB_s \right] \right) \\ &= \exp \left(-\frac{\sigma^2}{4\beta} \left[\sum_{j=1}^n \lambda_j e^{-\beta t_j} \right]^2 \right) \mathbb{E} \left(\exp \left[i \sigma \sum_{j=1}^n \eta_j Y_j \right] \right) \end{aligned}$$

where

$$\eta_j = \lambda_j e^{-\beta t_j}, \quad Y_j = \int_0^{t_j} e^{\beta s} dB_s, \quad t_0 = 0, \quad Y_0 = 0.$$

Moreover,

$$\sum_{j=1}^n \eta_j Y_j = \sum_{k=1}^n (Y_k - Y_{k-1}) \sum_{j=k}^n \eta_j$$

and

$$Y_k - Y_{k-1} = \int_{t_{k-1}}^{t_k} e^{\beta s} dB_s \sim \mathbf{N}(0, (2\beta)^{-1}(e^{2\beta t_k} - e^{2\beta t_{k-1}})) \quad \text{are independent.}$$

Consequently, we see that

$$\begin{aligned} & \mathbb{E} \left(\exp \left[i \sum_{j=1}^n \lambda_j X_{t_j} \right] \right) \\ &= \exp \left[-\frac{\sigma^2}{4\beta} \left(\sum_{j=1}^n \lambda_j e^{-\beta t_j} \right)^2 \right] \prod_{k=1}^n \exp \left[-\frac{\sigma^2}{4\beta} (e^{2\beta t_k} - e^{2\beta t_{k-1}}) \left(\sum_{j=k}^n \lambda_j e^{-\beta t_j} \right)^2 \right] \\ &= \exp \left[-\frac{\sigma^2}{4\beta} \left(\sum_{j=1}^n \lambda_j e^{-\beta t_j} \right)^2 \{1 + e^{2\beta t_1} - 1\} \right] \times \\ & \quad \times \prod_{k=2}^n \exp \left[-\frac{\sigma^2}{4\beta} (1 - e^{-2\beta(t_k - t_{k-1})}) \cdot \left(\sum_{j=k}^n \lambda_j e^{-\beta(t_j - t_k)} \right)^2 \right] \\ &= \exp \left[-\frac{\sigma^2}{4\beta} \left(\sum_{j=1}^n \lambda_j e^{-\beta(t_j - t_1)} \right)^2 \right] \times \\ & \quad \times \prod_{k=2}^n \exp \left[-\frac{\sigma^2}{4\beta} (1 - e^{-2\beta(t_k - t_{k-1})}) \cdot \left(\sum_{j=k}^n \lambda_j e^{-\beta(t_j - t_k)} \right)^2 \right]. \end{aligned}$$

Note: the distribution of $(X_{t_1}, \dots, X_{t_n})$ depends on the difference of the consecutive epochs $t_1 < \dots < t_n$.

e) We write for all $t \geq 0$

$$\tilde{X}_t = e^{\beta t} X_t \quad \text{and} \quad \tilde{U}_t = e^{\beta t} U_t$$

and we show that both processes have the same finite-dimensional distributions.

Clearly, both processes are Gaussian and both have independent increments. From

$$\tilde{X}_0 = X_0 = 0 \quad \text{and} \quad \tilde{U}_0 = U_0 = 0$$

and for $s \leq t$

$$\begin{aligned} \tilde{X}_t - \tilde{X}_s &= \sigma \int_s^t e^{\beta r} dB_r \\ &\sim \mathbf{N}(0, \frac{\sigma^2}{2\beta} (e^{2\beta t} - e^{2\beta s})), \\ \tilde{U}_t - \tilde{U}_s &= \frac{\sigma}{\sqrt{2\beta}} (B(e^{2\beta t} - 1) - B(e^{2\beta s} - 1)) \\ &\sim \frac{\sigma}{2\beta} B(e^{2\beta t} - e^{2\beta s}) \\ &\sim \mathbf{N}(0, \frac{\sigma^2}{2\beta} (e^{2\beta t} - e^{2\beta s})) \end{aligned}$$

we see that the claim is true.



Problem 21.6. Solution: We use the time-dependent Itô formula from Problem 18.7 (or the two-dimensional Itô-formula for the process (t, X_t)) with $f(t, x) = e^{ct} \int_0^x \frac{dy}{\sigma(y)}$. Note that the parameter c is still a free parameter.

Using Itô's multiplication rule— $(dt)^2 = dt dB_t = 0$ and $(dB_t)^2 = dt$ we get

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t \implies (dX_t)^2 = d\langle X \rangle_t = \sigma^2(X_t) dt.$$

Thus,

$$\begin{aligned} dZ_t &= df(t, X_t) = \partial_t f(t, X_t) dt + \partial_x f(t, X_t) dX_t + \frac{1}{2} \partial_x^2 f(t, X_t) (dX_t)^2 \\ &= ce^{ct} \int_0^{X_t} \frac{dy}{\sigma(y)} dt + e^{ct} \frac{1}{\sigma(X_t)} dX_t - \frac{1}{2} e^{ct} \frac{\sigma'(X_t)}{\sigma^2(X_t)} \sigma^2(X_t) dt \\ &= ce^{ct} \int_0^{X_t} \frac{dy}{\sigma(y)} dt + e^{ct} \frac{b(X_t)}{\sigma(X_t)} dt + e^{ct} dB_t - \frac{1}{2} e^{ct} \sigma'(X_t) dt \\ &= e^{ct} \left[c \int_0^{X_t} \frac{dy}{\sigma(dy)} - \frac{1}{2} \sigma'(X_t) + \frac{b(X_t)}{\sigma(X_t)} \right] dt + e^{ct} dB_t. \end{aligned}$$

Let us show that the expression in the brackets $[\dots]$ is constant if we choose c appropriately.

For this we differentiate this expression:

$$\begin{aligned} \frac{d}{dx} \left[c \int_0^x \frac{dy}{\sigma(dy)} - \frac{1}{2} \sigma'(x) + \frac{b(x)}{\sigma(x)} \right] &= \frac{c}{\sigma(x)} - \frac{d}{dx} \left[\frac{1}{2} \sigma'(x) - \frac{b(x)}{\sigma(x)} \right] \\ &= \frac{c}{\sigma(x)} - \left[\frac{1}{2} \sigma''(x) - \frac{d}{dx} \frac{b(x)}{\sigma(x)} \right] \\ &= \frac{1}{\sigma(x)} \left(c - \underbrace{\sigma(x) \left[\frac{1}{2} \sigma''(x) - \frac{d}{dx} \frac{b(x)}{\sigma(x)} \right]}_{=\text{const. by assumption}} \right) \end{aligned}$$

This shows that we should choose c in such a way that the expression $c - \sigma \cdot [\dots]$ becomes zero, i.e.

$$c = \sigma(x) \left[\frac{1}{2} \sigma''(x) - \frac{d}{dx} \frac{b(x)}{\sigma(x)} \right].$$



Problem 21.7. Solution: Set $f(t, x) = tx$. Then

$$\partial_t f(t, x) = x, \quad \partial_x f(t, x) = t, \quad \partial_x^2 f(t, x) = 0.$$

Using the time-dependent Itô formula (cf. Problem 18.7) or the two-dimensional Itô formula (cf. Theorem 18.8) for the process (t, B_t) we get

$$\begin{aligned} dX_t &= \partial_t f(t, B_t) dt + \partial_x f(t, B_t) dB_t + \frac{1}{2} \partial_x^2 f(t, B_t) dt \\ &= B_t dt + t dB_t \\ &= \frac{X_t}{t} dt + t dB_t. \end{aligned}$$

Together with the initial condition $X_0 = 0$ this is the SDE which has $X_t = tB_t$ as solution. The trouble is, that the solution is not unique! To see this, assume that X_t and Y_t are any two solutions. Then

$$dZ_t := d(X_t - Y_t) = dX_t - dY_t = \left(\frac{X_t}{t} - \frac{Y_t}{t} \right) dt = \frac{Z_t}{t} dt, \quad Z_0 = 0.$$

This is an ODE and all (deterministic) processes $Z_t = ct$ are solutions with initial condition $Z_0 = 0$. If we want to enforce uniqueness, we need a condition on Z'_0 . So

$$dX_t = \frac{X_t}{t} dt + t dB_t \quad \text{and} \quad \left. \frac{d}{dt} X_t \right|_{t=0} = x'_0$$

will do. (Note that tB_t is differentiable at $t = 0$!).

■ ■

Problem 21.8. Solution:

a) With the argument from Problem 21.7, i.e. Itô's formula, we get for $f(t, x) = x/(1+t)$

$$\partial_t f(t, x) = -\frac{x}{(1+t)^2}, \quad \partial_x f(t, x) = \frac{1}{1+t}, \quad \partial_x^2 f(t, x) = 0.$$

And so

$$\begin{aligned} dU_t &= -\frac{B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t \\ &= -\frac{U_t}{1+t} dt + \frac{1}{1+t} dB_t. \end{aligned}$$

The initial condition is $U_0 = 0$.

b) Using Itô's formula for $f(x) = \sin x$ we get, because of $\sin^2 x + \cos^2 x = 1$, that

$$\begin{aligned} dV_t &= \cos B_t dB_t - \frac{1}{2} \sin B_t dt \\ &= \sqrt{1 - \sin^2 B_t} dB_t - \frac{1}{2} \sin B_t dt \\ &= \sqrt{1 - V_t^2} dB_t - \frac{1}{2} V_t dt \end{aligned}$$

and the initial condition is $V_0 = 0$.

Attention: We loose all information on the sign of $\cos B_t$ when taking the square root $\sqrt{1 - \sin^2 B_t}$. This means that the SDE corresponds to $V_t = \sin B_t$ only when $\cos B_t$ is positive, i.e. for $t < \inf \{s > 0 : B_s \notin [-\frac{1}{2}\pi, \frac{1}{2}\pi]\}$.

c) Using Itô's formula in each coordinate we get

$$\begin{aligned} d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} -a \sin B_t \\ b \cos B_t \end{pmatrix} dB_t + \frac{1}{2} \begin{pmatrix} -a \cos B_t \\ -b \sin B_t \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{a}{b} b \sin B_t \\ \frac{b}{a} a \cos B_t \end{pmatrix} dB_t - \frac{1}{2} \begin{pmatrix} a \cos B_t \\ b \sin B_t \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{a}{b} Y_t \\ \frac{b}{a} X_t \end{pmatrix} dB_t - \frac{1}{2} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt. \end{aligned}$$

The initial condition is $(X_0, Y_0) = (a, 0)$.



Problem 21.9. Solution:

a) We use Example 21.7 (and 21.6) where we set

$$\alpha(t) \equiv b, \quad \beta(t) \equiv 0, \quad \gamma(t) \equiv 0, \quad \delta(t) \equiv \sigma.$$

Then we get

$$\begin{aligned} dX_t^\circ &= \sigma^2 X_t^\circ dt - \sigma X_t^\circ dB_t \\ dZ_t &= bX_t^\circ dt \end{aligned}$$

and, by Example 21.6 we see

$$\begin{aligned} X_t^\circ &= X_0^\circ \exp\left(\int_0^t (\sigma^2 - \frac{1}{2}\sigma^2) ds - \int_0^t \sigma dB_s\right) \\ &= X_0^\circ \exp\left(\frac{1}{2}\sigma^2 t - \sigma B_t\right) \\ Z_t &= \int_0^t bX_s^\circ ds \end{aligned}$$

Thus,

$$\begin{aligned} Z_t &= \int_0^t b e^{\frac{1}{2}\sigma^2 s - \sigma B_s} ds \\ X_t &= \frac{Z_t}{X_t^\circ} = b e^{-\frac{1}{2}\sigma^2 t + \sigma B_t} \int_0^t e^{\frac{1}{2}\sigma^2 s - \sigma B_s} ds. \end{aligned}$$

We finally have to adjust the initial condition by adding $X_0 = x_0$ to the X_t we have just found:

$$\implies X_t = X_0 + b e^{-\frac{1}{2}\sigma^2 t + \sigma B_t} \int_0^t e^{\frac{1}{2}\sigma^2 s - \sigma B_s} ds.$$

Alternative Solution (by R. Baumgarth, TU Dresden): This solution does not use Example 21.7 First, we solve the homogeneous SDE, i.e. $b = 0$:

$$dX_t = \sigma X_t dB_t.$$

Using $Z_t := \log X_t$ and Itô's formula (or simply Example we see that this equation has the unique solution

$$X_t = x_0 e^{-\frac{1}{2}\sigma^2 t + \sigma B_t},$$

which is of the form

$$\text{constant} = x_0 = X_t e^{\frac{1}{2}\sigma^2 t - \sigma B_t} = X_t X_t^\circ.$$

Because of the form of the homogeneous solution, we now use stochastic integration by parts¹

$$d(X_t X_t^\circ) = X_t dX_t^\circ + X_t^\circ dX_t + \underbrace{d\langle X, X^\circ \rangle_t}_{= dX_t dX_t^\circ}. \quad (*)$$

¹Note this formula can be shown by applying Itô's formula on $f(x, x^0) = x x^0$.

So we need to find dX_t^0 . Using $f(y) = e^y$ for the process $Y_t = \frac{1}{2}\sigma^2 t - \sigma B_t$ gives

$$\begin{aligned} dX_t^o &= df(Y_t) = f'(Y_t) dY_t + \frac{1}{2}f''(Y_t)(dY_t)^2 \\ &= X_t^o \left(\frac{1}{2}\sigma^2 dt - \sigma dB_t + \frac{1}{2}\sigma^2 dt \right) \\ &= \sigma^2 X_t^o dt - \sigma X_t^o dB_t. \end{aligned}$$

Inserting everything in (*) yields

$$\begin{aligned} d(X_t X_t^o) &= X_t X_t^o (\sigma^2 dt - \sigma dB_t) + X_t^o (b dt + \sigma X_t dB_t) \\ &= X_t^o b dt, \end{aligned}$$

and so the solution is

$$\begin{aligned} X_t - X_0 &= \frac{1}{X_t^o} b \int_0^t X_s^o ds \\ X_t &= X_0 + b e^{-\frac{1}{2}\sigma^2 t + \sigma B_t} \int_0^t e^{\frac{1}{2}\sigma^2 s - \sigma B_s} ds. \end{aligned}$$

b) We use Example 21.7 (and 21.6) where we set

$$\alpha(t) \equiv m, \quad \beta(t) \equiv -1, \quad \gamma(t) \equiv \sigma, \quad \delta(t) \equiv 0.$$

Then we get

$$\begin{aligned} dX_t^o &= X_t^o dt \\ dZ_t &= mX_t^o dt + \sigma X_t^o dB_t \end{aligned}$$

Thus,

$$\begin{aligned} X_t^o &= X_0^o e^t \\ Z_t &= \int_0^t m e^s ds + \sigma \int_0^t e^s dB_s \\ &= m(e^t - 1) + \sigma \int_0^t e^s dB_s \\ X_t &= \frac{Z_t}{X_t^o} = m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s \end{aligned}$$

and, if we take care of the initial condition $X_0 = x_0$, we get

$$\implies X_t = x_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s.$$

Alternative Solution (by R. Baumgarth, TU Dresden): This does not use Example 21.7. Consider the deterministic ODE $\sigma = 0$ and, to simplify the problem, $m = 0$.

$$\dot{x}(t) = -x(t)$$

has the unique solution

$$x(t) = x_0 e^{-t}.$$

Thus, the solution is of the form

$$\text{constant} = x_0 = x(t)e^t = f(t, x_t).$$

Now use Itô's formula for $Y_t = f(t, X_t)$:

$$\begin{aligned} df(t, X_t) &= \partial_t f(t, X_t) dt + \partial_x f(t, X_t) dX_t + \frac{1}{2} \partial_x^2 f(t, X_t) (dX_t)^2 \\ &= X_t e^t dt + e^t ((m - X_t) dt + \sigma dB_t) + 0 \\ &= e^t (m dt + \sigma dB_t). \end{aligned}$$

Hence,

$$X_t e^t - X_0 = m \underbrace{\int_0^t e^s ds}_{=(e^t-1)} + \sigma \int_0^t e^s dB_s$$

or

$$X_t = X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s.$$

Problem 21.10. Solution: Set

$$b(x) = \sqrt{1+x^2} + \frac{1}{2}x \quad \text{and} \quad \sigma(x) = \sqrt{1+x^2}.$$

Then we get (using the notation of Lemma 21.10)

$$\sigma'(x) = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \kappa(x) = \frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x) = 1.$$

Using the Ansatz of Lemma 21.10 we set

$$d(x) = \int_0^x \frac{dy}{\sigma(y)} = \operatorname{arsinh} x \quad \text{and} \quad Z_t = f(X_t) = d(X_t).$$

Using Itô's formula gives

$$\begin{aligned} dZ_t &= \partial_x f(X_t) dX_t + \frac{1}{2} \partial_x^2 f(X_t) \sigma^2(X_t) dt \\ &= \frac{1}{\sigma(X_t)} dX_t + \frac{1}{2} \left(\frac{1}{\sigma}\right)'(X_t) \sigma^2(X_t) dt \\ &= \left(1 + \frac{X_t}{2\sqrt{1+X_t^2}}\right) dt + dB_t + \frac{1}{2} \left(-\frac{X_t}{(1+X_t^2)^{3/2}}\right) (1+X_t^2) dt \\ &= dt + dB_t, \end{aligned}$$

and so $Z_t = Z_0 + t + B_t$. Finally,

$$X_t = \sinh(Z_0 + t + B_t) \quad \text{where} \quad Z_0 = \operatorname{arsinh} X_0.$$

Alternative Solution (by R. Baumgarth, TU Dresden): In order to guess the correct Ansatz, consider first the ODE

$$dx(t) = \left(\sqrt{1+x(t)^2} + \frac{1}{2} \frac{x(t)}{1+x(t)^2} \right) dt.$$

If we do not get rid of the second term, things get messy when integrating. Hence, this ODE is very easy to solve by separation of variables and we see

$$\operatorname{arsinh} x(t) = t + c,$$

so a suitable Ansatz for Itô's formula is $Y_t := f(t, X_t) = \operatorname{arsinh} X_t - t$.

$$\begin{aligned} dY_t &= f_t(t, X_t) dt + f_x(t, X_t) dX_t + \frac{1}{2} f_{xx}(t, X_t) (dX_t)^2 \\ &= -dt + \frac{1}{\sqrt{1+X_t^2}} dX_t + \frac{1}{2} \left(-\frac{X_t}{\sqrt{(1+X_t^2)^3}} \right) (1+X_t^2) dt \\ &= -dt + \frac{1}{\sqrt{1+X_t^2}} \left[\left(\sqrt{1+X_t^2} + \frac{1}{2} X_t \right) dt + \sqrt{1+X_t^2} dB_t \right] - \frac{X_t}{2\sqrt{1+X_t^2}} dt \\ &= -dt + dt + \frac{X_t}{2\sqrt{1+X_t^2}} dt + dB_t - \frac{X_t}{2\sqrt{1+X_t^2}} dt \\ &= dB_t, \end{aligned}$$

hence $Y_t = Y_0 + B_t$ and $X_t = \sinh(X_0 + B_t + t)$. ■ ■

Problem 21.11. Solution: Set $b = b(t, x)$, $b_0 = b(t, 0)$ etc. Observe that $\|b\| = (\sum_j |b_j(t, x)|^2)^{1/2}$ and $\|\sigma\| = (\sum_{j,k} |\sigma_{jk}(t, x)|^2)^{1/2}$ are norms; therefore, we get using the triangle estimate and the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$

$$\begin{aligned} \|b\|^2 + \|\sigma\|^2 &= \|b - b_0 + b_0\|^2 + \|\sigma - \sigma_0 + \sigma_0\|^2 \\ &\leq 2\|b - b_0\|^2 + 2\|\sigma - \sigma_0\|^2 + 2\|b_0\|^2 + 2\|\sigma_0\|^2 \\ &\leq 2L^2|x|^2 + 2\|b_0\|^2 + 2\|\sigma_0\|^2 \\ &\leq 2L^2(1+|x|)^2 + 2(\|b_0\|^2 + \|\sigma_0\|^2)(1+|x|)^2 \\ &\leq 2(L^2 + \|b_0\|^2 + \|\sigma_0\|^2)(1+|x|)^2. \end{aligned}$$

Problem 21.12. Solution:

a) If $b(x) = -e^x$ and $X_0^x = x$ we have to solve the following ODE/integral equation

$$X_t^x = x - \int_0^t e^{X_s^x} ds$$

and it is not hard to see that the solution is

$$X_t^x = \log\left(\frac{1}{t + e^{-x}}\right).$$

This shows that

$$\lim_{x \rightarrow \infty} X_t^x = \lim_{x \rightarrow \infty} \log\left(\frac{1}{t + e^{-x}}\right) = \log \frac{1}{t} = -\log t.$$

This means that Corollary 21.31 fails in this case since the coefficient of the ODE grows too fast.

b) Now assume that $|b(x)| + |\sigma(x)| \leq M$ for all x . Then we have

$$\left| \int_0^t b(X_s) ds \right| \leq Mt.$$

By Itô's isometry we get

$$\mathbb{E} \left[\left| \int_0^t \sigma(X_s^x) dB_s \right|^2 \right] = \mathbb{E} \left[\int_0^t |\sigma^2(X_s^x)| ds \right] \leq M^2 t.$$

Using $(a + b)^2 \leq 2a^2 + 2b^2$ we see

$$\begin{aligned} \mathbb{E}(|X_t^x - x|^2) &\leq 2 \mathbb{E} \left[\left| \int_0^t b(X_s) ds \right|^2 \right] + 2 \mathbb{E} \left[\left| \int_0^t \sigma(X_s^x) dB_s \right|^2 \right] \\ &\leq 2(Mt)^2 + 2M^2 t \\ &= 2M^2 t(t + 1). \end{aligned}$$

By Fatou's lemma

$$\mathbb{E} \left(\liminf_{|x| \rightarrow \infty} |X_t^x - x|^2 \right) \leq \liminf_{|x| \rightarrow \infty} \mathbb{E}(|X_t^x - x|^2) \leq 2M^2 t(t + 1)$$

which shows that $|X_t^x|$ cannot be bounded as $|x| \rightarrow \infty$.

c) Assume now that $b(x)$ and $\sigma(x)$ grow like $|x|^{p/2}$ for some $p \in (0, 2)$. A calculation as above yields

$$\left| \int_0^t b(X_s) ds \right|^2 \stackrel{\text{Cauchy-Schwarz}}{\leq} t \int_0^t |b(X_s)|^2 ds \leq c_p t \int_0^t (1 + |X_s|^p) ds$$

and, by Itô's isometry

$$\mathbb{E} \left[\left| \int_0^t \sigma(X_s^x) dB_s \right|^2 \right] = \mathbb{E} \left[\int_0^t |\sigma^2(X_s^x)| ds \right] \leq c' \int_0^t \mathbb{E}(1 + |X_s|^p) ds.$$

Using $(a + b)^2 \leq 2a^2 + 2b^2$ and Theorem 21.28 we get

$$\begin{aligned} \mathbb{E}|X_t^x - x|^2 &\leq 2c_p t \int_0^t (1 + \mathbb{E}(|X_s|^p)) ds + 2c' \int_0^t (1 + \mathbb{E}(|X_s|^p)) ds \\ &\leq c_{t,p} + c'_{t,p} \int_0^t |x|^p dt \\ &= c_{t,p} + t c'_{t,p} |x|^p. \end{aligned}$$

Again by Fatou's theorem we see that the left-hand side grows like $|x|^2$ (if X_t^x is bounded) while the (larger!) right-hand side grows like $|x|^p$, $p < 2$, and this is impossible.

Thus, $(X_t^x)_x$ is unbounded as $|x| \rightarrow \infty$.

■ ■

Problem 21.13. Solution: We have to show

$$\begin{aligned} & \frac{|x-y|}{(1+|x|)(1+|y|)} \stackrel{!}{\leq} \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| \\ \Leftrightarrow & \frac{|x-y|^2}{(1+|x|)^2(1+|y|)^2} \leq \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right|^2 \\ \Leftrightarrow & \frac{|x|^2 - 2\langle x, y \rangle + |y|^2}{(1+|x|)^2(1+|y|)^2} \leq \frac{|x|^2}{|x|^4} - \frac{2\langle x, y \rangle}{|x|^2|y|^2} + \frac{|y|^2}{|y|^4} \\ \Leftrightarrow & 2\langle x, y \rangle \left(\frac{1}{|x|^2|y|^2} - \frac{1}{(1+|x|)^2(1+|y|)^2} \right) \leq \frac{1}{|x|^2} + \frac{1}{|y|^2} - \frac{|x|^2 + |y|^2}{(1+|x|)^2(1+|y|)^2} \\ \Leftrightarrow & 2\langle x, y \rangle \left(\frac{1}{|x|^2|y|^2} - \frac{1}{(1+|x|)^2(1+|y|)^2} \right) \leq (|x|^2 + |y|^2) \left(\frac{1}{|x|^2|y|^2} - \frac{1}{(1+|x|)^2(1+|y|)^2} \right). \end{aligned}$$

By the Cauchy-Schwarz inequality we get $2\langle x, y \rangle \leq 2|x| \cdot |y| \leq |x|^2 + |y|^2$, and this shows that the last estimate is correct.

Alternative Solution (by R. Baumgarth, TU Dresden): We can also use the following direct calculation:

$$\begin{aligned} \frac{|x-y|^2}{(1+|x|)^2(1+|y|)^2} &= \frac{|x|^2 - 2\langle x, y \rangle + |y|^2}{(1+|x|)^2(1+|y|)^2} \\ &\leq \frac{|x|^2 - 2\langle x, y \rangle + |y|^2}{|x|^2|y|^2} \\ &= \frac{1}{|y|^2} - \frac{2\langle x, y \rangle}{|x|^2|y|^2} + \frac{1}{|x|^2} \\ &= \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right|^2. \end{aligned}$$

■ ■

Problem 21.14. Solution:

a) We have seen in Corollary 21.24 that the transition function is given by

$$p(t, x; B) = \mathbb{P}(X_t^x \in B), \quad t \geq 0, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R}).$$

Consequently,

$$T_t f(x) = \int f(y) p(t, x; dy) = \mathbb{E}(f(X_t^x)).$$

By Theorem 21.27 we know that $x \mapsto T_t f(x) = \mathbb{E}(f(X_t^x))$ is continuous for each $f \in \mathcal{C}_b(\mathbb{R})$. Since

$$|T_t f(x)| \leq \mathbb{E}|f(X_t^x)| \leq \|f\|_\infty$$

we conclude that T_t maps $\mathcal{C}_b(\mathbb{R})$ into itself.

b) Let $f \in \mathcal{C}_\infty(\mathbb{R})$, $t \geq 0$. By part (a), $T_t f \in \mathcal{C}_b(\mathbb{R})$. Therefore, it suffices to show

$$\lim_{|x| \rightarrow \infty} |T_t f(x)| = \lim_{|x| \rightarrow \infty} |\mathbb{E}(f(X_t^x))| = 0.$$

Since $f \in \mathcal{C}_\infty(\mathbb{R})$ we obtain by applying Corollary 21.31,

$$\lim_{|x| \rightarrow \infty} |f(X_t^x)| = 0 \quad \text{almost surely.}$$

The claim follows from the dominated convergence theorem.

c) Let $f \in \mathcal{C}_c^2(\mathbb{R})$, $x \in \mathbb{R}$. By Itô's formula,

$$\begin{aligned} f(X_t^x) - f(x) &= \int_0^t f'(X_s^x) dX_s^x + \frac{1}{2} \int_0^t f''(X_s^x) \sigma^2(X_s^x) ds \\ &= \int_0^t f'(X_s^x) \sigma(X_s^x) dB_s + \int_0^t \left(f'(X_s^x) b(X_s^x) + \frac{1}{2} f''(X_s^x) \sigma^2(X_s^x) \right) ds. \end{aligned}$$

The first term on the right is a martingale, its expectation equals 0. Thus,

$$\begin{aligned} \frac{\mathbb{E}(f(X_t^x)) - f(x)}{t} &= \frac{1}{t} \mathbb{E} \left[\int_0^t \left(f'(X_s^x) b(X_s^x) + \frac{1}{2} f''(X_s^x) \sigma^2(X_s^x) \right) ds \right] \\ &= \frac{1}{t} \int_0^t \mathbb{E} \left(f'(X_s^x) b(X_s^x) + \frac{1}{2} f''(X_s^x) \sigma^2(X_s^x) \right) ds. \end{aligned}$$

Using Theorem 7.22, we get

$$A f(x) = \lim_{t \rightarrow 0} \frac{T_t f(x) - f(x)}{t} = f'(x) b(x) + \frac{1}{2} f''(x) \sigma^2(x).$$

Since the right-hand side is in \mathcal{C}_∞ , Theorem 7.22 applies and shows that $\mathcal{C}_c^2(\mathbb{R}) \subset \mathfrak{D}(A)$. Moreover, the same calculation shows that

$$\mathfrak{D}(A) \supset \left\{ u \in \mathcal{C}_\infty(\mathbb{R}) : u', u'' \in \mathcal{C}(\mathbb{R}), bu' + \frac{1}{2} \sigma^2 u'' \in \mathcal{C}_\infty \right\}$$

i.e. the domain of A takes into account the growth of σ and b .

Since A is a second-order differential operator, it clearly has an extension onto $C_b^2(\mathbb{R})$.

d) Let $u \in C^{1,2}([0, \infty) \times \mathbb{R})$. By the time-dependent Itô formula we have

$$\begin{aligned} u(t, X_t^x) - u(0, x) &= \int_0^t \partial_x u(s, X_s^x) dX_s^x + \int_0^t \left(\partial_t u(s, X_s^x) + \frac{1}{2} \partial_x^2 u(s, X_s^x) \sigma^2(X_s^x) \right) ds \\ &= \int_0^t \partial_x u(s, X_s^x) \sigma(X_s^x) dB_s + \int_0^t \left(\partial_t u(s, X_s^x) + \partial_x u(s, X_s^x) b(X_s^x) + \right. \\ &\quad \left. + \frac{1}{2} \partial_x^2 u(s, X_s^x) \sigma^2(X_s^x) \right) ds. \end{aligned}$$

We have shown in part (c) that (for an extension of A)

$$A_x u(s, X_s^x) = b(X_s^x) \partial_x u(s, X_s^x) + \frac{1}{2} \sigma^2(X_s^x) \partial_x^2 u(s, X_s^x) \quad \text{for all fixed } s \geq 0.$$

Consequently,

$$u(t, X_t^x) - u(0, x) = \int_0^t \partial_x u(s, X_s^x) \sigma(X_s^x) dB_s + \int_0^t \left(\partial_t u(s, X_s^x) + A_x u(s, X_s^x) \right) ds.$$

In particular we find that

$$M_t^{u,x} := u(t, X_t^x) - u(0, x) - \int_0^t (\partial_t u(s, X_s^x) + A_x u(s, X_s^x)) ds$$

is a martingale. By our assumptions, it is a bounded martingale and so we can use Doob's optional stopping theorem for the stopping time $\tau \wedge n$

$$\mathbb{E} u(t, X_{\tau \wedge n}^x) = u(0, x) + \mathbb{E} \left(\int_0^{\tau \wedge n} (\partial_t u(s, X_s^x) + A_x u(s, X_s^x)) ds \right)$$

and, since everything is bounded and since $\mathbb{E} \tau < \infty$, dominated convergence proves the claim.

Remark: A close inspection of our argument reveals that we do not need boundedness of b, σ if we replace $\mathbb{E} \tau < \infty$ by

$$\mathbb{E} \left(\int_0^\tau \sigma^2(X_s^x) ds \right) + \mathbb{E} \left(\int_0^\tau |b(X_s^x)| ds \right) < \infty.$$

■ ■

22 Stratonovich's stochastic calculus

Problem 22.1. Solution: We have, using the time-dependent Itô formula (18.18) with $d = m = 1$,

$$\begin{aligned} df(t, X_t) &= \partial_t f(t, X_t) dt + \partial_x f(t, X_t) b(t) dt + \partial_x f(t, X_t) \sigma(t) dB_t + \frac{1}{2} \partial_x^2 f(t, X_t) \sigma^2(t) dt \\ &\stackrel{(18.18)}{=} \partial_t f(t, X_t) dt + \partial_x f(t, X_t) b(t) dt + \partial_x f(t, X_t) \sigma(t) \circ dB_t \end{aligned}$$

and this is exactly what we would get if we would use normal calculus rules and if $dB_t = \dot{\beta}_t dt$: By the usual chain rule

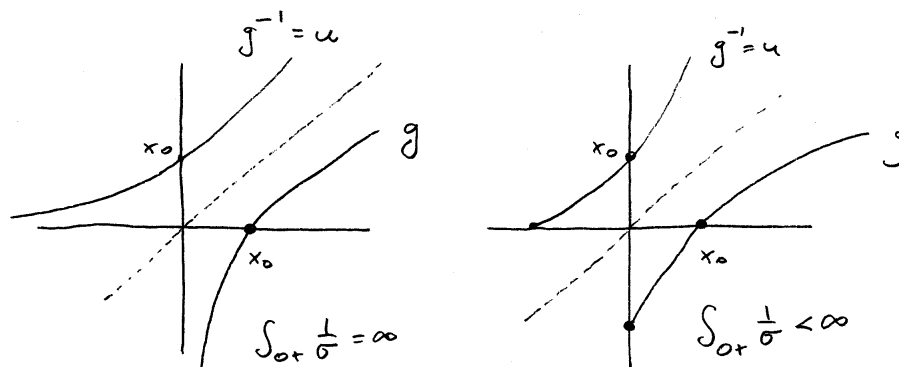
$$\begin{aligned} f(t, \xi_t) &= \partial_t f(t, \xi_t) + \partial_x f(t, \xi_t) \dot{\xi}_t \\ &= \partial_t f(t, \xi_t) + \partial_x f(t, \xi_t) \sigma(t) \dot{\beta}_t + \partial_x f(t, \xi_t) b(t) \end{aligned}$$

where we used that $\xi_t = \xi_0 + \int_0^t \sigma(s) \dot{\beta}(s) ds + \int_0^t b(s) ds$.

Problem 22.2. Solution: Then we have

$$g(x) = \begin{cases} \int_{x_0}^x \frac{d\xi}{\sigma(\xi)}, & \text{if } 0 < x_0 < x, \\ -\int_x^{x_0} \frac{d\xi}{\sigma(\xi)}, & \text{if } 0 < x \leq x_0. \end{cases}$$

The Function g and its inverse $u = g^{-1}$ are shown in the pictures below. Note the difference between the cases $\int_{0+} \frac{d\xi}{\sigma(\xi)} = \infty$ and $< \infty$.



The solution to the SDE is now given by

$$\tau := \inf \{t \geq 0 : u(B_t) = 0\},$$

$$X_t := u(B_t)\mathbb{1}_{\{\tau > t\}}.$$

Note that the fact that $\sigma(0) = 0$ means that in the SDE X_t cannot move once it reaches 0. The pictures above illustrate that

$$\int_{0+} \frac{d\xi}{\sigma(\xi)} = \infty \implies \tau = \infty$$

$$\int_{0+} \frac{d\xi}{\sigma(\xi)} < \infty \implies \tau < \infty.$$

■ ■

Problem 22.3. Solution: Denote by L_f, L_g the global Lipschitz constants and observe that the global Lipschitz property entails linear growth:

$$|g(x)| \leq |g(0)| + |g(x) - g(0)| \leq |g(0)| + L_g|x|.$$

Now let $-r \leq x, y \leq r$. Then

$$\begin{aligned} |h(x) - h(y)| &= |f(x)g(x) - f(y)g(y)| \\ &\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ &\leq |f(x) - f(y)||g(x)| + \|f\|_\infty |g(x) - g(y)| \\ &\leq L_f \sup_{|x| \leq r} |g(x)| |x - y| + \|f\|_\infty L_g |x - y|, \end{aligned}$$

and the local Lipschitz property follows. Finally,

$$|h(x)| = |f(x)||g(x)| \leq \|f\|_\infty (|g(0)| + L_g|x|),$$

and we are done.

■ ■

23 On diffusions

Problem 23.1. Solution: We have

$$Au = Lu = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j u + \sum_{i=1}^d b_i \partial_i u$$

and we know that $L : \mathcal{C}_c^\infty \rightarrow \mathcal{C}$. Fix $R > 0$ and $i, j \in \{1, \dots, d\}$ where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi|_{\mathbb{B}(0,R)} \equiv 1$.

For all $u, \chi \in \mathcal{C}^2$ we get

$$\begin{aligned} L(\phi u) &= \frac{1}{2} \sum_{i,j} a_{ij} \partial_i \partial_j (\phi u) + \sum_i b_i \partial_i (\phi u) \\ &= \frac{1}{2} \sum_{i,j} a_{ij} (\partial_i \partial_j \phi + \partial_i \partial_j u + \partial_i \phi \partial_j u + \partial_i u \partial_j \phi) + \sum_i b_i (u \partial_i \phi + \phi \partial_i u) \\ &= \phi Lu + uL\phi + \sum_{i,j} a_{ij} \partial_i \phi \partial_j u \end{aligned}$$

where we used the symmetry $a_{ij} = a_{ji}$ in the last step.

Now use $u(x) = x_i$ and $\phi(x) = \chi(x)$. Then $u\chi \in \mathcal{C}_c^\infty$, $L(u\chi) \in \mathcal{C}$ and so

$$L(u\chi)(x) = b_i(x) \quad \text{for all } |x| < R \implies b_i|_{\mathbb{B}(0,R)} \text{ continuous.}$$

Now use $u(x) = x_i x_j$ and $\phi(x) = \chi(x)$. Then $u\chi \in \mathcal{C}_c^\infty$, $L(u\chi) \in \mathcal{C}$ and so

$$L(u\chi)(x) = a_{ij} + x_j b_i(x) + x_i b_j(x) \quad \text{for all } |x| < R \implies a_{ij}|_{\mathbb{B}(0,R)} \text{ continuous.}$$

Since $R > 0$ is arbitrary, the claim follows. ■ ■

Problem 23.2. Solution: This is a straightforward application of the differentiation Lemma which is familiar from measure and integration theory, cf. Schilling [18, Theorem 12.5]: observe that by our assumptions

$$\left| \frac{\partial^2 p(t, x, y)}{\partial x_j \partial x_k} \right| \leq C(t) \quad \text{for all } x, y \in \mathbb{R}^d$$

which shows that for $u \in C_c^\infty(\mathbb{R}^d)$

$$\left| \frac{\partial^2 p(t, x, y)}{\partial x_j \partial x_k} u(y) \right| \leq C(t) |u(y)| \in L^1(\mathbb{R}^d) \quad (*)$$

for each $t > 0$. Thus we get

$$\frac{\partial^2}{\partial x_j \partial x_k} \int p(t, x, y) u(y) dy = \int \frac{\partial^2}{\partial x_j \partial x_k} p(t, x, y) u(y) dy.$$

Moreover, (*) and the fact that $p(t, \cdot, y) \in \mathcal{C}_\infty(\mathbb{R}^d)$ allow us to change limits and integrals to get for $x \rightarrow x_0$ and $|x| \rightarrow \infty$

$$\begin{aligned} \lim_{x \rightarrow x_0} \int \frac{\partial^2}{\partial x_j \partial x_k} p(t, x, y) u(y) dy &= \int \lim_{x \rightarrow x_0} \frac{\partial^2}{\partial x_j \partial x_k} p(t, x, y) u(y) dy \\ &= \int \frac{\partial^2}{\partial x_j \partial x_k} p(t, x_0, y) u(y) dy \\ &\implies T_t \text{ maps } \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ into } \mathcal{C}(\mathbb{R}^d); \\ \lim_{|x| \rightarrow \infty} \int \frac{\partial^2}{\partial x_j \partial x_k} p(t, x, y) u(y) dy &= \underbrace{\int \lim_{|x| \rightarrow \infty} \frac{\partial^2}{\partial x_j \partial x_k} p(t, x, y) u(y) dy}_{=0} = 0 \\ &\implies T_t \text{ maps } \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ into } \mathcal{C}_\infty(\mathbb{R}^d). \end{aligned}$$

Addition: With a standard uniform boundedness and density argument we can show that T_t maps \mathcal{C}_∞ into \mathcal{C}_∞ : fix $u \in \mathcal{C}_\infty(\mathbb{R}^d)$ and pick a sequence $(u_n)_n \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{n \rightarrow \infty} \|u - u_n\|_\infty = 0.$$

Then we get

$$\|T_t u - T_t u_n\|_\infty = \|T_t(u - u_n)\|_\infty \leq \|u - u_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$$

which means that $T_t u_n \rightarrow T_t u$ uniformly, i.e. $T_t u \in \mathcal{C}_\infty$ as $T_t u_n \in \mathcal{C}_\infty$. ■ ■

Problem 23.3. Solution: Let $u \in \mathcal{C}_c^2$. Then there is a sequence of test functions $(u_n)_n \subset \mathcal{C}_c^\infty$ such that $\|u_n - u\|_{(2)} \rightarrow 0$. Thus, $u_n \rightarrow u$ uniformly and $A(u_n - u_m) \rightarrow 0$ uniformly. The closedness now gives $u \in \mathfrak{D}(A)$. ■ ■

Problem 23.4. Solution: Let $u, \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned} \langle Lu, \phi \rangle_{L^2} &= \sum_{i,j} \int_{\mathbb{R}^d} a_{ij} \partial_i \partial_j u \cdot \phi dx + \sum_j \int_{\mathbb{R}^d} b_j \partial_j u \cdot \phi dx + \int_{\mathbb{R}^d} cu \cdot \phi dx \\ &\stackrel{\text{int. by parts}}{=} \sum_{i,j} \int_{\mathbb{R}^d} u \cdot \partial_i \partial_j (a_{ij} \phi) dx - \sum_j \int_{\mathbb{R}^d} u \cdot \partial_j (b_j \phi) dx + \int_{\mathbb{R}^d} u \cdot c \phi dx \\ &= \langle u, L^* \phi \rangle_{L^2} \end{aligned}$$

where

$$L^*(x, D_x)\phi(x) = \sum_{i,j} \partial_i \partial_j (a_{ij}(x)\phi(x)) - \sum_j \partial_j (b_j(x)\phi(x)) + c(x)\phi(x).$$

Now assume that we are in $(t, x) \in [0, \infty) \times \mathbb{R}^d$ —the case $\mathbb{R} \times \mathbb{R}^d$ is easier, as we have no boundary term. Consider $L + \partial_t = L(x, D_x) + \partial_t$ for sufficiently smooth $u = u(t, x)$ and $\phi = \phi(t, x)$ with compact support in $[0, \infty) \times \mathbb{R}^d$. We find

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^d} (L + \partial_t)u(t, x) \cdot \phi(t, x) \, dx \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} Lu(t, x) \cdot \phi(t, x) \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^d} \partial_t u(t, x) \cdot \phi(t, x) \, dx \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} Lu(t, x) \cdot \phi(t, x) \, dx \, dt + \int_{\mathbb{R}^d} \int_0^\infty \partial_t u(t, x) \cdot \phi(t, x) \, dt \, dx \\ &= \int_0^\infty \int_{\mathbb{R}^d} u(t, x) \cdot L^* \phi(t, x) \, dx \, dt + \int_{\mathbb{R}^d} \left(u(t, x) \phi(t, x) \Big|_{t=0}^\infty - \int_0^\infty u(t, x) \cdot \partial_t \phi(t, x) \, dt \right) dx \\ &= \int_0^\infty \int_{\mathbb{R}^d} u(t, x) \cdot L^* \phi(t, x) \, dx \, dt - \int_{\mathbb{R}^d} \left(u(0, x) \phi(0, x) + \int_0^\infty u(t, x) \cdot \partial_t \phi(t, x) \, dt \right) dx. \end{aligned}$$

This shows that $(L(x, D_x) + \partial_t)^* = L^*(x, D_x) - \partial_t - \delta_{(0,x)}$.

Problem 23.5. Solution: Using Lemma 7.10 we get for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} & \frac{d}{dt} T_t u(x) = T_t L(\cdot, D)u(x) \\ \implies & \frac{d}{dt} \int p(t, x, y) u(y) \, dy = \int p(t, x, y) L(y, D_y)u(y) \, dy \\ \implies & \int \frac{d}{dt} p(t, x, y) u(y) \, dy = \int p(t, x, y) L(y, D_y)u(y) \, dy. \end{aligned}$$

The change of differentiation and integration can easily be justified by a routine application of the differentiation lemma (e.g. Schilling [18, Theorem 12.5]): under our assumptions we have for all $\epsilon \in (0, 1)$ and $R > 0$

$$\sup_{t \in [\epsilon, 1/\epsilon]} \sup_{|x| \leq R} \left| \frac{d}{dt} p(t, x, y) u(y) \right| \leq C(\epsilon, R) |u(y)| \in L^1(\mathbb{R}^d).$$

Inserting the expression for the differential operator $L(y, D_y)$, we find for the right-hand side

$$\begin{aligned} & \int p(t, x, y) L(y, D_y)u(y) \, dy \\ &= \frac{1}{2} \sum_{j,k=1}^d \int p(t, x, y) \cdot a_{jk}(y) \frac{\partial^2 u(y)}{\partial y_j \partial y_k} \, dy + \sum_{j=1}^d \int p(t, x, y) \cdot b_j(y) \frac{\partial u(y)}{\partial y_j} \, dy \\ &\stackrel{\text{int. by parts}}{=} \frac{1}{2} \sum_{j,k=1}^d \int \frac{\partial^2}{\partial y_j \partial y_k} (a_{jk}(y) \cdot p(t, x, y)) u(y) \, dy + \sum_{j=1}^d \int \frac{\partial}{\partial y_j} (b_j(y) \cdot p(t, x, y)) u(y) \, dy \\ &= \int L^*(y, D_y) p(t, x, y) u(y) \, dy \end{aligned}$$

and the claim follows since $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is arbitrary.

Problem 23.6. Solution: Problem 6.5 shows that X_t is a Markov process. The continuity of the sample paths is obvious and so is the Feller property (using the form of the transition function found in the solution of Problem 6.5).

Let us calculate the generator. Set $I_t = \int_0^t B_s ds$. The semigroup is given by

$$T_t u(x, y) = \mathbb{E}^{x,y} u(B_t, I_t) = \mathbb{E} u(B_t + x, \int_0^t (B_s + x) ds + y) = \mathbb{E} u(B_t + x, I_t + tx + y).$$

If we differentiate the expression under the expectation with respect to t , we get with the help of Itô's formula

$$\begin{aligned} du(B_t + x, I_t + tx + y) &= \partial_x u(B_t + x, I_t + tx + y) dB_t \\ &\quad + \partial_y u(B_t + x, I_t + tx + y) d(I_t + tx) \\ &\quad + \frac{1}{2} \partial_x^2 u(B_t + x, I_t + tx + y) dt \\ &= \partial_x u(B_t + x, I_t + tx + y) dB_t \\ &\quad + \partial_y u(B_t + x, I_t + tx + y) (B_t + x) dt \\ &\quad + \frac{1}{2} \partial_x^2 u(B_t + x, I_t + tx + y) dt \end{aligned}$$

since $dB_s dI_s = 0$. So,

$$\begin{aligned} \mathbb{E} u(B_t + x, I_t + tx + y) - u(x, y) &= \int_0^t \mathbb{E} [\partial_y u(B_s + x, I_s + sx + y) (B_s + x)] ds \\ &\quad + \frac{1}{2} \int_0^t \mathbb{E} [\partial_x^2 u(B_s + x, I_s + sx + y)] ds. \end{aligned}$$

Dividing by t and letting $t \rightarrow 0$ we get

$$Lu(x, y) = x \partial_y u(x, y) + \frac{1}{2} \partial_x^2 u(x, y).$$

Problem 23.7. Solution: We assume for a) and b) that the operator L is more general than written in (23.1), namely

$$Lu(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial u(x)}{\partial x_j} + c(x)u(x)$$

where all coefficients are continuous functions.

a) If u has compact support, then Lu has compact support. Since, by assumption, the coefficients of L are continuous, Lu is bounded, hence M_t^u is square integrable.

Obviously, M_t^u is \mathcal{F}_t measurable. Let us establish the martingale property. For this we fix $s \leq t$. Then

$$\begin{aligned} \mathbb{E}^x (M_t^u \mid \mathcal{F}_s) &= \mathbb{E}^x \left(u(X_t) - u(X_0) - \int_0^t Lu(X_r) dr \mid \mathcal{F}_s \right) \\ &= \mathbb{E}^x \left(u(X_t) - u(X_s) - \int_s^t Lu(X_r) dr \mid \mathcal{F}_s \right) \\ &\quad + u(X_s) - u(X_0) - \int_0^s Lu(X_r) dr \\ &= \mathbb{E}^x \left(u(X_t) - u(X_s) - \int_0^{t-s} Lu(X_{r+s}) dr \mid \mathcal{F}_s \right) + M_s^u \end{aligned}$$

$$\stackrel{\substack{\text{Markov} \\ \text{property}}}{=} \mathbb{E}^{X_s} \left(u(X_{t-s}) - u(X_0) - \int_0^{t-s} Lu(X_r) dr \right) + M_s^u.$$

Observe that $T_t u(y) = \mathbb{E}^y u(X_t)$ is the semigroup associated with the Markov process. Then

$$\begin{aligned} & \mathbb{E}^y \left(u(X_{t-s}) - u(X_0) - \int_0^{t-s} Lu(X_r) dr \right) \\ &= T_{t-s} u(y) - u(y) - \int_0^{t-s} \mathbb{E}^y (Lu(X_r)) dr = 0 \end{aligned}$$

by Lemma 7.10, see also Theorem 7.30. This shows that $\mathbb{E}^x (M_t^u \mid \mathcal{F}_s) = M_s^u$, and we are done.

- b) Fix $R > 0$, $x \in \mathbb{R}^d$, and pick a smooth cut-off function $\chi = \chi_R \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi|_{\mathbb{B}(x, R)} \equiv 1$. Then for all $f \in \mathcal{C}^2(\mathbb{R}^d)$ we have $\chi f \in \mathcal{C}_c^2(\mathbb{R}^d)$ and it is not hard to see that the calculation in part a) still holds for such functions.

Set $\tau = \tau_R^x = \inf\{t > 0 : |X_t - x| \geq R\}$. This is a stopping time and we have

$$f(X_t^\tau) = \chi(X_t^\tau) f(X_t^\tau) = (\chi f)(X_t^\tau).$$

Moreover,

$$\begin{aligned} L(\chi f) &= \frac{1}{2} \sum_{i,j} a_{ij} \partial_i \partial_j (\chi f) + \sum_i b_i \partial_i (\chi f) + c \chi f \\ &= \frac{1}{2} \sum_{i,j} a_{ij} (f \partial_i \partial_j \chi + \chi \partial_i \partial_j f + \partial_i \chi \partial_j f + \partial_i f \partial_j \chi) + \sum_i b_i (f \partial_i \chi + \chi \partial_i f) + c \chi f \\ &= \chi Lf + f L\chi + \sum_{i,j} a_{ij} \partial_i \chi \partial_j f - c \chi f \end{aligned}$$

where we used the symmetry $a_{ij} = a_{ji}$ in the last step.

This calculation shows that $L(\chi f) = Lf$ on $\mathbb{B}(x, R)$.

By optional stopping and part a) we know that $(M_{t \wedge \tau_R}^{\chi f}, \mathcal{F}_t)_{t \geq 0}$ is a martingale. Moreover, we get for $s \leq t$

$$\begin{aligned} \mathbb{E}^x \left(M_{t \wedge \tau_R}^f \mid \mathcal{F}_s \right) &= \mathbb{E}^x \left(M_{t \wedge \tau_R}^{\chi f} \mid \mathcal{F}_s \right) \\ &= M_{s \wedge \tau_R}^{\chi f} \\ &= M_{s \wedge \tau_R}^f. \end{aligned}$$

Since $(\tau_R)_R$ is a localizing sequence, we are done.

- c) A diffusion operator L satisfies that $c = 0$. Thus, the calculation for $L(\chi f)$ in part b) shows that

$$L(u\phi) - uL\phi - \phi Lu = \sum_{ij} a_{ij} \partial_i u \partial_j \phi = \nabla u(x) \cdot a(x) \nabla \phi(x).$$

This proves the second equality in the formula of the problem.

For the first we note that $d\langle M^u, M^\phi \rangle_t = dM_t^u dM_t^\phi$ (by the definition of the bracket process) and the latter we can calculate with the rules for Itô differentials. We have

$$dX_t^j = \sum_k \sigma_{jk}(X_t) dB_t^k + b_j(X_t) dt$$

and, by Itô's formula,

$$du(X_t) = \sum_j \partial_j u(X_t) dX_t^j + dt\text{-terms} = \sum_{j,k} \partial_j u(X_t) \sigma_{jk}(X_t) dB_t^k + dt\text{-terms}.$$

By definition,

$$dM_t^u = du(X_t) - Lu(X_t) dt = \sum_{j,k} \partial_j u(X_t) \sigma_{jk}(X_t) dB_t^k + dt\text{-terms}.$$

Thus, using that all terms containing $(dt)^2$ and $dB_t^k dt$ are zero, we get

$$\begin{aligned} dM_t^u dM_t^\phi &= \sum_{j,k} \sum_{l,m} \partial_j u(X_t) \partial_l \phi(X_t) \sigma_{jk}(X_t) \sigma_{lm}(X_t) dB_t^k dB_t^m \\ &= \sum_{j,k} \sum_{l,m} \partial_j u(X_t) \partial_l \phi(X_t) \sigma_{jk}(X_t) \sigma_{lm}(X_t) \delta_{km} dt \\ &= \sum_{j,l} \partial_j u(X_t) \partial_l \phi(X_t) \sum_k \sigma_{jk}(X_t) \sigma_{lk}(X_t) dt \\ &= \sum_{j,l} \partial_j u(X_t) \partial_l \phi(X_t) a_{jl} dt \\ &= \nabla u(X_t) \cdot a(X_t) \nabla \phi(X_t) \end{aligned}$$

where $a_{jl} = \sum_k \sigma_{jk}(X_t) \sigma_{lk}(X_t) = (\sigma \sigma^\top)_{jl}$. ($x \cdot y$ denotes the Euclidean scalar product and $\nabla = (\partial_1, \dots, \partial_d)^\top$.)

Alternative proof of the first equality: Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. Without loss of generality we may assume $u(x) = 0$. The equality

$$u^2(X_t) = \left(M_t^u + \int_0^t Lu(X_r) dr \right)^2$$

implies

$$(M_t^u)^2 = u^2(X_t) - 2M_t^u \int_0^t Lu(X_r) dr - \left(\int_0^t Lu(X_r) dr \right)^2.$$

Part a) shows that

$$u^2(X_t) - \int_0^t L(u^2)(X_r) dr$$

is a martingale. Moreover, since $(M_t^u)_{t \geq 0}$ is a martingale, we obtain by the tower property

$$\begin{aligned} &\mathbb{E}^x \left(2M_t^u \int_0^t Lu(X_r) dr \mid \mathcal{F}_s \right) \\ &= 2 \mathbb{E}^x (M_t^u \mid \mathcal{F}_s) \int_0^s Lu(X_r) dr + 2 \int_s^t \mathbb{E}^x \left(\mathbb{E}^x (M_t^u Lu(X_r) \mid \mathcal{F}_r) \mid \mathcal{F}_s \right) dr \\ &= 2M_s^u \int_0^s Lu(X_r) dr + 2 \mathbb{E}^x \left(\int_s^t M_r^u Lu(X_r) dr \mid \mathcal{F}_s \right). \end{aligned} \quad (*)$$

By the definition of M_t^u ,

$$\begin{aligned}
 & 2 \int_s^t M_r^u Lu(X_r) dr \\
 &= 2 \int_s^t u(X_r) Lu(X_r) dr - 2 \int_s^t \int_0^r Lu(X_v) dv Lu(X_r) dr \\
 &= 2 \int_s^t u(X_r) Lu(X_r) dr - 2 \int_s^t \int_0^s Lu(X_v) Lu(X_r) dv dr \\
 &\quad - 2 \int_s^t \int_s^r Lu(X_v) Lu(X_r) dv dr \\
 &= 2 \int_s^t u(X_r) Lu(X_r) dr - \left(\int_0^t Lu(X_r) dr \right)^2 + \left(\int_0^s Lu(X_r) dr \right)^2 \quad (**)
 \end{aligned}$$

using that

$$2 \int_s^t \int_s^r Lu(X_v) Lu(X_r) dv dr = \left(\int_s^t Lu(X_r) dr \right)^2.$$

Combining (*) and (**), we see that

$$2M_t^u \int_0^t Lu(X_r) dr + \left(\int_0^t Lu(X_r) dr \right)^2 - 2 \int_0^t u(X_r) Lu(X_r) dr$$

is a martingale. Consequently,

$$\langle M^u \rangle_t = \int_0^t (L(u^2) - 2uLu)(X_r) dr.$$

This proves the first equality for $u = \phi$. The formula for the quadratic covariation $\langle M^u, M^\phi \rangle$ follows by using polarization, i.e.

$$\langle M^u, M^\phi \rangle_t = \frac{1}{4} (\langle M^u + M^\phi \rangle_t - \langle M^u - M^\phi \rangle_t) = \frac{1}{4} (\langle M^{u+\phi} \rangle_t - \langle M^{u-\phi} \rangle_t).$$

■ ■

Problem 23.8. Solution: We assume, without loss of generality, that $d = 1$. If $d > 1$ this is just a matter of notation and the use of the chain rule. Thus, we assume that $|b(x)| + |b'(x)| + |b''(x)| \leq c(1 + |x|)$.

We begin with $\xi(t, x)$. By assumption, we get for all $t \in [0, T]$

$$\begin{aligned}
 |\xi(t, x)| &\leq |x| + \int_0^t |b(\xi(s, x))| ds \\
 &\leq |x| + c \int_0^t (1 + |(\xi(s, x))|) ds \\
 &\leq (cT + |x|) + c \int_0^t |(\xi(s, x))| ds
 \end{aligned}$$

From Gronwall's inequality (Theorem A.42) we see that

$$|\xi(t, x)| \leq (cT + |x|)e^{cT} \leq C_T(1 + |x|).$$

Define $\eta(t, x) := \frac{d}{dx}\xi(t, x)$. Since $\eta(t, x)$ satisfies the formally differentiated ODE, we know that

$$\eta(t, x) = 1 + \int_0^t b'(\xi(s, x))\eta(s, x) dt,$$

and so

$$|\eta(t, x)| \leq 1 + \int_0^t \|b'\|_\infty |\eta(s, x)| ds \leq 1 + c \int_0^t |\eta(s, x)| ds.$$

Again with Gronwall we get $|\eta(s, x)| \leq C_T(1 + |x|)$. Finally, set $\theta(t, x) := \frac{d^2}{dx^2} \xi(t, x) = \frac{d}{dx} \eta(t, x)$. This function satisfies the ODE

$$\theta(t, x) = \int_0^t (b''(\xi(s, x))\eta^2(s, x) + b'(\xi(s, x))\theta(s, x)) ds$$

which we can estimate by

$$\begin{aligned} |\theta(t, x)| &\leq \int_0^t (\|b''\|_\infty (1 + |x|)^2 + \|b'\|_\infty |\theta(s, x)|) ds \\ &\leq c_T(1 + |x|^2) + c \int_0^t |\theta(s, x)| ds. \end{aligned}$$

This leads to the estimate $|\theta(t, x)| \leq C_T(1 + |x|^2)$.

Remark: We have to assume Lipschitz continuity of the second derivatives in order to guarantee that the formally differentiated solution of order two satisfies the formally differentiated ODE, see e.g. Coddington & Levinson [40, Theorem I.7.5].

Problem 23.9. Solution:

- a) If we start with the expression on the r.h.S. of the equality in the statement, then we use indeed Itô's formula for $c(B_t)$ etc. etc. In fact, this is a bit unfair as it presupposes the knowledge of the r.h.S., arguing top-down. In this solution we prefer the following bottom-up approach.

We have $X_t = B_t + \int_0^t \nabla c(X_s) ds$. Using Girsanov's theorem (and the notation used there, use $f = -\nabla c$ and $t = T$, we have

$$q_t = \exp\left(-\int_0^t \nabla c(X_s) dB_s - \frac{1}{2} \int_0^t |\nabla c(X_s)|^2 ds\right)$$

(attention: vector notation: $\nabla c(X_s) dB_s = \sum_{j=1}^d \partial_j c(X_s) B_s^{(j)}$!!) and Girsanov's theorem says that X is under the measure $\mathbb{E}^{\mathbb{Q}}(\dots) = \mathbb{E}(e^{q_t} \dots)$ a Brownian motion. Thus,

$$\begin{aligned} \mathbb{E} \phi(X_t) &= \mathbb{E}(q_t q_t^{-1} \phi(X_t)) \\ &= \mathbb{E}^{\mathbb{Q}}(q_t^{-1} \phi(X_t)) \\ &= \mathbb{E}^{\mathbb{Q}}\left(\exp\left[\int_0^t \nabla c(X_s) dB_s + \frac{1}{2} \int_0^t |\nabla c(X_s)|^2 ds\right] \phi(X_t)\right) \end{aligned}$$

We want to express everything under $\mathbb{E}^{\mathbb{Q}}$ by x , so we need to get rid of the B . To do so, use Itô's formula for $c(X_t)$ — this is the difference to the above-mentioned top-down approach, i.e.

$$c(X_t) - c(x) = \int_0^t \nabla c(X_s) dX_s + \frac{1}{2} \int_0^t \Delta c(X_s) d\langle X \rangle_s$$

$$= \int_0^t \nabla c(X_s) dB_s + \int_0^t |\nabla c(X_s)|^2 ds + \frac{1}{2} \int_0^t \Delta c(X_s) ds$$

and we can solve this for the stochastic integral and replace the integral in the first calculation. This shows

$$\begin{aligned} & \mathbb{E} \phi(X_t) \\ &= \mathbb{E}^{\mathbb{Q}} \left(\exp \left[c(X_t) - c(x) - \frac{1}{2} \int_0^t \Delta c(X_s) ds - \int_0^t |\nabla c(X_s)|^2 ds + \frac{1}{2} \int_0^t |\nabla c(X_s)|^2 ds \right] \phi(X_t) \right) \\ &= \mathbb{E}^{\mathbb{Q}} \left(\exp \left[c(X_t) - c(x) - \frac{1}{2} \int_0^t \Delta c(X_s) ds - \frac{1}{2} \int_0^t |\nabla c(X_s)|^2 ds \right] \phi(X_t) \right) \\ &= \mathbb{E}^{\mathbb{Q}} \left(\exp \left[-\frac{1}{2} \int_0^t V(X_s) ds \right] \phi(X_t) e^{c(X_t) - c(x)} \right) \end{aligned}$$

and now replace $\mathbb{E}^{\mathbb{Q}}$ by \mathbb{E} and X by a Brownian motion B to get

$$= \mathbb{E} \left(\exp \left[-\frac{1}{2} \int_0^t V(B_s) ds \right] \phi(B_t) e^{c(B_t) - c(x)} \right).$$

- b) This part is more of an interpretation issue. We know from the Feynman–Kac formula (see § 8.3) that

$$P_t \phi(x) = w(t, x) = \mathbb{E}^x \phi(B_t) e^{\int_0^t \gamma(B_s) ds}$$

is a semigroup solving the PDE

$$\partial_t w(t, x) - \frac{1}{2} \Delta_x w(t, x) = \gamma(x) w(t, x), \quad w(0, x) = \phi(x).$$

The semigroup P_t defines a stochastic process. In our case we have

$$T_t^X \phi(x) = P_t(\phi e^c)(x) = T_t^Y(\phi e^c)(x) = e^{-c(x)} \mathbb{E}^x \left(\phi(B_t) e^{c(B_t)} e^{\int_0^t (-V(B_s)) ds} \right).$$

Problem 23.10. Solution: By stopping we can assume that M is a vector of L^2 -martingales. By definition, $(M^{(j)})^2 - \langle M^{(j)} \rangle$ is a martingale, and so $\mathbb{E}[(M^{(j)})_t^2 - \langle M^{(j)} \rangle_t] = \mathbb{E}[(M^{(j)})_0^2 - \langle M^{(j)} \rangle_0] = 0$. This proves that $M = 0$ a.s.

Problem 23.11. Solution: Throughout the proof we fix $t \leq T$ and we take a partition $\Pi = \{0 = s_0 < \dots < s_n = T\}$ of $[0, T]$; By Taylor's series we have

$$\begin{aligned} u(N_t) - u(N_0) &= \sum_{\Pi} [u(N_{s_k \wedge t}) - u(N_{s_{k-1} \wedge t})] \\ &= \underbrace{\sum_{\Pi} u'(N_{s_{k-1} \wedge t}) (N_{s_k \wedge t} - N_{s_{k-1} \wedge t})}_{=:\mathbb{I}(\Pi)} + \underbrace{\frac{1}{2} \sum_{\Pi} u''(\xi_k) (N_{s_k \wedge t} - N_{s_{k-1} \wedge t})^2}_{=:\frac{1}{2} \mathbb{J}(\Pi)}. \end{aligned}$$

The Taylor remainder term $u''(\xi_k) = \int_0^1 u''(N_{s_{k-1} \wedge t} + r(N_{s_k \wedge t} - N_{s_{k-1} \wedge t}))(1-r) dr$ is clearly measurable.

From Problem 17.6 we know that

$$I(\Pi) \xrightarrow[|\Pi| \rightarrow 0]{\text{ucp}} \int_0^t u'(N_s) dN_s.$$

Let us now deal with $J(\Pi)$. We write

$$\begin{aligned} J(\Pi) &= \sum_{\Pi} u''(N_{s_{k-1} \wedge t}) (N_{s_k \wedge t} - N_{s_{k-1} \wedge t})^2 + \sum_{\Pi} (u''(\xi_k) - u''(N_{s_{k-1} \wedge t})) (N_{s_k \wedge t} - N_{s_{k-1} \wedge t})^2 \\ &=: J_1(\Pi) + J_2(\Pi). \end{aligned}$$

We begin with J_2 . Set $m(\Pi) := \max_k |u''(\xi_k) - \partial_i \partial_j u(N_{s_{k-1} \wedge t})| \xrightarrow[|\Pi| \rightarrow 0]{} 0$ a.s. and in \mathbb{P} because of the continuity of $t \mapsto u(N_t)$. Thus,

$$J_2(\Pi) \leq m(\Pi) \sum_{\Pi} (N_{s_k \wedge t} - N_{s_{k-1} \wedge t})^2 \xrightarrow[\text{Problem 17.3}]{\text{ucp}} 0 \times \langle N \rangle_t = 0;$$

alternatively you could refer to Theorem 17.1 and use a stopping argument.

Write

$$f^{\Pi}(t) = \sum_{\Pi} f(s_{k-1}) \mathbb{1}_{[s_{k-1}, s_k)}(t) \quad \text{and} \quad \langle U, V \rangle_t^{\Pi} = \sum_{\Pi} (U_{s_k \wedge t} - U_{s_{k-1} \wedge t})(V_{s_k \wedge t} - V_{s_{k-1} \wedge t}).$$

With these notations we see that

$$J_1(\Pi) = \langle (u''(N))^{\Pi} \bullet N, N \rangle_t^{\Pi} \xrightarrow[|\Pi| \rightarrow 0]{} \langle (u''(N)) \bullet N, N \rangle_t = \int_0^t u''(N_s) d\langle N, N \rangle_s$$

where we argue as in the second part of the proof of Theorem 17.15. Since $\langle N, N \rangle = \langle N \rangle$ and since there is (see calculation above) also a factor $\frac{1}{2}$ in front of $J = J_1 + J_2$, the proof is complete. ■■

Problem 23.12. Solution: Obviously, the continuity of the sample paths ensures that $|N_{t \wedge \tau_n}| \leq n$. We have to show the localizing property. Since the paths $t \mapsto N_t$ are continuous, they are locally bounded, and so $\lim_n \tau_n = \infty$. To see the martingale property, fix $s < t$ and $F \in \mathcal{F}_s$ and pick a localizing sequence σ_k (which must exist, by definition). Then, by optional stopping,

$$\mathbb{E}[N_{t \wedge \tau_n \wedge \sigma_k} \mathbb{1}_F] = \mathbb{E}[N_{s \wedge \tau_n \wedge \sigma_k} \mathbb{1}_F].$$

Both sides are bounded for fixed n . Thus we can let $k \rightarrow \infty$ to get

$$\mathbb{E}[N_{t \wedge \tau_n} \mathbb{1}_F] = \mathbb{E}[N_{s \wedge \tau_n} \mathbb{1}_F].$$

and this means that the sequence $(\tau_n)_n$ is already localizing. ■■

Problem 23.13. Solution: Without loss of generality we take $d = 2$ and assume that $(B_t, W_t)_t$, $B_0 = W_0 = 0$, is a two-dimensional Brownian motion. Note that $B \perp W$. From the one-dimensional theory we know already that $B_t^2 - t$ and $W_t^2 - t$ are martingales. Fix $s < t$ and $F \in \mathcal{F}_s$. By independence,

$$\begin{aligned} & \mathbb{E}[(W_t B_t - W_s B_s) \mathbf{1}_F] \\ &= \mathbb{E}[(W_t - W_s)(B_t - B_s) \mathbf{1}_F] + \mathbb{E}[(B_t - B_s) W_s \mathbf{1}_F] + \mathbb{E}[(W_t - W_s) B_s \mathbf{1}_F] \\ &= \mathbb{E}(W_t - W_s) \mathbb{E}(B_t - B_s) \mathbb{E} \mathbf{1}_F + \mathbb{E}(B_t - B_s) \mathbb{E}[W_s \mathbf{1}_F] + \mathbb{E}(W_t - W_s) \mathbb{E}[B_s \mathbf{1}_F] \end{aligned}$$

where we use that $(W_t - W_s), (B_t - B_s) \perp F$, $W_t - W_s \perp B_t - B_s$ for the first term and $(B_t - B_s) \perp W_s, F$, resp., $(W_t - W_s) \perp B_s, F$ for the second and third terms, respectively. This shows that $(W_t B_t)_{t \geq 0}$ is a martingale, hence $\langle W, B \rangle_t = 0$ b/o the uniqueness of the compensator.

Alternatively, we could use polarization and check $4\langle W, B \rangle_t = \langle B+W \rangle_t - \langle B-W \rangle_t = 2t - 2t = 0$. But this is, essentially, the same calculation.

■■

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