Preface

Brownian motion is arguably the single most important stochastic process. Historically it was the first stochastic process in continuous time and with a continuous state space, and thus it shaped the study of Gaussian processes, martingales, Markov processes, diffusions and random fractals. Its central position within mathematics is matched by numerous applications in science, engineering and mathematical finance.

The present book grew out of several courses, which I taught at the University of Marburg and TU Dresden, and it owes a great debt to the lecture notes [199] by Lothar Partzsch who was my co-author in the first two editions. Many students are interested in applications of probability theory and it is important to teach Brownian motion and stochastic calculus at an early stage of the curriculum. Such a course is very likely the first encounter with stochastic processes in continuous time, following directly on an introductory course on rigorous (i.e. measure-theoretic) probability theory. Typically, students would be familiar with the classical limit theorems of probability theory and basic discrete-time martingales, as it is treated, for example, by Jacod & Protter *Probability Essentials* [124], Williams *Probability with Martingales* [270], or in the more voluminous textbooks by Billingsley [15] and Durrett [67].

General textbooks on probability theory cover, if at all, Brownian motion only briefly. On the other hand, there is a quite substantial gap to more specialized texts on Brownian motion which is not so easy to overcome for the novice. Our aim was to write a book which can be used in the classroom as an introduction to Brownian motion and stochastic calculus, and as a first course in continuous-time and continuous-state Markov processes. My aim was to have a text which is both a self-contained back-up and self-study text for contemporary applications (such as mathematical finance) and a foundation to more advanced monographs, e.g. Ikeda & Watanabe [113], Revuz & Yor [216] or Rogers & Williams [222] (for stochastic calculus), Marcus & Rosen [178] (for Gaussian processes), Peres & Mörters [186] (for random fractals), Chung [33] or Port & Stone [209] (for potential theory) or Blumenthal & Getoor [18] (for Markov processes) to name but a few.

Things the readers are expected to know. Our presentation is basically self-contained, starting from scratch with continuous-time stochastic processes. We do, however, assume some basic measure theory (as in [232]) and a first course on probability theory and discrete-time martingales (as in [124] or [270]). This material can also be found in my lecture notes [235].

How to read this book. Of course, nothing prevents you from reading it linearly. But there is more material here than one could cover in a one-semester course. Depending on your needs and likings, there are at least three possible selections: *BM and Itô calculus*, *BM and its sample paths* and *BM as a Markov process*. The diagram on page XI will give you some ideas how things depend on each other and how to construct your own "Brownian sample path" through this book.

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! Whenever special attention is needed and to point out traps & pitfalls, we have used the **!** sign in the margin. Also in the margin, there are cross-references to exercises Ex. n.m at the end of each chapter which we think fit (and are sometimes needed) at that point.[1] They are not just drill problems but contain variants, excursions from and extensions of the material presented in the text. The proofs of the core material do not seriously depend on any of the problems.

Changes to the third edition. After the publication of the first two editions, I got many positive responses from colleagues and students alike, and quite a few led to improvements in this new edition. I took the opportunity to correct misprints and make many changes and additions scattered throughout the text. The most significant changes are the completely rewritten Chapters 8 and § 23.3 on PDEs, the new sections on orthogonal random measures and white noise § 15.6, local times and Brownian excursions § 19.8 and the martingale problem § 23.4. I have also expanded the presentation of stochastic integrals driven by (local) martingales (Chapters 16 and 17) and added a proof of the Wiener chaos expansion via iterated Itô integrals (Chapter 20). Throughout the book there are exercises which come with a detailed online solution manual www.motapa.de/brownian_motion. Despite of all of these changes I did not try to make the book into an encyclopedia, but I wanted to keep the easily accessible, introductory character of the exposition, and there are still plenty of topics deliberately left out.

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Dresden, summer of 2021 *René L. Schilling*

[1] For the readers' convenience there is a web page where additional material and solutions are available. The URL is www.motapa.de/brownian_motion.

René L. Schilling

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With a Chapter on Simulation by Björn Böttcher

3rd Edition

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Online Resources

www.motapa.de/brownian_motion

Book Cover

The cover shows a photograph of the *Quantum Cloud* sculpture by Antony Gormley in London, almost directly on the Greenwich Meridian (51° 30' 6.48" N and 0° 00' 32.76" E). It is approximately 30 metres high and portrays a figure appearing in a cloud of tetrahedron-shaped metal pieces; the cloud around the figure was constructed with the help of a random walk algorithm.

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