

# Index

The index should be used in conjunction with the Bibliography and the Index of Notation. Numbers following entries are page numbers which, if accompanied by (Pr.  $n$ ), refer to Problem  $n$  on that page. Within the index we use the abbreviations “BM” for Brownian motion and “LIL” for law of iterated logarithm.

- absorbing point, 116
- acceptance–rejection method, 462
- action functional, 221, 228
- adapted, 48
- additive functional, 351, 360 (Pr. 17)
- additive noise, 439
  - cancels out, 443
- admissible filtration, 48, 62, 80, 243 (Pr. 1)
- almost positive, 123 (Pr. 21)
- arc-sine law
  - for largest zero, 79, 193
  - for the time in  $(0, \infty)$ , 137
- Banach–Steinhaus theorem, 496
- Blumenthal’s 0-1-law, 82, 143
- Borel–Cantelli lemma, 204
- bounded variation, 152, 217, 244, 283, 316 (Pr. 17), 494
- Box–Muller-method, 464
- Brownian bridge, 18 (Pr. 9), 46 (Pr. 6)
- Brownian motion, 1, 4
  - canonical model, 39, 41
  - characteristic function, 7, 11–13
  - characterization, 10–13, 50
  - Chung’s LIL, 208
  - Ciesielski’s construction, 23, 467
  - conformal invariance, 144
  - covariance, 8
  - dimension of graph, 183
  - dimension of image, 182, 186
  - dimension of zero set, 192
  - distribution of zeros, 193–195
  - Donsker’s construction, 35, 237, 467
  - with drift, 213, 323, 324, 402, 432, 439
  - embedding of random variable, 232
  - excursion, 195, 347, 350
  - excursion measure, 354
  - exit from annulus, 74
  - exit from ball, 75
  - exit from interval, 56, 74
  - exponential moments, 8
    - fast point, 201
    - first passage time, 56, 57, 70
    - is Gaussian process, 9, 10
    - generator, 92–93, 101–103
    - geometric, 386
    - Hölder continuity, 154, 170, 175, 180
    - independent components, 13
    - interpolation, 28–30, 466
    - invariance principle, 35, 237, 467
    - Khintchine’s LIL, 199, 212
    - killing, 458 (Pr. 9)
    - Kolmogorov’s construction, 36
    - large deviations, 223, 224
    - law of  $(\min, \max, B)$ , 77, 207
    - law of  $(|B|, L)$ ,  $(M - B, M)$ , 349
    - law of first passage time, 70
    - law of large numbers, 19 (Pr. 27), 198
    - law of running min/max, 70, 349
    - level set, 189, 190
    - Lévy’s construction, 28–30, 466
    - Lévy’s modulus of continuity, 172
    - LIL, 199, 208, 212, 225
    - LIL fails, 201
    - local modulus of continuity, 201
    - local time, 312, 399
    - Markov property, 15, 62, 63
    - martingale characterization, 163, 321, 451
    - maximal inequality, 50
    - modulus of non-differentiability, 209
    - moments, 8
    - nowhere differentiable, 170
    - nowhere monotone, 187
    - $\mathcal{P}$  measurable, 83
    - projective reflection, 16, 42
    - quadratic variation, 153, 165 (Pr. 5)
    - quadratic variation = 0, 160
    - quadratic variation, strong =  $\infty$ , 158
    - record set, 192
    - reflected at  $b$ , 73
    - reflection, 14, 42
    - reflection principle, 69–70, 77

- renewal, 14, 42
- resolvent kernel, 99
- return to zero, 75
- scaling, 15, 42
- shift, 47 (Pr. 7), 84 (Pr. 4), 351
- slow point, 201
- started at  $x$ , 4, 63
- Strassen’s LIL, 212, 225
- strong Markov property, 65, 67, 69, 71
- strong variation, 176 (Pr. 5)
- tail estimate, 172, 198
- time inversion, 15, 42
- transition density, 9, 36
- unbounded, 54
- unique maximum, 188
- white noise, 264
- Wiener’s construction, 26
- with filtration, 48
- zero set, 192, 348
- Brownian scaling, 15, 42
- Burkholder inequalities, 335
- Burkholder–Davis–Gundy inequalities, 334, 338
  
- $\mathcal{C}_{(0)}$  is not measurable, 42
- Cameron–Martin
  - formula, 215, 218, 224
  - space, 213, 219
- canonical model, 39, 41
- canonical process, 43, 90
- carré-du-champ operator, 457 (Pr. 7)
- Cauchy problem, 128, 129
- central limit method, 465
- chaos decomposition, 375
- Chapman–Kolmogorov equations, 65, 121 (Pr. 6)
- characteristic operator, 117
- Chung’s LIL, 208
- closed operator, 95
- completion, 80
- composition method, 463
- conditional expectation
  - under change of measure, 324
- cone condition, 143
  - flat cone condition, 150 (Pr. 12)
- conformal map, 144
- consistent probability measures, 43
- control measure, 263
- convex function (representation), 339
- covariance matrix, 7
- cylinder set, 40, 474
  
- diffusion coefficient/matrix, 3, 297, 390, 427, 428, 430, 431, 437
- diffusion process, 428, 429
  - backward equation, 430
  - construction by SDE, 437
  - defined by SDE, 435
  - forward equation, 431, 456
  - generator, 428
- Dirichlet problem, 139, 148
  - in dimension 2, 144
- dissipative operator, 103
- Doléans–Dade exponential, 317, 319
- dominated convergence (stoch. int.), 278
- Donsker’s invariance principle, 35, 237, 467
- Doob
  - decomposition, 244
  - maximal inequality, 484
  - optional stopping theorem, 488
- Doob–Meyer decomposition, 283
- Doss–Sussmann theorem, 423
- drift coefficient/vector, 297, 323, 390, 402, 428, 430, 431, 437
- drift transformation, 390, 402, 442
- Duhamel principle, 129
- Dynkin system, 492
- Dynkin’s formula, 115
- Dynkin–Kinney criterion, 428
  
- energy, 180, 497
- Euler scheme, 469, 472
  - strong/weak order, 470
- excursion, 347, 350
- excursion measure, 354
- exponential martingale, 49
  
- Feller process, 91, 491
  - starting at  $x$ , 90
  - strong Markov property, 491
- Feller property, 88
  - strong Feller property, 88
- Feller’s minimum principle, 107
- Feynman–Kac formula, 131, 458 (Pr. 9)
- Fick’s law, 427
- finite variation, *see* bounded variation
- first entrance time, *see* first hitting time
- first hitting time, 53
  - expected value, 56, 116, 140
  - of balls, 74, 75
  - properties, 60 (Pr. 13)

- is stopping time, 53, 54
- first passage time, 54, 56, 70
- Fokker–Planck equation, 431, 432
- formal adjoint operator, 431
- Friedrichs mollifier, 121 (Pr. 8), 311, 315 (Pr. 13)
- Frostman’s theorem, 180, 498, 499
- Fubini thm. (stoch. integral), 279
- fundamental solution, 432
  - of diffusion equation, 432
  - existence, 434
  
- Gaussian
  - density (n-dimensional), 9
  - process, 8
  - random variable, 7
  - tail estimate, 172, 198
- generalized inverse, 332–333
- generator, 92
  - of Brownian motion, 92–93, 101–103
  - is closed operator, 95
  - is dissipative, 103
  - Dynkin’s generator, 117
  - Feller’s minimum principle, 107
  - is local operator, 118, 119
  - no proper extension, 100
  - pointwise definition, 107
  - positive maximum principle, 106
  - of the solution of an SDE, 435
  - weak generator, 107
- geometric BM, 386
- Girsanov transform, 219, 325, 359 (Pr. 5), 402, 442
- Gronwall’s lemma, 500
  
- Haar functions, 22
  - completeness, 23, 501
  - Haar–Fourier series, 23
- harmonic function, 140
  - and martingales, 140
  - maximum principle, 142
- mean value property, 140
- Hausdorff dimension, 179
  - of a Brownian graph, 183
  - of a Brownian path, 182, 186
  - of Brownian zeros, 192
  - vs. capacitary dimension, 500
- Hausdorff measure, 177
- heat equation, 125, 127
- Hermite polynomial, 367
  
- heuristics
  - for the Cauchy problem, 129
  - for Dirichlet’s problem, 139
  - for Itô’s formula, 299
  - for Kolmogorov’s equations, 433
  - for an SDE, 383
  - for Strassen’s LIL, 225
  - for Stratonovich SDE, 421
- Hille–Yosida theorem, 103
- Hölder continuity, 154
  - of a BM, 154, 170, 175, 180
- holding point, 116
  
- independent increments (B1), 4
- infinitesimal generator, *see* generator
- instantaneous point, 116
- inverse transform method, 461
- Itô
  - differential *see* stoch. differential,
  - integral *see* stochastic integral,
  - isometry, 250, 251, 254, 267, 290
  - process, 297, 298, 306
- Itô’s formula
  - for  $BM^1$ , 297
  - for  $BM^d$ , 307
  - convex function, 340
  - for martingales, 459 (Pr. 11)
  - Itô–Meyer formula, 340
  - for Itô process, 306
  - and local time, 313, 399
  - Tanaka’s formula, 313, 399
  - time-dependent, 310
- Itô’s synthesis theorem, 355
- iterated Itô integral, 363, 366
  - and Hermite polynomials, 373, 377
  - multiplication formula, 381 (Pr. 9)
  
- Kaufman’s theorem, 186
- Khintchine’s LIL, 199, 212
- Kolmogorov
  - backward equation, 430
  - consistency condition, 43
  - construction of BM, 36, 44–45
  - continuity criterion, 45, 167
  - existence theorem, 475
  - forward equation, 431, 456
- Kolmogorov’s test
  - upper function, 201
- Kolmogorov–Chentsov thm, 45, 167

- Lamperti transform, 390
- Langevin SDE, 387
- Laplace operator, 50, 59 (Pr. 8), 106, 125, 128, 139
- large deviation
  - lower bound, 224
  - upper bound, 223
- Lebesgue’s spine, 145–146
- level set, 189, 190
- Lévy’s modulus of continuity, 172
- LIL, 199, 201, 206, 208, 212, 225
- linear growth condition, 394
- Lipschitz condition, 393
- local martingale, 274
  - bounded variation, 316 (Pr. 17)
  - is martingale, 319
- local operator, 118
  - and continuous paths, 118, 119
  - is differential operator, 118
  - and diffusions, 428
- local time, 312, 399
  - approximation by excursion, 357
  - continuity, 342
  - inverse of, 351
  - is additive functional, 351
  - joint law with BM, 349
  - moments, 342
  - occupation time formula, 341
  - quadratic variation, 343
  - support, 348
- localizing sequence, 273, 274, 459 (Pr. 12)
  
- Malliavin derivative, 378
- Markov kernel, 240
- Markov process, 64
  - homogeneous Markov process, 64
  - starting at  $x$ , 64
  - strong Markov process, 69
- Markov property, 15, 62, 63
  - solution of SDE, 403, 404
- Markov semigroup, 88
- martingale, 48
  - backwards convergence theorem, 482
  - backwards martingale, 157, 158, 482
  - bounded variation, 283
  - characterization of BM, 163, 321, 451
  - closure, 482
  - constancy intervals, 288
  - convergence theorem, 480
  - Doob–Meyer decomposition, 283
  - embedding into BM, 241, 333
  - exit from interval, 74
  - exponential, 49
  - and generator, 51, 114, 124
  - $L^2$  martingale, 244, 249
  - local martingale, 274
  - maximal inequality, 484
  - norm in  $\mathcal{M}_T^{2,c}$ , 252, 271 (Pr. 4)
  - quadratic variation, 163, 244–245, 283
  - upcrossing, 479
- martingale problem, 446
  - and SDEs, 453, 455
  - equivalent conditions, 448
  - existence, 453
  - uniqueness, 455
- martingale representation
  - as time-changed BM, 333
  - if the BM given, 328, 329
  - if the filtration is given, 330, 331
- martingale transform, 245
  - characterization by bracket, 248
- Maruyama’s drift transformation, 402, 442
- mass distribution principle, 179
- maximum principle, 142
- mean value property, 140
- minimum principle, 107
- modification, 25
- modulus of continuity, 171–172
- modulus of non-differentiability, 209
- monotone class theorem, 492, 493
- Monte Carlo simulation, 473
- multiple stochastic integral, *see iterated* – –
  
- natural filtration, 48, 53
- Novikov condition, 57, 320
  
- occupation time formula, 341
- optional stopping/sampling, 488
- Ornstein–Uhlenbeck process, 18 (Pr. 11), 387
  
- Paley–Wiener–Zygmund integral, 217–218, 272 (Pr. 17), 361
- path space, 41
- Poisson point process, 354–356
- Poisson process, 176 (Pr. 1), 352, 354, 356, 359 (Pr. 2)
- positive maximum principle, 106, 123 (Pr. 21)
- potential
  - linear potential, 111

- logarithmic potential, 111
- Newton potential, 111
- potential operator, *see* resolvent, 108, 111, 113
- progressive/progressively/  $\mathscr{P}$
- measurable, 83, 272 (Pr. 22), 281 (Pr. 2)
- $\sigma$ -algebra, 260, 272 (Pr. 21)
- projective family, 43, 475
- projective limit, 475
  
- Q-Brownian motion, 13
- quadratic covariation, 247, 293, 371, 457 (Pr. 7)
- quadratic variation, 254, 283, 297
  - a.s. convergence, 155, 157
  - of BM, 153, 165 (Pr. 5)
  - is zero, 160
  - $L^2$ -convergence, 153
  - of martingale, 163
  - of local time, 343
  - strong q.v. =  $\infty$ , 158
- quasi left continuous, 60 (Pr. 16)
  
- random measure, 37 (Pr. 2)
- random orthogonal measure, 263
  - Itô isometry, 267
  - martingales, 268
  - stochastic integral, 265, 267
- random variable
  - embedded into BM, 232
- random walk, 2–3, 34
  - Chung’s LIL, 206
  - embedded into BM, 234
  - Khinchine’s LIL, 235–236
- recurrence, 76
- reflection principle, 69–70, 77
- regular point, 142
  - criterion for, 143, 150 (Pr. 12)
  - dimension 2, 144
- resolvent, 87, 96
  - resolvent equation, 97, 122 (Pr. 15)
- Riemann mapping theorem, 144
- Riemann–Stieltjes integral, 218, 244, 494–496
- Riesz representation theorem, 496
- right continuous filtration, 81
  
- sample path, 4
- sample value, 460
- Schauder functions, 22
  - completeness, 23, 501
- Schilder’s theorem, 222, 224
  
- semigroup, 87
  - conservative, 88, 92
  - contraction, 88
  - Feller, 88
  - Markov, 88
  - positive, 88
  - strong Feller, 88
  - strongly continuous, 88
  - sub-Markov, 88
- shift operator, 47 (Pr. 7), 84 (Pr. 4), 351
- simple function (off-diagonal), 362
- simple process, 249
  - closure, 262, 290
- simply connected, 144
- simulation of BM, 466, 467
- singular point, 142
  - examples, 144–147
- Skorokhod decomposition, 360 (Pr. 16)
- Skorokhod’s embedding theorem, 233
- Slutsky’s theorem, 5 (Pr. 2), 5 (Pr. 3)
- state space, 4
  - one-point compactification, 92, 428
- stationary increments (B2), 4
- stationary process, 19 (Pr. 23)
- stochastic differential, 299, 303
- stochastic differential equation, 383
  - continuity of solution, 408, 411
  - counterexamples, 398–402
  - dependence on initial value, 408, 411, 412
  - deterministic coefficients, 385
  - differentiability of solution, 411
  - of diffusion process, 435
  - drift transformation, 390, 402
  - examples, 385–389, 398–402, 422–423
  - existence of solution, 396
  - Feller continuity, 407
  - for given generator, 437
  - heuristics, 383
  - homogeneous linear SDE, 388
  - Langevin equation, 387
  - linear growth condition, 394
  - linear SDE, 389
  - Lipschitz condition, 393
  - local existence and uniqueness, 405
  - localization, 404
  - Markov property of solutions, 403
  - and martingale problem, 453, 455
  - measurability of solution, 398
  - moment estimate, 408, 410

- no solution, 398–402
- numerical solution, 469, 472
- Picard iteration, 396
- solution map, 423
- solution of SDE, 383
- stability of solutions, 394
- Stratonovich SDE, 421
- transformation into linear SDE, 392
- transformation of SDE, 385, 390–391
- transition function of solutions, 404
- uniqueness of solutions, 396, 402
- variance transformation, 390
- weak solution, 400
- stochastic dominated conv., 278
- stochastic exponential, 317, 319
- stochastic Fubini thm., 279
- stochastic integral, 253, 265, 267
  - characterization, 294
  - dominated convergence, 278
  - Fubini, 279
  - Itô isometry, 254
  - iterated, 363, 366
  - localization, 254
  - is martingale, 253
  - maximal inequalities, 254
  - mean-square cont. integrand, 259
  - no pathwise meaning, 259–260
  - Riemann approximation, 259, 276
  - Stratonovich integral, 419
  - stochastic integral (for  $\mathcal{M}_T^{2,c}$ ), 289
    - Itô isometry, 290
    - localization, 290
    - is martingale, 289
    - maximal inequalities, 290
  - stochastic integral (localized), 274, 291
    - characterization, 294
    - localization, 275
    - is martingale, 275
    - Riemann approximation, 276
  - stochastic integral (simple proc.), 249
  - Itô isometry, 250, 251
  - localization, 251
  - is martingale, 251
- stochastic process, 4
  - equivalent, 25
  - independence of two processes, 11
  - indistinguishable, 24
  - modification, 25
  - sample path, 4
  - simple, 249
  - state space, 4
  - stopping time, 53, 485
  - approximation from above, 487
  - Strassen’s LIL, 212
  - Stratonovich differential equation, 421
  - Stratonovich integral, 419
    - vs. Itô integral, 420
  - strong Feller property, 88
  - strong Markov property, 65, 67, 69, 71, 491
    - martingale proof, 66
  - strong order of convergence, 470
  - strong variation, 152, 244, 283, 493
  - sub-Markov semigroup, 88
    - vs. Markov semigroup, 92
  - symmetrization (function), 364
  - Tanaka’s formula, 313, 399
  - Taylor–Stroock formula, 379
  - trajectory, *see* sample path
  - transience, 76
  - transition kernel, 90
  - transition semigroup, *see* semigroup
  - true sample, 460
  - ucp-convergence, 274
    - and localization, 293
  - uniform convergence in probability, 274
  - upcrossing estimate, 480
  - upper function, 201
  - usual conditions, 82
  - variation
    - *see also* bounded v., quadratic, v., strong v.,
    - $p$ -variation, 152
    - variation sum, 152, 493
  - Wald’s identity, 55, 74
    - exponential Wald identity, 57
  - weak order of convergence, 470
  - white noise, 264
    - is not  $\sigma$ -additive, 264
  - Wiener chaos, 375
  - Wiener measure, 41, 213
  - Wiener process, *see* Brownian motion
  - Wiener space, 41, 213
  - Wiener–Fourier series, 26
  - Wiener–Itô decomposition, 375

Zaremba

– cone condition, 143

– deleted ball, 144

– needle, 147

René L. Schilling

# **Brownian Motion**

---

A Guide to Random Processes and Stochastic Calculus

With a Chapter on Simulation by Björn Böttcher

3rd Edition

**DE GRUYTER**



**Mathematics Subject Classification 2020**

Primary: 60-01, 60J65; Secondary: 60H05, 60H10, 60J35, 60G46, 60J60, 60J25.

**Author**

Prof. Dr. René L. Schilling  
Technische Universität Dresden  
Institut für Mathematische Stochastik  
D-01062 Dresden  
Germany  
rene.schilling@tu-dresden.de  
<https://tu-dresden.de/mn/math/stochastik/schilling>

**Online Resources**

[www.motapa.de/brownian\\_motion](http://www.motapa.de/brownian_motion)

**Book Cover**

The cover shows a photograph of the *Quantum Cloud* sculpture by Antony Gormley in London, almost directly on the Greenwich Meridian (51° 30' 6.48" N and 0° 00' 32.76" E). It is approximately 30 metres high and portrays a figure appearing in a cloud of tetrahedron-shaped metal pieces; the cloud around the figure was constructed with the help of a random walk algorithm.

ISBN 978-3-11-074125-4

e-ISBN (PDF) 978-3-11-074127-8

e-ISBN (EPUB) 978-3-11-074149-0

**Library of Congress Cataloging-in-Publication Data**

A CIP catalog record for this book has been applied for at the Library of Congress.

**Bibliographic information published by the Deutsche Nationalbibliothek**

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at <http://dnb.dnb.de>.

© 2021 Walter de Gruyter GmbH, Berlin/Boston

Cover image: Anthony Gormley Figure; Tony Hisgett from Birmingham, UK, CC BY 2.0

Typesetting: Typeset by the author

Printing and binding: CPI buch bücher.de GmbH, Birkach

☼ Printed on acid-free paper

Printed in Germany

[www.degruyter.com](http://www.degruyter.com)