Preface

Bernstein functions and the important subclass of complete Bernstein functions appear in various fields of mathematics—often with different definitions and under different names. Probabilists, for example, know Bernstein functions as Laplace exponents, and in harmonic analysis they are called negative definite functions. Complete Bernstein functions are used in complex analysis under the name Pick or Nevanlinna functions, while in matrix analysis and operator theory, the name operator monotone function is more common. When studying the positivity of solutions of Volterra integral equations, various types of kernels appear which are related to Bernstein functions. There exists a considerable amount of literature on each of these classes, but only a handful of texts observe the connections between them or use methods from several mathematical disciplines.

This book is about these connections. Although many readers may not be familiar with the name *Bernstein function*, and even fewer will have heard of *complete Bernstein functions*, we are certain that most have come across these families in their own research. Most likely only certain aspects of these classes of functions were important for the problems at hand and they could be solved on an *ad hoc* basis. This explains quite a few of the rediscoveries in the field, but also that many results and examples are scattered throughout the literature; the exceedingly rich structure connecting this material got lost in the process. Our motivation for writing this book was to point out many of these connections and to present the material in a unified way. We hope that our presentation is accessible to researchers and graduate students with different backgrounds. The results as such are mostly known, but our approach and some of the proofs are new: we emphasize the structural analogies between the function classes which we believe is a very good way to approach the topic. Since it is always important to know explicit examples, we took great care to collect many of them in the tables which form the last part of the book.

Completely monotone functions—these are the Laplace transforms of measures on the half-line $[0,\infty)$—and Bernstein functions are intimately connected. The derivative of a Bernstein function is completely monotone; on the other hand, the primitive of a completely monotone function is a Bernstein function if it is positive. This observation leads to an integral representation for Bernstein functions: the Lévy-Khintchine formula on the half-line

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt), \quad \lambda > 0.$$

Although this is familiar territory to a probabilist, this way of deriving the Lévy-Khintchine formula is not the usual one in probability theory. There are many more connections between Bernstein and completely monotone functions. For example, f is a Bernstein function if, and only if, for all completely monotone functions g the composition $g \circ f$ is completely monotone. Since g is a Laplace transform, it is enough to check this for the kernel of the Laplace transform, i.e. the basic completely monotone functions $g(\lambda) = e^{-t\lambda}$, $t > 0$.

A similar connection exists between the Laplace transforms of completely monotone functions, that is, *double Laplace* or *Stieltjes transforms*, and *complete* Bernstein functions. A function f is a complete Bernstein function if, and only if, for each $t > 0$ the composition $(t + f(\lambda))^{-1}$ of the Stieltjes kernel $(t + \lambda)^{-1}$ with f is a Stieltjes function. Note that $(t + \lambda)^{-1}$ is the Laplace transform of $e^{-t\lambda}$ and thus the functions $(t + \lambda)^{-1}$, $t > 0$, are the basic Stieltjes functions. With some effort one can check that complete Bernstein functions are exactly those Bernstein functions where the measure μ in the Lévy-Khintchine formula has a completely monotone density with respect to Lebesgue measure. From there it is possible to get a surprising geometric characterization of these functions: they are non-negative on $(0, \infty)$, have an analytic extension to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$ and preserve upper and lower half-planes. A familiar sight for a classical complex analyst: these are the Nevanlinna functions. One could go on with such connections, delving into continued fractions, continue into interpolation theory and from there to operator monotone functions ...

Let us become a bit more concrete and illustrate our approach with an example. The fractional powers $\lambda \mapsto \lambda^\alpha$, $\lambda > 0$, $0 < \alpha < 1$, are easily among the most prominent (complete) Bernstein functions. Recall that

$$f_\alpha(\lambda) := \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-\lambda t}\right) t^{-\alpha-1}\, dt. \tag{1}$$

Depending on your mathematical background, there are many different ways to derive and to interpret (1), but we will follow probabilists' custom and call (1) the Lévy-Khintchine representation of the Bernstein function f_α. At this point we do not want to go into details, instead we insist that one should read this formula as an integral representation of f_α with the kernel $(1 - e^{-\lambda t})$ and the measure $c_\alpha\, t^{-\alpha-1}\, dt$.

This brings us to negative powers, and there is another classical representation

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^{\beta-1}\, dt, \quad \beta > 0, \tag{2}$$

showing that $\lambda \mapsto \lambda^{-\beta}$ is a completely monotone function. It is no accident that the reciprocal of the Bernstein function λ^α, $0 < \alpha < 1$, is completely monotone, nor is it an accident that the representing measure $c_\alpha\, t^{-\alpha-1}\, dt$ of λ^α has a completely monotone density. Inserting the representation (2) for $t^{-\alpha-1}$ into (1) and working out the double integral and the constant, leads to the second important formula for the fractional powers,

$$\lambda^\alpha = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{\lambda}{\lambda + t}\, t^{\alpha-1}\, dt. \tag{3}$$

We will call this representation of λ^α the Stieltjes representation. To explain why this is indeed an appropriate name, let us go back to (2) and observe that $t^{\alpha-1}$ is a Laplace transform. This shows that $\lambda^{-\alpha}$, $\alpha > 0$, is a double Laplace or Stieltjes transform. Another non-random coincidence is that

$$\frac{f_\alpha(\lambda)}{\lambda} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{1}{\lambda + t}\, t^{\alpha-1}\, dt$$

is a Stieltjes transform and so is $\lambda^{-\alpha} = 1/f_\alpha(\lambda)$. This we can see if we replace $t^{\alpha-1}$ by its integral representation (2) and use Fubini's theorem:

$$\frac{1}{f_\alpha(\lambda)} = \lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{1}{\lambda + t}\, t^{-\alpha}\, dt. \tag{4}$$

It is also easy to see that the fractional powers $\lambda \mapsto \lambda^\alpha = \exp(\alpha \log \lambda)$ extend analytically to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$. Moreover, z^α maps the upper half-plane into itself; actually it contracts all arguments by the factor α. Apart from some technical complications this allows to surround the singularities of f_α—which are all in $(-\infty, 0)$—by an integration contour and to use Cauchy's theorem for the half-plane to bring us back to the representation (3).

Coming back to the fractional power λ^α, $0 < \alpha < 1$, we derive yet another representation formula. First note that $\lambda^\alpha = \int_0^\lambda \alpha s^{-(1-\alpha)}\, ds$ and that the integrand $s^{-(1-\alpha)}$ is a Stieltjes function which can be expressed as in (4). Fubini's theorem and the elementary equality

$$\int_0^\lambda \frac{1}{t+s}\, ds = \log\left(1 + \frac{\lambda}{t}\right)$$

yield

$$\lambda^\alpha = \frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \log\left(1 + \frac{\lambda}{t}\right)\, t^{\alpha-1}\, dt. \tag{5}$$

This representation will be called the Thorin representation of λ^α. Not every complete Bernstein function has a Thorin representation. The critical step in deriving (5) was the fact that the derivative of λ^α is a Stieltjes function.

What has been explained for fractional powers can be extended in various directions. On the level of functions, the structure of (1) is characteristic for the class $\mathcal{BF}$ of Bernstein functions, (3) for the class $\mathcal{CBF}$ of complete Bernstein functions, and (5) for the Thorin-Bernstein functions $\mathcal{TBF}$. If we consider $\exp(-tf)$ with f from $\mathcal{BF}$, $\mathcal{CBF}$ or $\mathcal{TBF}$, we are led to the corresponding families of completely monotone functions and measures. Apart from some minor conditions, these are the infinitely divisible distributions ID, the Bondesson class of measures BO and the generalized Gamma convolutions GGC. The diagrams in Remark 9.17 illustrate these connections. If we replace (formally) λ by $-A$, where A is a negative semi-definite matrix or a dissipative closed operator, then we get from (1) and (2) the classical formulae for fractional powers, while (3) turns into Balakrishnan's formula. Considering $\mathcal{BF}$ and $\mathcal{CBF}$ we obtain a fully-fledged functional calculus for generators and potential operators. Since complete Bernstein functions are operator monotone functions we can even recover the famous Heinz-Kato inequality.

Let us briefly describe the content and the structure of the book. It consists of three parts. The first part, Chapters 1–10, introduces the basic classes of functions: the positive definite functions comprising the completely monotone, Stieltjes and Hirsch functions, and the negative definite functions which consist of the Bernstein functions and their subfamilies—special, complete and Thorin-Bernstein functions. Two probabilistic intermezzi explore the connection between Bernstein functions and certain classes of probability measures. Roughly speaking, for every Bernstein function f the functions $\exp(-tf)$, $t > 0$, are completely monotone, which implies that $\exp(-tf)$ is the Laplace transform of an infinitely divisible sub-probability measure. This part of the book is essentially self-contained and should be accessible to non-specialists and graduate students.

In the second part of the book, Chapter 11 through Chapter 14, we turn to applications of Bernstein and complete Bernstein functions. The choice of topics reflects our own interests and is by no means complete. Notable omissions are applications in integral equations and continued fractions.

Among the topics are the spectral theorem for self-adjoint operators in a Hilbert space and a characterisation of all functions which preserve the order (in quadratic form sense) of dissipative operators. Bochner's subordination plays a fundamental role in Chapter 12 where also a functional calculus for subordinate generators is developed. This calculus generalizes many formulae for fractional powers of closed operators. As another application of Bernstein and complete Bernstein functions we establish estimates for the eigenvalues of subordinate Markov processes. This is continued in Chapter 13 which contains a detailed study of excessive functions of killed and subordinate killed Brownian motion. Finally, Chapter 14 is devoted to two results in the theory of generalized diffusions, both related to complete Bernstein functions through Kreĭn's theory of strings. Many of these results appear for the first time in a monograph.

The third part of the book is formed by extensive tables of complete Bernstein functions. The main criteria for inclusion in the tables were the availability of explicit representations and the appearance in mathematical literature.

In the Appendix we collect, for the readers' convenience, some supplementary results.

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