# Part I

# An Introduction to Lévy and Feller Processes

By René L. Schilling

# Preface

These lecture notes are an extended version of my lectures on Lévy and Lévy-type processes given at the *Second Barcelona Summer School on Stochastic Analysis* organized by the *Centre de Recerca Matemàtica* (CRM). The lectures are aimed at advanced graduate and PhD students. In order to read these notes, one should have sound knowledge of measure theoretic probability theory and some background in stochastic processes, as it is covered in my books *Measures, Integals and Martingales* [54] and *Brownian Motion* [56].

My purpose in these lectures is to give an introduction to Lévy processes, and to show how one can extend this approach to space inhomogeneous processes which behave locally like Lévy processes. After a brief overview (Chapter 1) I introduce Lévy processes, explain how to characterize them (Chapter 2) and discuss the quintessential examples of Lévy processes (Chapter 3). The Markov (loss of memory) property of Lévy processes is studied in Chapter 4. A short analytic interlude (Chapter 5) gives an introduction to operator semigroups, resolvents and their generators from a probabilistic perspective. Chapter 6 brings us back to generators of Lévy processes which are identified as pseudo-differential operators whose symbol is the characteristic exponent of the Lévy process. As a by-product we obtain the Lévy–Khintchine formula.

Continuing this line, we arrive at the first construction of Lévy processes in Chapter 7. Chapter 8 is devoted to two very special Lévy processes: (compound) Poisson processes and Brownian motion. We give elementary constructions of both processes and show how and why they are special Lévy processes, indeed. This is also the basis for the next chapter (Chapter 9) where we construct a random measure from the jumps of a Lévy process. This can be used to provide a further construction of Lévy processes, culminating in the famous Lévy–Itô decomposition and yet another proof of the Lévy–Khintchine formula.

A second interlude (Chapter 10) embeds these random measures into the larger theory of random orthogonal measures. We show how we can use random orthogonal measures to develop an extension of Itô's theory of stochastic integrals for square-integrable (not necessarily continuous) martingales, but we restrict ourselves to the bare bones, i.e., the $L^2$-theory. In Chapter 11 we introduce Feller processes as the proper spatially inhomogeneous brethren of Lévy processes, and we show how our proof of the Lévy–Khintchine formula carries over to this setting. We will see, in particular, that Feller processes have a **symbol** which is the state-space-dependent analogue of the characteristic exponent of a Lévy process. The symbol describes the process and its generator. A probabilistic way to calculate the symbol and some first consequences (in particular the semimartingale decomposition of Feller processes) is discussed in Chapter 12; we also show that the symbol contains information on global properties of the process, such as conservativeness. In the final Chapter 13, we summarize (mostly without proofs) how other path properties of a Feller process can be obtained via the symbol. In order to make these notes self-contained, we collect in the appendix some material which is not always included in standard graduate probability courses.

It is now about time to thank many individuals who helped to bring this enterprise on the way. I am grateful to the scientific board and the organizing committee for the kind invitation to deliver these lectures at the *Centre de Recerca Matemàtica* in Barcelona. The CRM is a wonderful place to teach and to do research, and I am very happy to acknowledge their support and hospitality. I would like to thank the students who participated in the CRM course as well as all students and readers who were exposed to earlier (temporally & spatially inhomogeneous…) versions of my lectures; without your input these notes would look different!

I am greatly indebted to Ms. Franziska Kühn for her interest in this topic; her valuable comments pinpointed many mistakes and helped to make the presentation much clearer.

And, last and most, I thank my wife for her love, support and forbearance while these notes were being prepared.

Dresden, September 2015 *René L. Schilling*

## Part II

# Invariance and Comparison Principles for Parabolic Stochastic Partial Differential Equations

By Davar Khoshnevisan

# Preface

These notes aim to introduce the reader to aspects of the theory of parabolic stochastic partial differential equations (SPDEs, for short). As an example of the type of object that we wish to study, let us consider the following boundary value problem: we aim to find a real-valued space-time function $(t\,,x) \mapsto u_t(x)$, where $t \geqslant 0$ and $x \in [0\,,1]$, such that

$$\left[\begin{array}{lr} \dot{u}_t(x) = u_t''(x) + \sigma(u_t(x))\xi_t(x) & \text{for } t > 0 \text{ and } 0 < x < 1, \\ u_0(x) = \sin(2\pi x) & \text{for } 0 < x < 1, \\ u_t(0) = u_t(1) = 0 & \text{for all } t > 0. \end{array}\right. \tag{13.1}$$

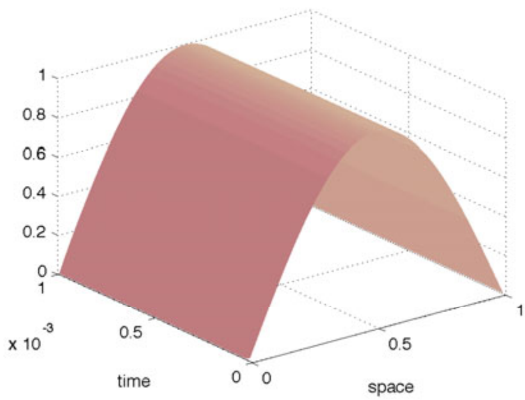
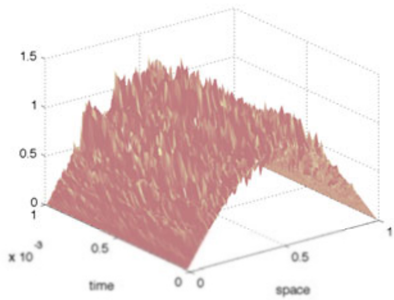


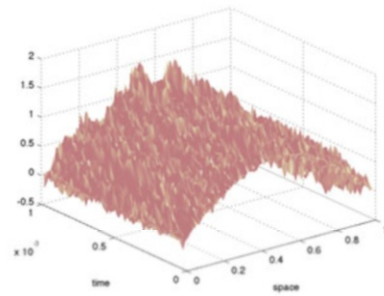

Figure 13.1: A numerical evaluation of the heat equation, where $\sigma(u) \equiv 0$.

We have written $u_t(x)$ in place of the more commonplace notation $u(t\,,x)$, as it is more natural in the probabilistic context. Thus, $u_t$ designates the map $t \mapsto u$ and not the time derivative $\partial u/\partial t$.

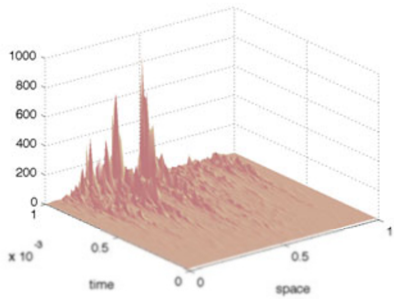
If $\sigma\colon \mathbb{R} \to \mathbb{R}$ and $\xi\colon [0\,,\infty)\times\mathbb{R} \to \mathbb{R}$ are sufficiently smooth then the preceding is a classical problem of the theory of heat flow, the solution exists, is unique, and has good regularity properties; see, for instance, Evans [17, Chapter 2, §2.3].



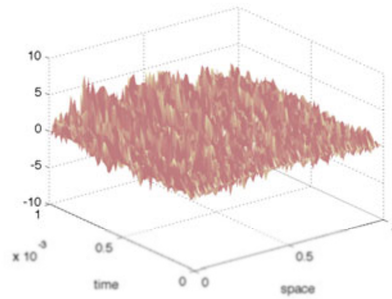

(a) A simulation of the stochastic heat equation where $\sigma(u) = u$.



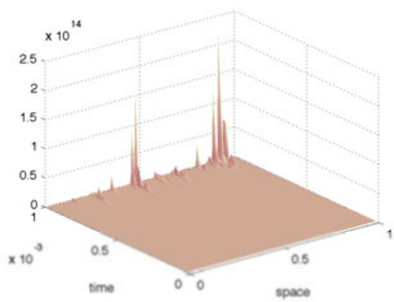

(b) A simulation of the stochastic heat equation where $\sigma(u) = 1$.



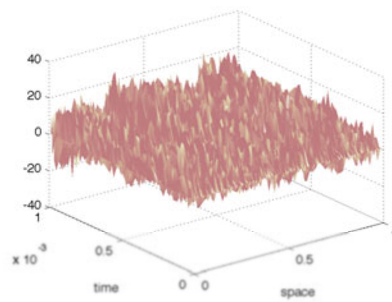

(c) A simulation of the stochastic heat equation where $\sigma(u) = 10u$.




(d) A simulation of the stochastic heat equation where $\sigma(u) = 10$.




(e) A simulation of the stochastic heat equation where $\sigma(u) = 50u$.




(f) A simulation of the stochastic heat equation where $\sigma(u) = 50$.

Figure 13.2: The left column consists of simulations of (13.1) where $\sigma(u) = \lambda u$, and the right column is for $\sigma(u) = \lambda$, as $\lambda$ ranges in $\{1\,,10\,,50\}$.

Consider an ideal rod of length one unit, and identify the rod with the interval $[0\,,1]$. Suppose the rod is heated at time $t = 0$ such that the heat density at every point $x \in [0\,,1]$ ($x$ units along the rod) is $\sin(2\pi x)$. Then it can be argued using Fourier's law of thermal conduction that, under ideal conditions, the heat density $u_t(x)$ at place $x \in [0\,,1]$ and at time $t > 0$ solves the linear heat equation $\dot{u} = \nu u''$, subject to $u_0(x) = \sin(2\pi x)$. Here, $\nu$ is a physical constant and is sometimes called "thermal conductivity", in this context.

We can always scale the problem so that $\nu = 1$. Indeed, if $\dot{u} = \nu u''$ then $F_t(x) := u_{t/\nu}(x)$ solves the heat equation $\dot{F} = F''$, subject to the same initial and boundary conditions as $u$. In this way, we arrive at (13.1) with $\sigma := \xi := 0$.

Suppose that the rod also feels external density $\xi_t(x)$ of heat (or cold, if $\xi_t(x) < 0$) at the point $(t\,,x)$. Then, the heat density solves $\dot{u} = u'' + \xi$. That is, (13.1) with $\sigma(u) \equiv 1$.

The general form of (13.1) arises when the external heating/cooling source interacts with the heat flow on the rod due to the presence of one or more feedback systems. In that case, the function $\sigma$ models the nature of the feedback mechanism.

The main goal of these notes is to study the heat-flow problem (13.1) in the case where$\xi$ denotes "space-time white noise" (a notion defined carefully below). For the time being, we can think of the $\xi_t(x)$'s as a collection of independent mean-zero normal random variables. In this sense, (13.1) describes heat flow in a random environment.

We plan to study how the solution depends on the nonlinearity $\sigma$. In order to motivate this, consider the simplest case that $\sigma \equiv 0$. In that case, we can solve $u$ explicitly, and find that $u_t(x) = \exp(-4\pi^2 t)\sin(2\pi x)$ for all $t \geqslant 0$ and $x \in [0\,,1]$, when $\sigma(u) \equiv 0$.

Figure 13.1 shows a numerical evaluation of the solution for time values $t \in [0\,,10^{-3}]$. Figures 13.2(a) and 13.2(b) show typical simulations of the solution for $\sigma(u) = u$ and $\sigma(u) = 1$, respectively. Figures 13.2(c) and 13.2(d) do the same thing for $\sigma(u) = 10u$ and $\sigma(u) = 10$, respectively. And Figures 13.2(e) and 13.2(f) for $\sigma(u) = 50u$ and $\sigma(u) = 50$. A quick inspection of these suggests that the behavior of the solution to (13.1) depends critically on the properties of the nonlinearity $\sigma$. In the last chapter of these notes, an answer on how this phenomenon can arise will be provided.

These notes are based on lectures given in the summer of 2014 at the *Second Summer School on Stochastic Analysis* held at the *Centre de Recerca Matemàtica* (CRM) in Barcelona. I would like to thank the CRM for their generous hospitality. Many hearty thanks are owed to the organizing and scientific committee, David Applebaum, Robert Dalang, Lłuis Quer-Sardanyons, Marta Sanz-Solé, Frederic Utzet, and Josep Vives for their kind invitation.

The material of these notes is based on my collaborations with Kunwoo Kim [33, 34], as well as Mathew Joseph and Carl Mueller [30]. I thank all three for many years of extremely enjoyable scientific discourse. Many thanks are due to

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Salt Lake City (Utah), August, 2014 *Davar Khoshnevisan*